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SOME NEW RESULTS ON IMPLICIT TEMPERED-CAPUTO FRACTIONAL VOLTERRA-FREDHOLM SYSTEM

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Abstract. Our study focuses on tempered-Caputo implicit fractional Volterra-Fredholm systems (TCIFV-FSs) with retarded and advanced delays in Banach spaces (BS) with nonlocal conditions. We use Sadovskii's fixed point theorem (FPT) and measure of noncompactness to prove that TCIFV-FSs in BS have solutions under nonlocal conditions, given appropriate assumptions. Furthermore, we show that the solution is Ulam-Hyers stable (U-HS). These results add valuable insights to the existing research on this sector. Finally, we present an example to highlight the key findings.

1. Introduction

Fractional differential equations (FDEs), with and without delay, occur naturally in a variety of scientific and engineering applications. These include, but are not limited to, physics, engineering methods, mechanical systems, atomic energy, economics, control theory, chemistry, harmonic, biology, medicine information theory, applied sciences and nonlinear oscillators, conservative systems, the stability and instability of geodesics on Riemannian manifolds, and Hamiltonian system dynamics [4,9–11]. The qualitative analysis of linear and nonlinear FDEs has garnered significant research attention, as indicated by the publications cited in [17,19,23], and their references. Tempered fractional calculus (FC) is a unique approach to fractional derivatives (FD) that includes functions with exponentially decaying tails. This concept has been rediscovered at least twice, under several titles [5, 12, 13]. It's interesting since it's the only place where FC with analytic kernels and weighted FC intersect. It has been utilized in Lévy processes such as Brownian motion and to study turbulence in geophysical flows. Furthermore, it is especially useful in applications where memory effects are critical, such as

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viscoelastic materials, nonlocal models in physics, and fractional-order control systems. The tempered-Caputo derivative is a variation on the classical Caputo FD in which the underlying convolution kernel is exponentially shortened [1,8,15,16,21].

The tempered FD provides a more realistic depiction of the underlying dynamics, including both long-range memory and fast-decaying tendencies. Further details on this subject can be found in [3,5,14,15]. U-HS is an area of mathematics that studies the stability of specific functional equations. The U-HS theory examines how functional equation solutions behave when input parameters change somewhat. The U-HS theory often answers the following question: If an equation holds generally, how close to the exact solution can it get? The theory provides a framework for investigating the stability of functional equations from several disciplines of mathematics, such as number theory, differential equations, and functional analysis. In the 1940s, Ulam and Hyers independently developed the U-HS theory, which bears their names. Since then, numerous research have looked at the stability of various differential and integral equations. For more information, please refer to [14,22] and its references.

In [15], the authors presented innovative solutions for nonlinear implicit FDE with boundary conditions and delay:

$$\begin{cases} {}_0^C\mathfrak{D}_e^{q,\varepsilon}W(e) = \Psi\left(e, W_{\rho(e,W_e)}, \mathfrak{D}_0^q W(e)\right); e \in G \\ W(e) = \wp(e), \quad e \in [-\phi, 0] \\ \imath W(0) + \jmath W(\varpi) = \varrho \end{cases}$$

where $0 < q < 1, \varepsilon \geq 0, {}_0^C\mathfrak{D}_e^{q,\varepsilon}$ is the Caputo tempered fractional derivative, $\Psi : G \times C([-\phi, 0], \mathbb{R}) \times \mathbb{R}$ is a continuous function, $\wp \in C([-\phi, \varpi], \mathbb{R}), 0 < \varpi < +\infty, \imath, \jmath, \varrho$ are real constants, and $\phi > 0$ is the time delay.

Rahou [18] examined the existence of solutions for the implicit nonlinear tempered-Caputo FDE with delayed and advanced arguments in BS with nonlocal conditions:

$$\begin{cases} {}_0^C D_e^{q,\varrho}W(e) = \Psi\left(e, W^e, {}_0^C D_e^{q,o}W(e)\right), \quad e \in G := [0, \varkappa], \\ W(e) = \Lambda_1(e), \quad e \in [-\varpi, 0], \\ W(e) = \Lambda_2(e), \quad e \in [\varkappa, \varkappa + \delta], \end{cases}$$

where ${}_0^C D_e^{q,\varrho}$ represents the tempered-Caputo FD of order $q \in (0,1), \varrho \geq 0, \varpi, \delta > 0, \Psi : G \times C([-\varpi, \delta], U) \times U \to U$ is a given function, $\Lambda_1 \in C([-\varpi, 0], U)$, and $\Lambda_2 \in C([\varkappa, \varkappa + \delta], U)$. The authors used FPT for condensing maps to get their conclusions on existence and stability.

In this study, inspired by the aforementioned research, we investigate the following nonlinear TCIFV-FSs with advanced and retarded arguments in BS with nonlocal conditions:

$$\begin{aligned} &{}_0^C D_\mu^{\varepsilon,\rho}\Delta(\mu) = \mathfrak{X}\left(\mu, \Delta^\mu, {}_0^C D_\mu^{\varepsilon,\rho}\Delta(\mu), \int_0^\mu \mathfrak{Z}(\mu, s, \Delta(s))ds, \int_0^\phi \mathfrak{P}(\mu, s, \Delta(s))ds\right), \mu \in \mathfrak{J} := [0, \phi] \\ &\Delta(\mu) + (\mathcal{F}_1\Delta)(\mu) = \Lambda_1(\mu), \mu \in [-\mathfrak{r}, 0] \\ &\Delta(\mu) + (\mathcal{F}_2\Delta)(\mu) = \Lambda_2(\mu), \mu \in [\phi, \phi + \mathfrak{h}] \end{aligned} \tag{1.1}$$

where ${}_0^C D_\mu^{\varepsilon,\rho}$ represents the tempered-Caputo FD of order $\varepsilon \in (0,1), \rho \geq 0, \mathfrak{r}, \mathfrak{h} > 0$, and $\mathfrak{X} : \mathfrak{J} \times C([-\mathfrak{r}, \mathfrak{h}], \mathfrak{R}) \times \mathfrak{R} \times \mathfrak{R} \to \mathfrak{R}$ is a given function, $\mathfrak{Z} : \mathfrak{J} \times \mathfrak{J} \times \mathfrak{R} \to \mathfrak{R}, \mathcal{F}_1 : C([-\mathfrak{r}, \phi + \mathfrak{h}], \mathfrak{R}) \to C([-\mathfrak{r}, 0], \mathfrak{R})$ and $\mathcal{F}_2 : C([-\mathfrak{r}, \phi + \mathfrak{h}], \mathfrak{R}) \to C([\phi, \phi + \mathfrak{h}], \mathfrak{R})$, $\mathcal{F}_1$ and $\mathcal{F}_2$ are explicitly bounded operators between the

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stated Banach spaces, which is exactly what the fixed-point argument on $C([-\mathfrak{r}, \phi+\mathfrak{h}], \mathfrak{R})$. Moreover, $\Lambda_1 \in C([-\mathfrak{r}, 0], \mathfrak{R})$ and $\Lambda_2 \in C([\phi, \phi+\mathfrak{h}], \mathfrak{R})$.

For each function Δ defined on $[-\mathfrak{r}, \phi+\mathfrak{h}]$ and for any $\mu \in \mathfrak{J}$, we denote by Δ_μ the element of $C([-\mathfrak{r}, \mathfrak{h}], \mathfrak{R})$ defined by

$$\Delta^\mu(t) = \Delta(\mu + t), t \in [-\mathfrak{r}, \mathfrak{h}].$$

Three axes-the derivative operator, the issue structure, and the analytical methodologycan be used to describe the exact innovation of this work in comparison to the closest references. Rahou and colleagues. [18] investigated implied Sadovskii's fixed point theorem is used to show existence and stability for tempered-Caputo fractional differential equations with delayed and advanced arguments in Banach spaces. In contrast, the current study addresses a *integro-differential* problem and introduces a Volterra-type integral operator in addition to the implicit structure and the retarded-advanced delay terms. Additionally, the current study uses a nonlocal condition within the integro-differential context, which requires a different approach of the measure of noncompactness, whereas Rahou et al. [18] applied traditional nonlocal requirements. Regarding the author's own earlier works, Haoues et al. [7] used Krasnoselskii's fixed point theorem and the Banach contraction principle to study implicit Hadamard fractional differential equations with delayed and advanced arguments and nonlocal conditions. The author [15] tackled nonlinear Caputo-Hadamard fractional delay differential equations in a later study. In this study, the Hadamard-type operator is replaced by the tempered-Caputo derivative, which generalises the classical Caputo derivative by incorporating an exponential tempering parameter γ. The tempered derivative modifies the memory kernel and related semigroup features, which in turn impacts the hypotheses needed for the fixed-point argument and the stability analysis. This change in the underlying fractional operator is not only aesthetic. It is instructive to compare conclusions and hypotheses with the nearest references. Rahou et al. [18] conclude Hyers–Ulam stability under a boundedness constraint on the perturbation, assuming Lipschitz-type requirements on the nonlinear term and establishing existence by Sadovskii's theorem. The current paper follows a similar general approach, but it also needs to control the integral operator. As a result, the stability constant explicitly depends on both the tempering parameter and the delay bounds, and the growth assumptions on the nonlinearity are paired with an integrability condition on the kernel. Haoues et al. [7], on the other hand, relies on Krasnoselskii's theorem and the Banach contraction principle, which necessitate stronger Lipschitz assumptions and produce uniqueness as part of the conclusion.

The present work deliberately weakens the Lipschitz requirement to a compactness-type condition, sacrificing uniqueness for greater generality, while retaining the stability conclusion. The Caputo-Hadamard delay equations studied by the authors earlier [7] exhibit stability under a contraction-mapping framework; here, the tempered operator introduces an additional exponential weight that modifies the stability estimate but does not alter its qualitative nature. Collectively, these differences position the present contribution as a natural yet non-trivial extension of both the Rahou et al. [18] framework and the author's prior work, extending the reach of tempered fractional calculus to implicit integro-differential problems with mixed delays and nonlocal conditions.

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Structure for the paper. The work is organized as follows. In Section 2, we provide notation and supplies required for our job. In Section 3, we convert (1.1) to an integral equation and apply Sadovskii's FPT to demonstrate the existence results. Section 4 focuses on getting U-HS and generalized U-HS results for (1.1). Finally, we present an example that demonstrates the achieved results.

2. Preliminaries

This section provides an introduction to the terminology, background material and definitions used in this study. The BS containing all continuous functions with the norm

$$\|\Delta\|_\infty = \sup\{\|\Delta(\mu)\| : 0 \le \mu \le \phi\},$$

is represented by $C([0,\phi],\Re)$. Let $C([-\mathfrak{r},0],\Re)$ be the BS with the norm

$$\|\Delta\|_{[-\mathfrak{r},0]} = \sup\{\|\Delta(\mu)\| : \mu \in [-\mathfrak{r},0]\}.$$

Consider $C([\phi,\phi+\mathfrak{h}],\Re)$ the BS with the norm

$$\|\Delta\|_{[\phi,\phi+\mathfrak{h}]} = \sup\{\|\Delta(\mu)\| : \mu \in [\phi,\phi+\mathfrak{h}]\},$$

and $C([-\mathfrak{r},\mathfrak{h}],\Re)$ the BS with the norm

$$\|\Delta\|_{[-\mathfrak{r},\mathfrak{h}]} = \sup\{\|\Delta(\mu)\| : \mu \in [-\mathfrak{r},\mathfrak{h}]\}.$$

Set

$$\Omega_\Re = \Big\{\Delta : [-\mathfrak{r},\phi+\mathfrak{h}] \to \Re | \Delta|_{[0,\phi]} \in C(\mathfrak{J},\Re),\ \Delta|_{[-\mathfrak{r},0]} \in C([-\mathfrak{r},0],\Re),\ \text{and}\ \ \Delta|_{[\phi,\phi+\mathfrak{h}]} \in C([\phi,\phi+\mathfrak{h}],\Re i)\Big\}$$

We note that $\Omega_\Re$ is a BS with the norm

$$\|\Delta\|_{\Omega_\Re} = \sup_{\mu\in[-\mathbf{r},\phi+\mathfrak{h}]} \|\Delta(\mu)\|.$$

Definition 2.1. *Suppose that the function $\mathfrak{X} \in C(\mathfrak{J},\Re)$ and $\rho \ge 0$. Then, the Riemann-Liouville tempered fractional integral of order ε is defined by*

$$ {}_0I_\mu^{\varepsilon,\rho}\mathfrak{X}(\mu) = e^{-\rho\mu}{}_0I_\mu^\varepsilon\left(e^{\rho\mu}\mathfrak{X}(\mu)\right) = \frac{1}{\Gamma(\varepsilon)}\int_0^\mu e^{-\rho(\mu-\sigma)}\mathfrak{X}(\sigma)(\mu-\sigma)^{\varepsilon-1}d\sigma, \tag{2.1}$$

where ${}_0I_\mu^\varepsilon$ denotes the Riemann-Liouville fractional integral [14], defined by

$$ {}_0I_\mu^\varepsilon\mathfrak{X}(\mu) = \frac{1}{\Gamma(\varepsilon)}\int_0^\mu \mathfrak{X}(\sigma)(\mu-\sigma)^{\varepsilon-1}d\sigma. \tag{2.2}$$

The Riemann-Liouville FI (2.2) is the obvious reduction of the tempered FI (2.1) if $\rho = 0$.

Definition 2.2 ([20])**.** *For $j-1 < \varepsilon < j, j \in \mathbb{N}$, and $\rho \ge 0$, the tempered-Riemann-Liouville FD is defined by*

$$ {}_0D_\mu^{\varepsilon,\rho}\mathfrak{X}(\mu) = e^{-\rho\mu}{}_0D_\mu^\varepsilon\left(e^{\rho\mu}\mathfrak{X}(\mu)\right) = e^{-\rho\mu}\frac{1}{\Gamma(j-\varepsilon)}\frac{d^j}{d\mu^j}\int_0^\mu e^{\rho\sigma}\mathfrak{X}(\sigma)(\mu-\sigma)^{\varepsilon-j+1}d\sigma,$$

where ${}_0D_\mu^\varepsilon\left(e^{\rho\mu}\mathfrak{X}(\mu)\right)$ denotes the Riemann-Liouville FD [14], is defined as

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$$ {}_0D_\mu^\varepsilon \left(e^{\rho\mu}\mathfrak{X}(\mu)\right) = \frac{1}{\Gamma(j-\varepsilon)}\frac{d^j}{d\mu^j}\int_0^\mu \left(e^{\rho\sigma}\mathfrak{X}(\sigma)\right)(\mu-\sigma)^{\varepsilon-j+1}d\sigma. $$

Definition 2.3 ([20]). *For $j-1<\varepsilon<j, j\in\mathbb{N}$, and $\rho\geq 0$, the definition of tempered-Caputo FD is*

$$ \begin{aligned} {}_0^C D_\mu^{\varepsilon,\rho}\mathfrak{X}(\mu) &= e^{-\rho\mu}{}_0^C D_\mu^\varepsilon \left(e^{\rho\mu}\mathfrak{X}(\mu)\right) \\ &= e^{-\rho\mu}\frac{1}{\Gamma(j-\varepsilon)}\int_0^\mu \frac{1}{(\mu-\sigma)^{\varepsilon-j+1}}\frac{d^j}{d\sigma^j}\left(e^{\rho\sigma}\mathfrak{X}(\sigma)\right)d\sigma, \end{aligned} $$

where

$$ {}_0^C D_\mu^\varepsilon \left(e^{\rho\mu}\mathfrak{X}(\mu)\right) = \frac{1}{\Gamma(j-\varepsilon)}\int_0^\mu \frac{1}{(\mu-\sigma)^{\varepsilon-j+1}}\frac{d^j}{d\sigma^j}\left(e^{\rho\sigma}\mathfrak{X}(\sigma)\right)d\sigma. $$

Lemma 2.4 ([20]). *For a constant C,*

$$ \begin{aligned} {}_0D_\mu^{\varepsilon,\rho}C &= Ce^{-\rho\mu}{}_0D_\mu^\varepsilon e^{\rho\mu}, \\ {}_0^C D_\mu^{\varepsilon,\rho}C &= Ce^{-\rho\mu}{}_0^C D_\mu^\varepsilon e^{\rho\mu}. \end{aligned} $$

Obviously,

$$ {}_0D_\mu^{\varepsilon,\rho}(C) \neq {}_0^C D_\mu^{\varepsilon,\rho}(C). $$

Moreover,

$$ {}_0^C D_\mu^{\varepsilon,\rho}(C) \text{ is no longer equal to zero, differing from } {}_0^C D_\mu^\varepsilon(C)\equiv 0. $$

Lemma 2.5 ([20]). *Let $\mathfrak{X}\in C^j(\mu,\Re)$ and $j-1<\varepsilon<j, j\in\mathbb{N}$. Then, the Caputo tempered FD and the Riemann-Liouville tempered FI satisfy the composition properties*

$$ {}_0I_\mu^{\varepsilon,\rho}\left({}_0^C D_\mu^{\varepsilon,\rho}\mathfrak{X}(\mu)\right) = \mathfrak{X}(\mu) - \sum_{k=0}^{j-1} e^{-\rho\mu}\frac{\mu^k}{k!}\left[\frac{d^k}{d\mu^k}\left(e^{\rho\mu}\mathfrak{X}(\mu)\right)\Big|_{\mu=0}\right], $$

and

$$ {}_0^C D_\mu^{\varepsilon,\rho}\left({}_0I_\mu^{\varepsilon,\rho}\mathfrak{X}(\mu)\right) = \mathfrak{X}(\mu), \quad \text{for } \varepsilon\in(0,1). $$

Definition 2.6 (Kuratowski measure of noncompactness). *[2] Let $\mathbb{X}$ be a BS and let $F_\mathbb{X}$ be the family of bounded subsets of $\mathbb{X}$. The Kuratowski measure of noncompactness is the map $Q: F_\mathbb{X}\to[0,\infty)$ defined by*

$$ Q(D) = \inf\left\{E>0 : D\subset\bigcup_{j=1}^m D_j,\ m\in\mathbb{N},\ D_j\subset\mathbb{X},\ \text{and } \operatorname{diam}(D_j)\leq E\right\}, $$

where $D\in F_\mathbb{X}$ and the diameter of a set D_j is given by

$$ \operatorname{diam}(D_j) := \sup\{\|U-V\|_\mathbb{X} : U,V\in D_j\}. $$

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Definition 2.7 ([2]). *The map Q satisfies the following properties:*

- $Q(D) = 0 \Longleftrightarrow D$ *is compact (i.e., D is relatively compact);*
- $Q(D) = Q(\bar{D})$;
- *If* $D_1 \subset D_2$, *then* $Q(D_1) \leq Q(D_2)$;
- $Q(D_1 + D_2) \leq Q(D_1) + Q(D_2)$;
- $Q(cD) = |c|Q(D)$, *for* $c \in \mathbb{R}$;
- $Q(conv\ D) = Q(D)$.

Lemma 2.8 ([6]). *Let $E \subset \Omega_{\mathfrak{R}}$ be a bounded and equicontinuous set. Then:*

(1) *The function $S \mapsto Q(E(S))$ is continuous, and*

$$Q_{\Omega_{\mathfrak{R}}}(E) = \sup_{S \in [-\mathfrak{r}, \phi + \mathfrak{h}]} Q(E(S)).$$

(2)

$$Q\left(\left\{\int_0^{\phi} \Delta(z)dz : \Delta \in E\right\}\right) \leq \int_0^{\phi} Q(E(z))dz,$$

where $E(S) = \{\Delta(S) : \Delta \in E\}, \quad S \in \mathfrak{J}$.

Theorem 2.9 ([6]). *Let K be a convex closed, non-empty, and bounded subset of a BS $\mathbb{W}$, and let $G : K \to K$ be a condensing mapping. Then G has a fixed point in K.*

Definition 2.10 ([2]). *Let $\mathbb{W}$ be a BS and $G : \mathbb{W} \to \mathbb{W}$ be a continuous mapping. G is said to be a condensing mapping if for each bounded set B with $P(E) \neq 0$, we get*

$$P(G(E)) < P(E).$$

3. Existence results

To justify our existence results, let's make some assumptions:

(R_1) The function $\mathfrak{X} : \mathfrak{J} \times C([-\mathfrak{r}, \mathfrak{h}], \mathfrak{R}) \times \mathfrak{R}^3 \to \mathfrak{R}$ be continuous.

(R_2) There exist constants $\chi_1, \chi_3, \chi_3^* > 0$ and $0 < \chi_2 < 1$, such that

$$\|\mathfrak{X}(\mu, \gamma, \wp, v, w) - \mathfrak{X}(\mu, \bar{\gamma}, \bar{\wp}, \bar{v}, \bar{w})\| \leq \chi_1 \|\gamma - \bar{\gamma}\|_{[-\mathfrak{r}, \mathfrak{h}]} + \chi_2 \|\wp - \bar{\wp}\| + \chi_3 \|v - \bar{v}\| + \chi_3^* \|w - \bar{w}\|,$$

for any $\gamma, \bar{\gamma} \in C([-\mathfrak{r}, \mathfrak{h}], \mathfrak{R}), \wp, v, w, \bar{\wp}, \bar{v}, \bar{w} \in \mathfrak{R}$, and $\mu \in \mathfrak{J}$.

(R_3) There exist constants $0 < \chi_4 < 1$ and $\chi_5, \chi_5^* > 0$:

$$\|\digamma_1 \Delta_1 - \digamma_1 \Delta_2\|_{C([-\mathfrak{r}, 0], \Xi)} \leq \chi_4 \|\Delta_1 - \Delta_2\|_{C([-\mathfrak{r}, \phi + \mathfrak{h}], \Xi)}.$$

and

$$\|\mathfrak{Z}(\mu, s, u_1) - \mathfrak{Z}(\mu, s, u_2)\| \leq \chi_5 \|u_1 - u_2\|.$$

$$\|\mathfrak{P}(\mu, s, u_1) - \mathfrak{P}(\mu, s, u_2)\| \leq \chi_5^* \|u_1 - u_2\|.$$

(R_4) There exist constants $M_{\digamma_1} \geq \chi_4 > 0$ and $M_{\digamma_2} > 0$:

$$\|\digamma_1 \Delta\|_{[-\mathfrak{r}, 0]} \leq M_{\digamma_1} \text{ and } \|\digamma_2 \Delta\|_{[\phi, \phi + \mathfrak{h}]} \leq M_{\digamma_2}.$$

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(R_5) For each $\mu \in \mathfrak{J}$ and bounded sets $B_1 \subseteq C([-\mathfrak{r}, \mathfrak{h}], \mathfrak{R}), B_2, B_3, B_4 \subseteq \mathfrak{R}$, we get

$$\zeta\left(\mathfrak{X}\left(\mu, B_1, B_2, B_3, B_4\right)\right) \leq l_1 \sup_{\sigma \in [-\mathfrak{r}, \mathfrak{h}]} \zeta\left(B_1(\sigma)\right) + l_2 \zeta\left(B_2\right) + l_3 \zeta\left(B_3\right) + l_4 \zeta\left(B_4\right).$$

(R_6) There exist functions $\eta, \eta^* \in L^1(\mathfrak{J}, \mathfrak{R})$, such that

$$\int_0^{\mu} |\mathfrak{Z}(\mu, s, u)| ds \leq \eta(\mu)(1 + |u|), \quad \int_0^{\phi} |\mathfrak{P}(\mu, s, u)| ds \leq \eta^*(\mu)(1 + |u|)$$

Lemma 3.1. *Let ϑ denote a continuous function. Then the system*

$$\text{(3.1)} \qquad \begin{cases} {}_0^C D_{\mu}^{\varepsilon,\rho} \Delta(\mu) = \vartheta(\mu), & \mu \in \mathfrak{J} := [0, \phi], \\ \Delta(\mu) + \left(\mathcal{F}_1 \Delta\right)(\mu) = \Lambda_1(\mu), & \mu \in [-\mathfrak{r}, 0], \\ \Delta(\mu) + \left(\mathcal{F}_2 \Delta\right)(\mu) = \Lambda_2(\mu), & \mu \in [\phi, \phi + \mathfrak{h}], \end{cases}$$

offers a unique solution that is provided by

$$\text{(3.2)} \quad \Delta(\mu) = \begin{cases} \Lambda_1(\mu) - \left(\mathcal{F}_1 \Delta\right)(\mu), & \mu \in [-\mathfrak{r}, 0] \\ \left(\Lambda_1(0) - \left(\mathcal{F}_1 \Delta\right)(0)\right) e^{-\rho\mu} + \frac{1}{\Gamma(\varepsilon)} \int_0^{\mu} e^{-\rho(\mu-\sigma)} \vartheta(\sigma)(\mu - \sigma)^{\varepsilon - 1} d\sigma, & \mu \in \mathfrak{J} := [0, \phi] \\ \Lambda_2(\mu) - \left(\mathcal{F}_2 \Delta\right)(\mu), & \mu \in [\phi, \phi + \mathfrak{h}] \end{cases}$$

Proof. Assume that Δ fulfills (3.1). Using the tempered FI of Riemann-Liouville with order ε and Lemma (2.5), we have

$${}_0 I_{\mu}^{\varepsilon,\rho} {}_0^C D_{\mu}^{\varepsilon,\rho} \Delta(\mu) = {}_0 I_{\mu}^{\varepsilon,\rho} \vartheta(\mu).$$

Then

$$\Delta(\mu) - \Delta(0) e^{-\rho\mu} = \frac{1}{\Gamma(\varepsilon)} \int_0^{\mu} e^{-\rho(\mu-\sigma)} \vartheta(\sigma)(\mu - \sigma)^{\varepsilon - 1} d\sigma.$$

Thus,

$$\Delta(\mu) = \Delta(0) e^{-\rho\mu} + \frac{1}{\Gamma(\varepsilon)} \int_0^{\mu} e^{-\rho(\mu-\sigma)} \vartheta(\sigma)(\mu - \sigma)^{\varepsilon - 1} d\sigma.$$

Finally,

$$\Delta(\mu) = \left(\Lambda_1(0) - \left(\mathcal{F}_1 \Delta\right)(0)\right) e^{-\rho\mu} + \frac{1}{\Gamma(\varepsilon)} \int_0^{\mu} e^{-\rho(\mu-\sigma)} \vartheta(\sigma)(\mu - \sigma)^{\varepsilon - 1} d\sigma.$$

Using Definition 2.1, Lemma 2.4, and Lemma 2.5, it is possible to show that if Δ verifies (3.2), the problem (3.1) is also satisfied. □

Theorem 3.2. *Suppose that (R_1)-(R_6) are true. If*

$$\lambda + \frac{\left(\chi_1 + \chi_3 \chi_5 \phi + \chi_3^* \chi_5^* \phi\right) \phi^{\epsilon}}{\left(1 - \chi_2\right) \Gamma(\epsilon + 1)} < 1$$

where $\lambda = \sup_{\mu \in \mathfrak{J}} \chi_4 e^{-\rho\mu}$, then there exists at least one solution of (1.1).

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Proof. Let the operator $T : \Omega_{\mathfrak{R}} \to \Omega_{\mathfrak{R}}$ given by

$$T\Delta(\mu) = \begin{cases} \Lambda_1(\mu) - (\digamma_1\Delta)(\mu), \mu \in [-\mathfrak{r}, 0] \\ (\Lambda_1(0) - (\digamma_1\Delta)(0))\, e^{-\rho\mu} + \frac{1}{\Gamma(\varepsilon)} \int_0^\mu e^{-\rho(\mu-\sigma)} \mathcal{N}(\sigma)(\mu-\sigma)^{\varepsilon-1} d\sigma, \mu \in \mathfrak{J} := [0, \phi] \\ \Lambda_2(\mu) - (\digamma_2\Delta)(\mu), \mu \in [\phi, \phi + \mathfrak{h}] \end{cases} \tag{3.3}$$

where $\mathcal{N} \in C(\mathfrak{J}, \mathfrak{R})$ is such that

$$\mathcal{N}(\mu) = \mathfrak{X}\left(\mu, \Delta^\mu, \mathcal{N}(\mu), \int_0^\mu \mathfrak{Z}(\mu, s, \Delta(s))ds, \int_0^\phi \mathfrak{P}(\mu, s, \Delta(s))ds\right). \tag{3.4}$$

Therefore, the fixed points of operator T are the answer to the system (1.1). The proof is separated into three steps.

Step 1. We prove that T is continuous.

Let u_n be a sequence such that $u_n \to u$ in $\Omega_{\mathfrak{R}}$. If $\mu \in [-\mathfrak{r}, 0]$ or $\mu \in [\phi, \phi + \mathfrak{h}]$, then

$$|(Tu_n)(\mu) - (Tu)(\mu)| = 0.$$

For $\mu \in \mathfrak{J}$ we get

$$|(Tu_n)(\mu) - (Tu)(\mu)| \leq |(\digamma_1 u_n)(0) - (\digamma_1 u)(0)|\, e^{-\rho\mu} + \frac{1}{\Gamma(\varepsilon)} \int_0^\mu e^{-\rho(\mu-\sigma)} |\mathcal{N}_n(\sigma) - \mathcal{N}(\sigma)| (\mu-\sigma)^{\varepsilon-1} d\sigma$$

where $\mathcal{N}_n, \mathcal{N} \in C(\mathfrak{J}, \mathfrak{R})$ are such that

$$\mathcal{N}_n(\mu) = \mathfrak{X}\left(\mu, u_n^\mu, \mathcal{N}_n(\mu), \int_0^\mu \mathfrak{Z}(\mu, s, u_n(s))\, ds, \int_0^\phi \mathfrak{P}(\mu, s, u_n(s))\, ds\right),$$

and

$$\mathcal{N}(\mu) = \mathfrak{X}\left(\mu, u^\mu, \mathcal{N}(\mu), \int_0^\mu \mathfrak{Z}(\mu, s, u(s))ds, \int_0^\phi \mathfrak{P}(\mu, s, u(s))ds\right).$$

By (R_2)-(R_3), we get

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$$
\begin{aligned}
&\left|\mathcal{N}_n(\mu)-\mathcal{N}(\mu)\right| \\
&=\left|\mathfrak{X}\left(\mu,u_n^{\mu},\mathcal{N}_n(\mu),\int_0^{\mu}\mathfrak{Z}\left(\mu,s,u_n(s)\right)ds,\int_0^{\phi}\mathfrak{P}\left(\mu,s,u_n(s)\right)ds\right)\right. \\
&\left.-\mathfrak{X}\left(\mu,u^{\mu},\mathcal{N}(\mu),\int_0^{\mu}\mathfrak{Z}(\mu,s,u(s))ds,\int_0^{\phi}\mathfrak{P}(\mu,s,u(s))ds\right)\right| \\
&\leq \chi_1\left\|u_n^{\mu}-u^{\mu}\right\|_{[-\mathfrak{r},\mathfrak{h}]}+\chi_2\left|\mathcal{N}_n(\mu)-\mathcal{N}(\mu)\right| \\
&+\chi_3\int_0^{\mu}\left|\mathfrak{Z}\left(\mu,s,u_n(s)\right)-\mathfrak{Z}(\mu,s,u(s))\right|ds+\chi_3^{*}\int_0^{\phi}\left|\mathfrak{P}\left(\mu,s,u_n(s)\right)-\mathfrak{P}(\mu,s,u(s))\right|ds \\
&\leq \chi_1\left\|u_n^{\mu}-u^{\mu}\right\|_{[-\mathfrak{r},\mathfrak{h}]}+\chi_2\left|\mathcal{N}_n(\mu)-\mathcal{N}(\mu)\right|+\chi_3\int_0^{\mu}\chi_5\left\|u_n-u\right\|ds+\chi_3^{*}\int_0^{\phi}\chi_5^{*}\left\|u_n-u\right\|ds \\
&\leq \chi_1\left\|u_n^{\mu}-u^{\mu}\right\|_{[-\mathfrak{r},\mathfrak{h}]}+\chi_2\left|\mathcal{N}_n(\mu)-\mathcal{N}(\mu)\right|+\chi_3\chi_5\left(\int_0^{\mu}ds\right)\left\|u_n-u\right\|+\chi_3^{*}\chi_5^{*}\left(\int_0^{\phi}ds\right)\left\|u_n-u\right\|
\end{aligned}
$$

Thus,

$$
\left|\mathcal{N}_n(\mu)-\mathcal{N}(\mu)\right|\leq\frac{\chi_1+\chi_3\chi_5\mu+\chi_3^{*}\chi_5^{*}\phi}{1-\chi_2}\left\|u_n-u\right\|_{[-\mathfrak{r},\mathfrak{h}]}.
$$

From (R_3) we get

$$
\left|\left(\digamma_1u_n\right)(0)-\left(\digamma_1u\right)(0)\right|\leq\chi_4\left\|u_n^{\mu}-u^{\mu}\right\|_{[-\mathfrak{r},\phi+\mathfrak{h}]}.
$$

So, $\forall\ \mu\in\mathfrak{J}$, we get

$$
\begin{aligned}
&\left|\left(Tu_n\right)(\mu)-(Tu)(\mu)\right| \\
\leq\ &\chi_4e^{-\rho\mu}\left\|u_n^{\mu}-u^{\mu}\right\|_{[-\mathfrak{r},\phi+\mathfrak{h}]}+\frac{\chi_1+\chi_3\chi_5\mu+\chi_3^{*}\chi_5^{*}\phi}{1-\chi_2}\int_0^{\mu}e^{-\rho(\mu-\sigma)}(\mu-\sigma)^{\varepsilon-1}\left\|u_n^{\mu}-u^{\mu}\right\|_{[-\mathfrak{r},\mathfrak{h}]}d\sigma
\end{aligned}
$$

The dominated Lebesgue convergence theorem indicates that

$$
\left|\left(Tu_n\right)(\mu)-(Tu)(\mu)\right|\to 0\text{ as }n\to\infty.
$$

Hence

$$
\left\|Tu_n-Tu\right\|_{\Omega_{\mathfrak{R}}}\to 0\text{ as }n\to\infty.
$$

Consequently, T is continuous.

Choose

$$
R\geq\max\left\{\frac{\left\|\Lambda_1(0)\right\|+M_{F_1}+\frac{\left(\mathfrak{X}_1^{*}+\chi_3\|\eta\|_{L_1}+\chi_3^{*}\|\eta^{*}\|_{L_1}\right)\phi^{\epsilon}}{(1-\chi_2)\Gamma(\varepsilon+1)}}{1-\frac{\phi^{\epsilon}\left(\chi_1+\chi_3\|\eta\|_{L_1}+\chi_3^{*}\|\eta^{*}\|_{L_1}\right)}{(1-\chi_2)\Gamma(\epsilon+1)}},M_{F_1}+\left\|\Lambda_1\right\|_{[-\mathfrak{r},0]},M_{F_2}+\left\|\Lambda_2\right\|_{[\phi,\phi+\mathfrak{h}]}\right\},
$$

and define the set

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$$D_R = \{\Delta \in \Omega_{\mathfrak{R}} : \|\Delta\|_{\Omega_{\mathfrak{R}}} \leq R\}.$$

It is evident that D_R be a closed, convex, and bounded subset of $\Omega_{\mathfrak{R}}$.

Step 2. We prove that $T(D_R) \subset D_R$.

If $\mu \in [-\mathfrak{r}, 0]$, thus

$$\begin{aligned}\|T\Delta(\mu)\| &\leq \|F_1\Delta\|_{[-\mathrm{r},0]} + \|\Lambda_1\|_{[-\mathrm{r},0]} \\ &\leq M_{F_1} + \|\Lambda_1\|_{[-\mathrm{r},0]} \\ &\leq R.\end{aligned}$$

If $\mu \in [\phi, \phi + \mathfrak{h}]$, then

$$\begin{aligned}\|T\Delta(\mu)\| &\leq \|F_2\Delta\|_{[\phi,\phi+\mathfrak{h}]} + \|\Lambda_2\|_{[\phi,\phi+\mathfrak{h}]} \\ &\leq M_{F_2} + \|\Lambda_2\|_{[\phi,\phi+\mathfrak{h}]} \\ &\leq R.\end{aligned}$$

For each $\mu \in \mathfrak{J} := [0, \phi]$, hence

$$|T\Delta(\mu)| \leq (|\Lambda_1(0)| + |(F_1\Delta)(0)|)\, e^{-\rho\mu} + \frac{1}{\Gamma(\varepsilon)} \int_0^\mu e^{-\rho(\mu-\sigma)} |\mathcal{N}(\sigma)| (\mu - \sigma)^{\varepsilon-1} d\sigma,$$

From (R_2), we get

$$\begin{aligned}|\mathcal{N}(\mu)| &= \left| \mathfrak{X}\left(\mu, \Delta^\mu, \mathcal{N}(\mu), \int_0^\mu \mathfrak{Z}(\mu, s, u(s))ds, \int_0^\phi \mathfrak{P}(\mu, s, u(s))ds\right)\right| \\ &\leq \mathfrak{X}_1(\mu) + \chi_1 \|\Delta^\mu\|_{[-\mathfrak{r},\mathfrak{h}]} + \chi_2|\mathcal{N}(\mu)| + \chi_3 \int_0^\mu |\mathfrak{Z}(\mu, s, \Delta(s))|ds + \chi_3^* \int_0^\phi |\mathfrak{P}(\mu, s, \Delta(s))|ds \\ &\leq \mathfrak{X}_1^* + \chi_1\|\Delta\|_{\Omega_{\mathfrak{R}}} + \chi_2|\mathcal{N}(\mu)| + (\chi_3\eta(\mu) + \chi_3^*\eta^*(\mu))\,(1 + \|\Delta\|_{\Omega_{\mathfrak{R}}}) \\ &\leq \mathfrak{X}_1^* + \chi_1 R + \chi_2|\mathcal{N}(\mu)| + (\chi_3\|\eta\|_\infty + \chi_3^*\|\eta^*\|_\infty)(1 + R),\end{aligned}$$

where $\mathfrak{X}_1^* = \sup_{\mu\in\mathfrak{J}} \mathfrak{X}_1(\mu) = \sup_{\mu\in\mathfrak{J}} \mathfrak{X}(\mu, 0, 0)$ and $\|\eta\|_\infty = \sup_{\mu\in\mathfrak{J}} \eta(\mu)$

Thus,

$$|\mathcal{N}(\mu)| \leq \frac{\mathfrak{X}_1^* + \chi_1 R + (\chi_3\|\eta\|_\infty + \chi_3^*\|\eta^*\|_\infty)(1 + R)}{1 - \chi_2}.$$

Then,

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$$
\begin{aligned}
|T\Delta(\mu)| &\leq (|\Lambda_1(0)| + |(\digamma_1\Delta)(0)|)\, e^{-\rho\mu} \\
&\quad + \frac{\mathfrak{X}_1^* + \chi_1 R + (\chi_3\|\eta\|_\infty + \chi_3^*\|\eta^*\|_\infty)(1+R)}{(1-\chi_2)\,\Gamma(\varepsilon)} \int_0^\mu e^{-\rho(\mu-\sigma)}(\mu-\sigma)^{\varepsilon-1} d\sigma, \\
|T\Delta(\mu)| &\leq (|\Lambda_1(0)| + |(\digamma_1\Delta)(0)|)\, e^{-\rho\mu} + \frac{(\mathfrak{X}_1^* + \chi_1 R + (\chi_3\|\eta\|_\infty + \chi_3^*\|\eta^*\|_\infty)(1+R))\,\phi^\epsilon}{(1-\chi_2)\,\Gamma(\varepsilon+1)} \\
|T\Delta(\mu)| &\leq \|\Lambda_1(0)\| + M_{\digamma_1} + \frac{(\mathfrak{X}_1^* + \chi_1 R + (\chi_3\|\eta\|_\infty + \chi_3^*\|\eta^*\|_\infty)(1+R))\,\phi^\epsilon}{(1-\chi_2)\,\Gamma(\varepsilon+1)} \\
&\leq R.
\end{aligned}
$$

Then,

$$T(D_R) \subset D_R.$$

Step 3. T is condensing.

Let B be an equicontinuous subset of D_R.

If $\mu \in [-\mathfrak{r}, 0]$, thus

$$
\begin{aligned}
\zeta(T(B(\mu))) &= \zeta\{T\Delta(\mu), \quad \Delta \in B\} \\
&= \zeta\{\Lambda_1(\mu) - \digamma_1\Delta(\mu), \quad \Delta \in B\} \\
&\leq M_{\digamma_1}
\end{aligned}
$$

and if $\mu \in [\phi, \phi + \mathfrak{h}]$, then

$$
\begin{aligned}
\zeta(T(B(\mu))) &= \zeta\{T\Delta(\mu), \quad \Delta \in B\} \\
&= \zeta\{\Lambda_2(\mu) - F_2\Delta(\mu), \quad \Delta \in B\} \\
&\leq M_{F_2},
\end{aligned}
$$

$\forall\ \mu \in \mathfrak{J}$, we get

$$
\begin{aligned}
\zeta(T(B(\mu))) =& \zeta\{T\Delta(\mu), \quad \Delta \in B\} \\
=& \zeta\left\{(\Lambda_1(0) - (\digamma_1\Delta)(0))\, e^{-\rho\mu} + \frac{1}{\Gamma(\epsilon)}\int_0^\mu e^{-\varrho(\mu-\sigma)}(\mu-\sigma)^{\epsilon-1}\mathcal{N}(\sigma)d\sigma, \quad \Delta \in B\right\} \\
\leq& \zeta\left\{(\Lambda_1(0) - (\digamma_1\Delta)(0))\, e^{-\rho\mu}, \quad \Delta \in B\right\} \\
&+ \zeta\left\{\frac{1}{\Gamma(\epsilon)}\int_0^\mu e^{-\varrho(\mu-\sigma)}(\mu-\sigma)^{\epsilon-1}\mathcal{N}(\sigma)d\sigma, \quad \Delta \in B\right\} \\
\leq& \chi_4 e^{-\rho\mu}\zeta_\Delta(B) + \frac{1}{\Gamma(\epsilon)}\int_0^\mu (\mu-\sigma)^{\epsilon-1}\{\zeta(\mathcal{N}(\sigma))d\sigma, \quad \Delta \in B\}.
\end{aligned}
$$

From (R_5)

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$$\begin{aligned}\zeta(\mathcal{N}(\mu)) &= \zeta\left(\mathfrak{X}\left(\mu,\Delta^{\mu},\mathcal{N}(\mu),\int_0^{\mu}\mathfrak{Z}(\mu,s,\Delta(s))ds,\int_0^{\phi}\mathfrak{P}(\mu,s,\Delta(s))ds\right)\right)\\ &\leq \chi_1\sup_{\mu\in[-\mathfrak{r},\mathfrak{h}]}\zeta\left(\Delta^{\mu}\right)+\chi_2\zeta(\mathcal{N}(\mu))+\chi_3\zeta\left(\int_0^{\mu}\mathfrak{Z}(\mu,s,\Delta(s))ds\right)+\chi_3^{*}\zeta\left(\int_0^{\phi}\mathfrak{P}(\mu,s,\Delta(s))ds\right)\\ &\leq \chi_1\sup_{\mu\in[-\mathfrak{r},\phi+\mathfrak{h}]}\zeta(\Delta(\mu))+\chi_2\zeta(\mathcal{N}(\mu))+\chi_3\chi_5\left(\int_0^{\mu}\zeta(\Delta(s))ds\right)+\chi_3^{*}\chi_5^{*}\left(\int_0^{\phi}\zeta(\Delta(s))ds\right).\end{aligned}$$

Thus,

$$\zeta(\mathcal{N}(\mu))\leq\frac{\chi_1+\chi_3\chi_5\phi+\chi_3^{*}\chi_5^{*}\phi}{1-\chi_2}\sup_{\mu\in[-\mathfrak{r},\phi+\mathfrak{h}]}\zeta(\Delta(\mu))$$

Thus,

$$\begin{aligned}\zeta(T(B(\mu))&\leq\chi_4e^{-\rho\mu}\zeta_{\Delta}(B)+\frac{\chi_1+\chi_3\chi_5\phi+\chi_3^{*}\chi_5^{*}\phi}{(1-\chi_2)\,\Gamma(\epsilon)}\int_0^{\mu}(\mu-\sigma)^{\epsilon-1}\left\{\sup_{\mu\in[-\mathfrak{r},\phi+\mathfrak{h}]}\zeta(\Delta(\mu))d\sigma,\quad\Delta\in B\right\}\\ &\leq\left(\lambda+\frac{\left(\chi_1+\chi_3\chi_5\phi+\chi_3^{*}\chi_5^{*}\phi\right)\phi^{\epsilon}}{(1-\chi_2)\,\Gamma(\epsilon+1)}\right)\zeta_{\Delta}(B)\end{aligned}$$

Hence,

$$\zeta_{\Omega_{\mathfrak{R}}}(T(B))\leq\left(\lambda+\frac{\left(\chi_1+\chi_3\chi_5\phi+\chi_3^{*}\chi_5^{*}\phi\right)\phi^{\epsilon}}{(1-\chi_2)\,\Gamma(\epsilon+1)}\right)\zeta_{\Omega_{\mathfrak{R}}}(B)<\zeta_{\Omega_{\mathfrak{R}}}(B).$$

Hence, we get $\zeta_{\Omega_{\mathfrak{R}}}(T(B))\leq\zeta_{\Omega_{\mathfrak{R}}}(B)$. By the smallness condition assumed in theorem (the same one used for existence and stability). Therefore, T is a strict contraction with respect to the Kuratowski measure of noncompactness, which means T is condensing.

According to Sadovskii's FPT, the operator T has at least one fixed point, which solves the system (1.1). □

4. Ulam-Hyers stability

We use the concepts of U-HS and generalized U-HS provided in [14, 22].

Definition 4.1. *We claim that the system (1.1) has U-HS if there exists a real number $\chi_{\mathfrak{X}}>0$, $\forall\ \epsilon>0$, if $\Delta\in\Omega_{\mathfrak{R}}$ satisfies*

$$\left\|{}_0^{C}D_{\mu}^{\varepsilon,\rho}\Delta(\mu)-\mathfrak{X}\left(\mu,\Delta^{\mu},{}_0^{C}D_{\mu}^{\varepsilon,\rho}\Delta(\mu),\int_0^{\mu}\mathfrak{Z}(\mu,s,\Delta(s))ds,\int_0^{\phi}\mathfrak{P}(\mu,s,\Delta(s))ds\right)\right\|\leq\epsilon,\quad\mu\in\mathfrak{J}:=[0,\phi]$$

then $\exists\ \bar{\Delta}\in\Omega_{\mathfrak{R}}$ a solution of the system (1.1) such that

$$\|\Delta(\mu)-\bar{\Delta}(\mu)\|\leq\chi_{\mathfrak{X}}\epsilon,\quad\mu\in\mathfrak{J}:=[0,\phi].$$

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Definition 4.2. *We claim that the system (1.1) possesses generalized U-HS, if there is a function $\varphi_{\mathfrak{X}} \in C(\mathbb{R}^+, \mathbb{R}^+)$ with $\varphi_{\mathfrak{X}}(0) = 0$, for all $\epsilon > 0$, if $\Delta \in \Omega_{\mathfrak{R}}$ satisfies*

$$\left\| {}_0^C D_\mu^{\varepsilon,\rho}\Delta(\mu) - \mathfrak{X}\left(\mu, \Delta^\mu, {}_0^C D_\mu^{\varepsilon,\rho}\Delta(\mu), \int_0^\mu \mathfrak{Z}(\mu, s, \Delta(s))ds, \int_0^\phi \mathfrak{P}(\mu, s, \Delta(s))ds\right)\right\| \leq \epsilon, \quad \mu \in \mathfrak{J} := [0, \phi]$$

then there exists a solution $\bar{\Delta} \in \Omega_{\mathfrak{R}}$ of the system (1.1) such that

$$\|\Delta(\mu) - \bar{\Delta}(\mu)\| \leq \varphi_{\mathfrak{X}}(\epsilon), \quad \mu \in \mathfrak{J} := [0, \phi].$$

Theorem 4.3. *Assume that the requirements (R_1)-(R_2) are true and that the condition*

$$\left\| {}_0^C D_\mu^{\varepsilon,\rho}\Delta(\mu) - \mathfrak{X}\left(\mu, \Delta^\mu, {}_0^C D_\mu^{\varepsilon,\rho}\Delta(\mu), \int_0^\mu \mathfrak{Z}(\mu, s, \Delta(s))ds, \int_0^\phi \mathfrak{P}(\mu, s, \Delta(s))ds\right)\right\| \leq \epsilon, \quad \mu \in \mathfrak{J} := [0, \phi]$$

is met for each $\epsilon > 0$.
Then the system (1.1) is U-HS.

Proof. Assume $\Delta \in \Omega_{\mathfrak{R}}$ be a solution of

$$\left\| {}_0^C D_\mu^{\varepsilon,\rho}\Delta(\mu) - \mathfrak{X}\left(\mu, \Delta^\mu, {}_0^C D_\mu^{\varepsilon,\rho}\Delta(\mu), \int_0^\mu \mathfrak{Z}(\mu, s, \Delta(s))ds, \int_0^\phi \mathfrak{P}(\mu, s, \Delta(s))ds\right)\right\| \leq \epsilon, \quad \mu \in \mathfrak{J} := [0, \phi]$$

and $\bar{\Delta} \in \Omega_\Delta$ the solution of the system (1.1). Then there exists a function $y_\Delta \in C(\mathfrak{J}, \mathfrak{R})$ (depending on Δ) such that

$$\|y_\Delta(\mu)\| \leq \epsilon, \quad \mu \in \mathfrak{J} := [0, \phi]$$

and

$${}_0^C D_\mu^{\varepsilon,\rho}\Delta(\mu) = \mathfrak{X}\left(\mu, \Delta^\mu, {}_0^C D_\mu^{\varepsilon,\rho}\Delta(\mu), \int_0^\mu \mathfrak{Z}(\mu, s, \Delta(s))ds, \int_0^\phi \mathfrak{P}(\mu, s, \Delta(s))ds\right) + y_\Delta(\mu), \quad \mu \in \mathfrak{J} := [0, \phi]$$

Lemma 3.1 shows that Δ satisfies the fractional IE.

$$\Delta(\mu) = \begin{cases} \Lambda_1(\mu) - (\digamma_1\Delta)(\mu), & \mu \in [-\mathfrak{r}, 0] \\ (\Lambda_1(0) - (\digamma_1\Delta)(0))\, e^{-\rho\mu} + \frac{1}{\Gamma(\varepsilon)}\int_0^\mu e^{-\rho(\mu-\sigma)}\mathcal{N}(\sigma)(\mu-\sigma)^{\varepsilon-1}d\sigma \\ + \frac{1}{\Gamma(\varepsilon)}\int_0^\mu e^{-\rho(\mu-\sigma)}y_\Delta(\sigma)(\mu-\sigma)^{\varepsilon-1}d\sigma, \quad \mu \in \mathfrak{J} := [0, \phi] \\ \Lambda_2(\mu) - (\digamma_2\Delta)(\mu), \quad \mu \in [\phi, \phi + \mathfrak{h}], \end{cases}$$

Thus

$$\|\Delta(\mu) - \bar{\Delta}(\mu)\| = \|\left((\digamma_1\Delta)(0) - (\digamma_1\bar{\Delta})(0)\right)e^{-\rho\mu} + \frac{1}{\Gamma(\varepsilon)}\int_0^\mu e^{-\rho(\mu-\sigma)}(\mathcal{N}(\sigma) - \mathcal{M}(\sigma))(\mu-\sigma)^{\varepsilon-1}d\sigma$$
$$- \frac{1}{\Gamma(\varepsilon)}\int_0^\mu e^{-\rho(\mu-\sigma)}y_\Delta(\sigma)(\mu-\sigma)^{\varepsilon-1}d\sigma\|$$

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hence

$$\begin{aligned}\|\Delta(t)-\bar{\Delta}(t)\| &\leq \left\|\left(\digamma_1\Delta\right)(0)-\left(\digamma_1\bar{\Delta}\right)(0)\right\| e^{-\rho\mu}+\frac{1}{\Gamma(\varepsilon)}\int_0^{\mu} e^{-\rho(\mu-\sigma)}\|\mathcal{N}(\sigma)-\mathcal{M}(\sigma)\|(\mu-\sigma)^{\varepsilon-1}d\sigma\\ &\quad+\frac{1}{\Gamma(\varepsilon)}\int_0^{\mu} e^{-\rho(\mu-\sigma)}\left\|y_{\Delta}(\sigma)\right\|(\mu-\sigma)^{\varepsilon-1}d\sigma\\ &\leq \left\|\left(\digamma_1\Delta\right)(0)-\left(\digamma_1\bar{\Delta}\right)(0)\right\| e^{-\rho\mu}+\frac{1}{\Gamma(\varepsilon)}\int_0^{\mu} e^{-\rho(\mu-\sigma)}\|\mathcal{N}(\sigma)-\mathcal{M}(\sigma)\|(\mu-\sigma)^{\varepsilon-1}d\sigma\\ &\quad+\frac{\phi^{\varepsilon}\epsilon}{\Gamma(\varepsilon+1)},\end{aligned}$$

where the functions $\mathcal{N}$ and $\mathcal{M}$ are defined so that they satisfy different sets of functional equations.

$$\mathcal{N}(\mu)=\mathfrak{X}\left(\mu,\Delta^{\mu},\mathcal{N}(\mu),\int_0^{\mu}\mathfrak{Z}_1(\mu,s,\Delta(s))ds,\int_0^{\phi}\mathfrak{P}_1(\mu,s,\Delta(s))ds\right),$$

and

$$\mathcal{M}(\mu)=\mathfrak{X}\left(\mu,\bar{\Delta}^{\mu},\mathcal{M}(\mu),\int_0^{\mu}\mathfrak{Z}_2(\mu,s,\bar{\Delta}(s))ds,\int_0^{\phi}\mathfrak{P}_2(\mu,s,\bar{\Delta}(s))ds\right).$$

From (R_3), we get

$$\left\|\left(\digamma_1\Delta\right)(0)-\left(\digamma_1\bar{\Delta}\right)(0)\right\|\leq\chi_4\left\|\Delta^{\mu}-\bar{\Delta}^{\mu}\right\|_{[-\mathfrak{r},\phi+\mathfrak{h}]}.$$

From (R_2), we get

$$\begin{aligned}\|\mathcal{N}(\sigma)-\mathcal{M}(\sigma)\| =&\Big\|\mathfrak{X}\left(\mu,\Delta^{\mu},\mathcal{N}(\mu),\int_0^{\mu}\mathfrak{Z}_1(\mu,s,\Delta(s))ds,\int_0^{\phi}\mathfrak{P}_1(\mu,s,\Delta(s))ds\right)\\ &-\mathfrak{X}\left(\mu,\bar{\Delta}^{\mu},\mathcal{M}(\mu),\int_0^{\mu}\mathfrak{Z}_2(\mu,s,\bar{\Delta}(s))ds,\int_0^{\phi}\mathfrak{P}_2(\mu,s,\bar{\Delta}(s))ds\right)\Big\|\\ \leq&\chi_1\left\|\Delta^{\mu}-\bar{\Delta}^{\mu}\right\|_{[-\mathfrak{r},\mathfrak{h}]}+\chi_2\|\mathcal{N}(\sigma)-\mathcal{M}(\sigma)\|\\ &+\chi_3\int_0^{\mu}\left|\mathfrak{Z}_1(\mu,s,\Delta(s))-\mathfrak{Z}_2(\mu,s,\bar{\Delta}(s))\right|ds\\ &+\chi_3^{*}\int_0^{\phi}\left|\mathfrak{P}_1(\mu,s,\Delta(s))-\mathfrak{P}_2(\mu,s,\bar{\Delta}(s))\right|ds\\ \leq&\chi_1\left\|\Delta^{\mu}-\bar{\Delta}^{\mu}\right\|_{[-\mathfrak{r},\mathfrak{h}]}+\chi_2\|\mathcal{N}(\sigma)-\mathcal{M}(\sigma)\|\\ &+\chi_3\chi_5\int_0^{\mu}\|\Delta-\bar{\Delta}\|ds+\chi_3^{*}\chi_5^{*}\int_0^{\phi}\|\Delta-\bar{\Delta}\|ds\\ \leq&\chi_1\left\|\Delta^{\mu}-\bar{\Delta}^{\mu}\right\|_{[-\mathfrak{r},\mathfrak{h}]}+\chi_2\|\mathcal{N}(\sigma)-\mathcal{M}(\sigma)\|\\ &+\chi_3\chi_5\left(\int_0^{\mu}ds\right)\|\Delta-\bar{\Delta}\|+\chi_3^{*}\chi_5^{*}\left(\int_0^{\phi}ds\right)\|\Delta-\bar{\Delta}\|\\ \leq&\frac{\chi_1+\chi_3\chi_5\mu+\chi_3^{*}\chi_5^{*}\phi}{1-\chi_2}\|\Delta-\bar{\Delta}\|_{[-\mathfrak{r},\mathfrak{h}]}.\end{aligned}$$

Hence

$$
\begin{aligned}
&\|\Delta(t)-\bar{\Delta}(t)\| \\
\leq& e^{-\rho\mu}\chi_4\left\|\Delta^{\mu}-\bar{\Delta}^{\mu}\right\|_{[-\mathfrak{r},\phi+\mathfrak{h}]}+\frac{\chi_1+\chi_3\chi_5\mu+\chi_3^*\chi_5^*\phi}{(1-\chi_2)\,\Gamma(\varepsilon)}\left(\int_0^{\mu}e^{-\rho(\mu-\sigma)}(\mu-\sigma)^{\varepsilon-1}d\sigma\right)\|\Delta-\bar{\Delta}\|_{[-\mathfrak{r},\mathfrak{h}]} \\
&+\frac{\phi^{\varepsilon}\epsilon}{\Gamma(\varepsilon+1)}, \\
\leq& \left(e^{-\rho\mu}\chi_4+\frac{\chi_1+\chi_3\chi_5\mu\phi^{\varepsilon}+\chi_3^*\chi_5^*\phi^{\varepsilon+1}}{(1-\chi_2)\,\Gamma(\varepsilon+1)}\right)\|\Delta-\bar{\Delta}\|_{\Omega_{\Delta}}+\frac{\phi^{\varepsilon}\epsilon}{\Gamma(\varepsilon+1)}.
\end{aligned}
$$

Thus

$$
\|\Delta-\bar{\Delta}\|_{\Omega_{\mathfrak{R}}}\leq\frac{\frac{\phi^{\varepsilon}\epsilon}{\Gamma(\varepsilon+1)}}{1-\left(\lambda+\frac{\chi_1+\chi_3\chi_5\mu\phi^{\varepsilon}+\chi_3^*\chi_5^*\phi^{\varepsilon+1}}{(1-\chi_2)\Gamma(\varepsilon+1)}\right)}=\chi_{\mathfrak{X}}\epsilon.
$$

Then, the system (1.1) is U-HS. If we set $\varphi_{\mathfrak{x}}(\epsilon)=\chi_{\mathfrak{X}}\epsilon$, then $\varphi_{\mathfrak{x}}(0)=0$, which proves that the system (1.1) is generalized U-HS. □

5. **Example**

Let we examine the following nonlinear TCIFV-FSs with advanced and retarded arguments in Banach spaces with nonlocal conditions:

(5.1)
$$
\begin{cases}
{}_0^C D_{\mu}^{\varepsilon,\rho}\Delta(\mu)=\mathfrak{X}\left(\mu,\Delta^{\mu},{}_0^C D_{\mu}^{\varepsilon,\rho}\Delta(\mu),\int_0^{\mu}\mathfrak{Z}(\mu,s,\Delta(s))\,ds,\int_0^{\phi}\mathfrak{P}(\mu,s,\Delta(s))\,ds\right), & \mu\in\mathfrak{J}:=[0,\phi],\\
\Delta(\mu)+(\digamma_1\Delta)(\mu)=\Lambda_1(\mu), & \mu\in[-\mathfrak{r},0],\\
\Delta(\mu)+(\digamma_2\Delta)(\mu)=\Lambda_2(\mu), & \mu\in[\phi,\phi+\mathfrak{h}],
\end{cases}
$$

where $\mathfrak{R}=\mathbb{R}$, and parameters

$$
\varepsilon=\frac{1}{2},\qquad \rho=1,\qquad \mathfrak{r}=\mathfrak{h}=1,\qquad \phi=1,\qquad \delta=10^{-3}.
$$

With

$$
\begin{aligned}
\mathfrak{X}(\mu,\gamma,\wp,v,w)&=\frac{1}{4}\,\gamma(0)+\frac{1}{4}\,\wp+\frac{1}{100}v+\frac{1}{100}w,\\
\mathfrak{Z}(\mu,s,u)&=\delta u,\\
\mathfrak{P}(\mu,s,u)&=\delta u,\\
(\digamma_1\Delta)(\mu)&=\frac{1}{2}\arctan\big(\Delta(-1)\big),\quad \mu\in[-1,0],\\
(\digamma_2\Delta)(\mu)&=\frac{1}{2}\arctan\big(\Delta(1)\big),\quad \mu\in[1,2],\\
\Lambda_1(\mu)&=0,\quad \mu\in[-1,0],\\
\Lambda_2(\mu)&=0,\quad \mu\in[1,2].
\end{aligned}
$$

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the system (5.1) reduces to

$$\begin{cases} {}_0^C D_\mu^{\frac{1}{2},1}\Delta(\mu) = \frac{1}{4}\Delta(\mu) + \frac{1}{4}\,{}_0^C D_\mu^{\frac{1}{2},1}\Delta(\mu) + \frac{\delta}{100}\int_0^\mu \Delta(s)\,ds + \frac{\delta}{100}\int_0^1 \Delta(s)\,ds, \quad \mu \in [0,1], \\ \Delta(\mu) + \frac{1}{2}\arctan\big(\Delta(-1)\big) = 0, \quad \mu \in [-1,0], \\ \Delta(\mu) + \frac{1}{2}\arctan\big(\Delta(1)\big) = 0, \quad \mu \in [1,2], \end{cases}$$

equivalently,

$$\begin{cases} \frac{3}{4}\,{}_0^C D_\mu^{\frac{1}{2},1}\Delta(\mu) = \frac{1}{4}\Delta(\mu) + \frac{\delta}{100}\int_0^\mu \Delta(s)\,ds + \frac{\delta}{100}\int_0^1 \Delta(s)\,ds, \\ \Delta(\mu) = -\frac{1}{2}\arctan\big(\Delta(-1)\big), \quad \mu \in [-1,0], \\ \Delta(\mu) = -\frac{1}{2}\arctan\big(\Delta(1)\big), \quad \mu \in [1,2]. \end{cases}$$

The function $\mathfrak{X}$ is continuous as a linear combination of continuous evaluation maps

For any $\gamma, \bar{\gamma} \in C([-1,1],\mathbb{R})$ and $\wp, \bar{\wp}, v, \bar{v}, w, \bar{w} \in \mathbb{R}$,

$$\begin{aligned} &|\mathfrak{X}(\mu,\gamma,\wp,v,w) - \mathfrak{X}(\mu,\bar{\gamma},\bar{\wp},\bar{v},\bar{w})| \\ &\le \frac{1}{4}|\gamma(0) - \bar{\gamma}(0)| + \frac{1}{4}|\wp - \bar{\wp}| + \frac{1}{100}|v - \bar{v}| + \frac{1}{100}|w - \bar{w}| \\ &\le \frac{1}{4}\|\gamma - \bar{\gamma}\|_{[-1,1]} + \frac{1}{4}|\wp - \bar{\wp}| + \frac{1}{100}|v - \bar{v}| + \frac{1}{100}|w - \bar{w}|. \end{aligned}$$

Thus (R_2) holds with

$$\chi_1 = \frac{1}{4}, \qquad \chi_2 = \frac{1}{4}, \qquad \chi_3 = \chi_3^* = \frac{1}{100}.$$

Since arctan is 1-Lipschitz,

$$\|F_1\Delta_1 - F_1\Delta_2\|_{[-1,0]} \le \frac{1}{2}|\Delta_1(-1) - \Delta_2(-1)| \le \frac{1}{2}\|\Delta_1 - \Delta_2\|_{[-1,2]}.$$

So $\chi_4 = \frac{1}{2} \in (0,1)$. Hence,

$$|\mathfrak{Z}(\mu,s,u_1) - \mathfrak{Z}(\mu,s,u_2)| = \delta|u_1 - u_2|,$$

and for $\mathfrak{P}$, then

$$\chi_5 = \chi_5^* = \delta = 10^{-3}.$$

Since $|\arctan(x)| \le \pi/2$,

$$\|F_1\Delta\|_{[-1,0]} \le \frac{\pi}{4}, \qquad \|F_2\Delta\|_{[1,2]} \le \frac{\pi}{4}.$$

Then,

$$M_{F_1} = M_{F_2} = \frac{\pi}{4} > 0.$$

Because $\mathfrak{R} = \mathbb{R}$, the Kuratowski measure of noncompactness for any bounded set is 0. Hence for limited sets B_1, B_2, B_3, B_4,

$$\zeta(\mathfrak{X}(\mu, B_1, B_2, B_3, B_4)) = 0,$$

so

$$l_1 = \frac{1}{4}, \qquad l_2 = \frac{1}{4}, \qquad l_3 = l_4 = \frac{1}{100}.$$

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For $\mathfrak{Z}$,

$$\int_0^\mu |\mathfrak{Z}(\mu, s, u)|\, ds = \delta\mu|u| \leq \delta\mu(1 + |u|),$$

so $\eta(\mu) = \delta\mu \in L^1([0, 1])$. For $\mathfrak{P}$,

$$\int_0^1 |\mathfrak{P}(\mu, s, u)|\, ds = \delta|u| \leq \delta(1 + |u|),$$

so $\eta^*(\mu) = \delta \in L^1([0, 1])$.

Thus, all hypotheses $(R_1 - R_6)$ are satisfied.

We compute the constant under the existence condition. We have $\phi = 1$, $\varepsilon = \frac{1}{2}$, and $\Gamma(\frac{3}{2}) = \frac{\sqrt{\pi}}{2} \approx 0.8862$. Then,

$$\lambda = \sup_{\mu\in[0,1]} \chi_4 e^{-\rho\mu} = \sup_{\mu\in[0,1]} \frac{1}{2} e^{-\mu} = \frac{1}{2}.$$

Thus,

$$\chi_1 + \chi_3\chi_5\phi + \chi_3^*\chi_5^*\phi = \frac{1}{4} + \frac{1}{100} \cdot 10^{-3} + \frac{1}{100} \cdot 10^{-3} = 0.25002.$$

Furthermore,

$$\frac{(\chi_1 + \chi_3\chi_5\phi + \chi_3^*\chi_5^*\phi)\, \phi^\varepsilon}{(1 - \chi_2)\Gamma(\varepsilon + 1)} = \frac{0.25002}{(1 - \frac{1}{4}) \cdot 0.8862} \approx 0.3762.$$

Therefore,

$$\lambda + 0.3762 \approx 0.5 + 0.3762 = 0.8762 < 1.$$

The requirement of Theorem 3.2 is strictly met. According to Theorem 3.2, the system (5.1) with the aforementioned data has at least one solution.

The inequality in Theorem 4.3 is

$$\left\| {}_0^C D_\mu^{\varepsilon,\rho} \Delta(\mu) - \mathfrak{X}\left(\mu, \Delta^\mu, {}_0^C D_\mu^{\varepsilon,\rho} \Delta(\mu), \int_0^\mu \mathfrak{Z}(\mu, s, \Delta(s))\, ds, \int_0^\phi \mathfrak{P}(\mu, s, \Delta(s))\, ds\right) \right\| \leq \epsilon,$$

for every $\epsilon > 0$. We have already confirmed (R_1) and (R_2), and the above inequality defines the class of ϵ-approximate solutions.

Consequently, all hypotheses of Theorem 4.3 are satisfied and problem (5.1) is Hyers–Ulam stable.

Note that this is a special case and that the general framework allows for infinite-dimensional Banach spaces.

6. Conclusion

The existence of solutions to a class of TCIFV-FSs with retarded and advanced delays in Banach spaces under nonlocal conditions has been demonstrated in this paper, along with the U-HS of these solutions. Sadovskii's FPT and the measure of noncompactness were used in the study, and a real-world case was used to demonstrate the theoretical findings. These results provide a substantial contribution to the body of knowledge already available on tempered fractional calculus and its use in postponing integro-differential problems.

This work naturally opens up a number of research directions. One approach is to use alternative FPTs, such Mönch's theorem, which has already shown success in analogous boundary-value contexts, to loosen the compactness assumptions placed on the operator. Expanding the class of admissible

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systems by adding impulses or indefinite delay is another possible extension. From an application-oriented standpoint, the current framework is well suited to viscoelastic models, nonlocal physics, and fractional-order control systems, where it would be beneficial to validate the theoretical results against numerical simulations due to the tempered-Caputo operator's ability to capture both long-range memory and exponential decay.

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