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BEST PROXIMITY POINT RESULTS FOR GENERALIZED MULTIVALUED GERAGHTY-TYPE $\mathcal{F}$-CONTRACTION

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Abstract. In this work, we define the concept of generalized multivalued $\mathcal{F}$-contraction for two multivalued non-self-mappings and we prove the existence of a common best proximity point for such multivalued non-self-mappings in complete metric spaces. Then, we provide some interesting results and examples to illustrate our main theorem. We generalize the concept of Geraghty contraction and define generalized multivalued Geraghty-type $\mathcal{F}$-contraction. Finally, as a result of the main theorem, we prove that if a pair of multivalued non-self-mappings apply in such a contraction, then they will have a common best proximity point.

1. Introduction

Let $\mathcal{S}$ be a non-empty subset of a metric space $(\mathcal{X}, \delta)$, and let $\mathcal{K} : \mathcal{S} \to \mathcal{X}$ be a mapping. A point $s \in \mathcal{S}$ is called a fixed point of $\mathcal{K}$ if $\mathcal{K}s = s$, equivalently, $\delta(s, \mathcal{K}s) = 0$. When the fixed point equation $\mathcal{K}s = s$ admits no solution, we have $\delta(s, \mathcal{K}s) > 0$ for all $s \in \mathcal{S}$. In this case, the natural goal is to find an element $s \in \mathcal{S}$ that minimizes the distance $\delta(s, \mathcal{K}s)$ in some meaningful sense. This direction of research is precisely the focus of best approximation theory and the theory of best proximity points. The following classical best approximation theorem, due to Ky Fan [14], serves as a foundational result in this area.

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Theorem 1.1 ([14]). *Let $\mathcal{S}$ be a non-empty compact convex subset of a normed linear space $\mathcal{X}$, and let $\mathcal{K}:\mathcal{S}\to\mathcal{X}$ be a continuous mapping. Then there exists $s\in\mathcal{S}$ such that*

$$\|s-\mathcal{K}s\| = dist(\mathcal{K}s,\mathcal{S}) := \inf\{\|\mathcal{K}s-s^*\| : s^*\in\mathcal{S}\}.$$

Such a point s is called a best approximation point of $\mathcal{K}$ in $\mathcal{S}$.
It should be noted that if $s\in\mathcal{S}$ is a best approximation of a mapping $\mathcal{K}:\mathcal{S}\to\mathcal{X}$, then the quantity $\|s-\mathcal{K}s\|$ need not attain the global optimum. Best proximity point theorems are precisely designed to provide sufficient conditions under which the minimization problem

$$\min_{s\in\mathcal{S}}\|s-\mathcal{K}s\|$$

admits at least one solution. To establish a concrete lower bound, consider two non-empty subsets $\mathcal{S},\mathcal{T}$ of a metric space $(\mathcal{X},\delta)$ and a non-self-mapping $\mathcal{K}:\mathcal{S}\to\mathcal{T}$. The natural question is whether there exists an element $s_0\in\mathcal{S}$ such that

$$\delta(s_0,\mathcal{K}s_0) = \min_{s\in\mathcal{S}} d(s,\mathcal{K}s).$$

Since for every $s\in\mathcal{S}$, we have $\delta(s,\mathcal{K}s)\geq\delta(\mathcal{S},\mathcal{T})$, where

$$\delta(\mathcal{S},\mathcal{T}) := \inf\{\delta(s,t) : s\in\mathcal{S},\, t\in\mathcal{T}\},$$

the optimal solution to the minimization problem over the domain $\mathcal{S}$ is precisely the one for which the lower bound $\delta(\mathcal{S},\mathcal{T})$ is achieved.

Definition 1.2. Let $\mathcal{S}$ and $\mathcal{T}$ be non-empty subsets of a metric space $(\mathcal{X},\delta)$ and let $\mathcal{K}:\mathcal{S}\to\mathcal{T}$ be a non-self-mapping. A point $s\in\mathcal{S}$ is called a best proximity point of $\mathcal{S}$ if $\delta(s,\mathcal{S}s)=\delta(\mathcal{S},\mathcal{T})$.

Observe that when $\delta(\mathcal{S},\mathcal{T})=0$, a best proximity point coincides with a fixed point of $\mathcal{K}$, thereby generalizing the classical fixed point theory to the non-self-mapping.
Nadler [28] introduced a Hausdorff-type notion based on the distance between two arbitrary sets, and thereby extended the Banach fixed point theorem from singlevalued maps to multivalued contractive maps. Prior to presenting this fundamental result, we first establish some basic notation.
Let $(\mathcal{X},\delta)$ be a complete metric space and let $\mathrm{CB}(\mathcal{X})$ defines the category of all non-empty closed and bounded subsets of the non-empty set $\mathcal{X}$. For $\mathcal{S},\mathcal{T}\in\mathrm{CB}(\mathcal{X})$, the map $\mathrm{H}:\mathrm{CB}(\mathcal{X})\times\mathrm{CB}(\mathcal{X})\to[0,\infty)$ denote by

$$\mathrm{H}(\mathcal{S},\mathcal{T}) = \max\{\sup_{t\in\mathcal{T}}\Delta(t,\mathcal{S}),\sup_{s\in\mathcal{S}}\Delta(s,\mathcal{T})\}$$

where $\Delta(s,\mathcal{T})=\inf_{t\in\mathcal{T}}\delta(s,t)$, is a metric on $\mathrm{CB}(\mathcal{X})$. H is called the Hausdorff metric induced by δ. It is well Known that $(\mathrm{CB}(\mathcal{X}),\mathrm{H})$ is a complete metric space, whenever $(\mathcal{X},\delta)$ is a complete metric space.

Remark 1.3. *Let $\mathcal{K}:\mathcal{X}\to\mathrm{CB}(\mathcal{Y})$ be a multivalued mapping, where $(\mathcal{X},\delta)$, $(\mathcal{Y},\rho)$ are two metric spaces and* H *is the Hausdorff metric on* $\mathrm{CB}(\mathcal{Y})$. *The mapping $\mathcal{K}$ is continuous with respect to the Hausdorff metric* H *at $x\in\mathcal{X}$ if $\delta(x_n,x)\to0$ in $\mathcal{X}$, then $\mathrm{H}(\mathcal{K}x_n,\mathcal{K}x)\to0$ as $n\to\infty$.*

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Let $\mathcal{S}$ and $\mathcal{T}$ be any two non-empty subsets of the metric space $(\mathcal{X}, \delta)$. The following concepts will be used in all over of the paper:

$$\delta(\mathcal{S}, \mathcal{T}) := \inf\{\delta(s,t) : s \in \mathcal{S}, t \in \mathcal{T}\},$$

$$\mathcal{S}_0 := \{s \in \mathcal{S} : \delta(s,t) = \delta(\mathcal{S}, \mathcal{T}) \text{ for some } t \in \mathcal{T}\},$$

$$\mathcal{T}_0 := \{t \in \mathcal{T} : \delta(s,t) = \delta(\mathcal{S}, \mathcal{T}) \text{ for some } s \in \mathcal{S}\}.$$

For $\mathcal{S}, \mathcal{T} \in \mathrm{CB}(\mathcal{X})$, it is obvious that

$$\delta(\mathcal{S}, \mathcal{T}) \leq \mathrm{H}(\mathcal{S}, \mathcal{T}).$$

Definition 1.4. Let $\mathcal{K}, \mathcal{R} : \mathcal{S} \to \mathrm{CB}(\mathcal{T})$ be two multivalued functions. An element $s \in \mathcal{S}$ is said to be a common best proximity point of $\mathcal{K}$ and $\mathcal{R}$ if $\Delta(s, \mathcal{K}s) = \Delta(s, \mathcal{R}s) = \delta(\mathcal{S}, \mathcal{T})$.
If $\mathcal{S} = \mathcal{T}$, then $s \in \mathcal{K}s \cap \mathcal{R}s$, since $\Delta(s, \mathcal{K}s^*) = 0$, if and only if $s \in \mathcal{K}s^*$ for all $s, s^* \in \mathcal{S}$. Then, we say that s is a common fixed point of $\mathcal{K}, \mathcal{R} : \mathcal{S} \to \mathrm{CB}(\mathcal{S})$.

Important best proximity point theorems for singlevalued and multivalued mappings can be found in [1, 5, 10, 12, 13, 16, 17, 19, 20, 22–27, 29].
In [30], Sankar Raj introduced one of the basic concepts for proving the best proximity point theorems.

Definition 1.5 ([30]). Let $\mathcal{S}$ and $\mathcal{T}$ be two non-empty subsets of a metric space $(\mathcal{X}, \delta)$ with $\mathcal{S}_0 \neq \emptyset$. The pair $(\mathcal{S}, \mathcal{T})$ is said to have the P-property if for any $s_1, s_2 \in \mathcal{S}_0$ and $t_1, t_2 \in \mathcal{T}_0$

$$\delta(s_1, t_1) = \delta(\mathcal{S}, \mathcal{T}) = \delta(s_2, t_2) \implies \delta(s_1, s_2) = \delta(t_1, t_2)$$

The following important lemmas are used in proving the main theorems of this paper.

Lemma 1.6 ([7]). *Let $\mathcal{S}, \mathcal{T} \in \mathrm{CB}(\mathcal{X})$. Then*

i) $\Delta(s, \mathcal{T}) \leq \delta(s,t)$ *for all* $t \in \mathcal{T}$ *and* $s \in \mathcal{X}$;
ii) $\Delta(s, \mathcal{T}) \leq \mathrm{H}(\mathcal{S}, \mathcal{T})$ *for all* $s \in \mathcal{S}$.

Lemma 1.7 ([28]). *Let $\mathcal{S}, \mathcal{T} \in \mathrm{CB}(\mathcal{X})$ and let $s \in \mathcal{S}$. For all $\rho > 0$, then there exists $t \in \mathcal{T}$ such that*

$$\delta(s,t) \leq \mathrm{H}(\mathcal{S}, \mathcal{T}) + \rho.$$

On the other hand, we may not be able to find $t \in \mathcal{T}$ such that

$$\delta(s,t) \leq \mathrm{H}(\mathcal{S}, \mathcal{T})$$

Moreover, when $\mathcal{T}$ is compact, there exists $t \in \mathcal{X}$ such that $\delta(s,t) \leq \mathrm{H}(\mathcal{S}, \mathcal{T})$.

Wardowski gave the following definition of $\mathcal{F}$-contraction, see [31].

Definition 1.8 ([31]). Let $\mathcal{F} : (0, +\infty) \to (-\infty, +\infty)$ be given mapping such that:

($\mathcal{F}_1$) $\mathcal{F}$ is strictly increasing, i.e

$$s < t \Longrightarrow \mathcal{F}(s) < \mathcal{F}(t) \qquad \forall s, t \in (0, \infty),$$

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($\mathcal{F}_2$) For each sequence $\{s_n\}_{n\in\mathbb{N}}$ of positive numbers
$$\lim_{n\to\infty} s_n = 0 \Longleftrightarrow \lim_{n\to\infty} \mathcal{F}(s_n) = -\infty,$$
(F_3) There exists $\alpha \in (0,1)$ such that $\lim_{s\to 0^+} s^{\alpha}\mathcal{F}(s) = 0$.

Suppose $\mathbf{F}$ defines the set of all such mapping $\mathcal{F}$. IF $(\mathcal{X},\delta)$ is a metric space, then $\mathcal{G}:\mathcal{X}\to\mathcal{X}$ is said to be an $\mathcal{F}$-contraction, if for all $s,t\in\mathcal{X}$ there exists $\eta>0$ and $\mathcal{F}\in\mathbf{F}$ such that

$$\delta(\mathcal{G}s,\mathcal{G}t)>0 \quad \Longrightarrow \quad \eta+\mathcal{F}(\delta(\mathcal{G}s,\mathcal{G}t))\le\mathcal{F}(\delta(s,t)).$$

Lo′lo′ et al. [21] extended the $\mathcal{F}$-contraction for four mappings and proved the common best proximity point theorem for four non-self-mappings in complete metric space.
In [6], Altun et al. introduced the concept of multivalued $\mathcal{F}$-contraction.

Definition 1.9 ([6]). Let $(\mathcal{X},\delta)$ be a metric space. A multivalued function $\mathcal{K}:\mathcal{X}\to \mathrm{CB}(\mathcal{X})$ is said to be a multivalued $\mathcal{F}$-contraction if there exist $\eta>0$ and $\mathcal{F}\in\mathbf{F}$ such that

$$\text{for all } s,t\in\mathcal{X} \text{ with } \mathcal{K}s\neq\mathcal{K}t \Longrightarrow \eta+\mathcal{F}(\mathrm{H}(\mathcal{K}s,\mathcal{K}t))\le\mathcal{F}(\delta(s,t)).$$

Subsequently, Acar et al. [4] introduced the concept of generalized multivalued $\mathcal{F}$-contraction mappings and presented a fixed point result that constitutes a genuine generalization of several well-known multivalued fixed point theorems, including Nadler's theorem.

Definition 1.10 ([4]). Let $(\mathcal{X},\delta)$ be a metric space and $\mathcal{K}:\mathcal{X}\to\mathrm{CB}(\mathcal{X})$ be a mapping. Then $\mathcal{K}$ is said to be a generalized multivalued $\mathcal{F}$-contraction if there exists $\eta>0$ and $\mathcal{F}\in\mathbf{F}$ such that

$$\text{for all } s,t\in\mathcal{X} \text{ with } \mathcal{K}s\neq\mathcal{K}t \Longrightarrow \eta+\mathcal{F}(\mathrm{H}(\mathcal{K}s,\mathcal{K}t))\le\mathcal{F}(\mathcal{M}(s,t)).$$

where

$$\mathcal{M}(s,t)=\max\{\delta(s,t),\Delta(s,\mathcal{K}s),\Delta(t,\mathcal{K}t),\frac{1}{2}[\Delta(s,\mathcal{K}t)+\Delta(t,\mathcal{K}s)]\}$$

Then, Debnath et al. [11] generalized the multivalued $\mathcal{F}$-contraction for two functions as follows.

Definition 1.11 ([11]). Let $(\mathcal{X},\delta)$ be a metric space. The pair of multivalued functions $\mathcal{K},\mathcal{R}:\mathcal{X}\to\mathrm{CB}(\mathcal{X})$ is said to be a multivalued $\mathcal{F}$-contraction if there exist $\eta>0$ and $\mathcal{F}\in\mathbf{F}$ such that

$$\text{for all } s,t\in\mathcal{X} \text{ with } \mathcal{K}s\neq\mathcal{R}t \Longrightarrow \eta+\mathcal{F}(\mathrm{H}(\mathcal{K}s,\mathcal{R}t))\le\mathcal{F}(\delta(s,t)).$$

Geraghty [18] introduced the concepts of a Geraghty function and a Geraghty contraction as follows.

Definition 1.12 ([18]). A function $\gamma:[0,\infty)\to[0,1)$ is called a Geraghty function, if $\lim_{n\to\infty}\gamma(p_n)=1$ implies $\lim_{n\to\infty} p_n=0$.
Clearly $\gamma(p)=e^{-2p}$ for $t>0$ and $\gamma(0)\in[0,1)$ is a Geraghty function.

Definition 1.13. Let $(\mathcal{X},\delta)$ be a metric space and $\mathcal{G}:\mathcal{X}\to\mathcal{X}$ be a self-map. If there exists a Geraghty function γ such that

$$\delta(\mathcal{G}s,\mathcal{G}t)\le\gamma(\delta(s,t))\delta(s,t),$$

for all $s,t\in\mathcal{X}$, then $\mathcal{G}$ is called a Geraghty contraction.

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Many researchers generalized the Geraghty contraction and proved the fixed point theorems, see [2, 3, 8, 9, 15].

In this article, we introduce the notions of generalized multivalued $\mathcal{F}$-contraction and generalized multivalued Geraghty-type $\mathcal{F}$-contraction for pairs of multivalued non-self-mappings. Building on these concepts, we establish several new common best proximity point theorems that not only unify many known results in the literature but also provide significant generalizations within the setting of complete metric spaces.

2. Main result

Definition 2.1. Let $\mathcal{S}$ and $\mathcal{T}$ be two non-empty subsets of a metric space $(\mathcal{X}, \delta)$. The pair of mappings $\mathcal{K}, \mathcal{R} : \mathcal{S} \to \mathrm{CB}(\mathcal{T})$ is said to be a generalized multivalued $\mathcal{F}$-contraction pair if there exist $\eta > 0$ and $\mathcal{F} \in \mathbf{F}$ such that

$$\text{for all } s, t \in \mathcal{S} \text{ with } \mathcal{K}s \neq \mathcal{R}t \Longrightarrow \eta + \mathcal{F}(\mathrm{H}(\mathcal{K}s, \mathcal{R}t)) \leq \mathcal{F}(\Omega(s, t)),$$

and

$$\Omega(s,t) = \max\{\delta(s,t), \Delta(s, \mathcal{K}s) - \delta(\mathcal{S}, \mathcal{T}), \Delta(t, \mathcal{R}t) - \delta(\mathcal{S}, \mathcal{T}), \frac{\Delta(s, \mathcal{R}t) + \Delta(t, \mathcal{K}s)}{2} - \delta(\mathcal{S}, \mathcal{T})\}.$$

Note that while some individual terms in $\Omega(s,t)$, such as $\Delta(s, \mathcal{K}s) - \delta(\mathcal{S}, \mathcal{T})$, may be negative, the maximum operator guarantees that $\Omega(s,t) \geq 0$, as $\delta(s,t) \geq 0$ is always included. This ensures that the expression is well-defined for the function $\mathcal{F}$.

Theorem 2.2. *Let $\mathcal{S}$ and $\mathcal{T}$ be non-empty closed subsets of a complete metric space $(\mathcal{X}, \delta)$ with $\mathcal{S}_0 \neq \emptyset$. Let the multivalued mappings $\mathcal{K}, \mathcal{R} : \mathcal{S} \to \mathrm{CB}(\mathcal{T})$ satisfy the following conditions:*

i) *the pair $\{\mathcal{S}, \mathcal{T}\}$ has the P-property;*
ii) *$\mathcal{K}$ and $\mathcal{R}$ satisfy the generalized multivalued $\mathcal{F}$-contraction;*
iii) *$\mathcal{K}s \subset \mathcal{T}_0$ and $\mathcal{R}s \subset \mathcal{T}_0$ for all $s \in \mathcal{S}_0$;*
iv) *$\mathcal{K}s$ and $\mathcal{R}s$ are compact for all $s \in \mathcal{S}$;*
v) *$\mathcal{K}$ and $\mathcal{R}$ are continuous with respect to the Hausdorff metric* H *on* $\mathrm{CB}(\mathcal{T})$.

Then $\mathcal{K}$ and $\mathcal{R}$ have a common best proximity point.

Proof. Let $s_0 \in \mathcal{S}_0$ and choose $t_0 \in \mathcal{K}s_0 \subseteq \mathcal{T}_0$. Then there exists $s_1 \in \mathcal{S}_0$ such that

$$\delta(s_1, t_0) = \delta(\mathcal{S}, \mathcal{T}). \tag{2.1}$$

If $t_0 \in \mathcal{K}s_1 \cap \mathcal{R}s_1$, then

$$\delta(\mathcal{S}, \mathcal{T}) \leq \Delta(s_1, \mathcal{K}s_1) \leq \delta(s_1, t_0) = \delta(\mathcal{S}, \mathcal{T}), \qquad \text{since } t_o \in \mathcal{K}s_1,$$

$$\delta(\mathcal{S}, \mathcal{T}) \leq \Delta(s_1, \mathcal{R}s_1) \leq \delta(s_1, t_0) = \delta(\mathcal{S}, \mathcal{T}), \qquad \text{since } t_o \in \mathcal{R}s_1.$$

Therefore $\delta(\mathcal{S}, \mathcal{T}) = \Delta(s_1, \mathcal{K}s_1) = \Delta(s_1, \mathcal{R}s_1)$, then s_1 is a best proximity point of $\mathcal{K}$ and $\mathcal{R}$. Thus, assume that $t_0 \notin \mathcal{K}s_1 \cap \mathcal{R}s_1$.

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Let $t_0 \notin \mathcal{R}s_1$. Since $\mathcal{R}s_1$ is compact, Lemma 1.7, there exists $t_1 \in \mathcal{R}s_1 \subset \mathcal{T}_0$ such that

$$0 < \delta(t_0, t_1) \leq \mathrm{H}(\mathcal{K}s_0, \mathcal{R}s_1). \tag{2.2}$$

Since $\mathcal{F}$ is strictly increasing, by (2.2), we have

$$\begin{aligned} \mathcal{F}(\delta(t_0, t_1)) &\leq \mathcal{F}(\mathrm{H}(\mathcal{K}s_0, \mathcal{R}s_1)) \\ &\leq \mathcal{F}(\Omega(s_0, s_1)) - \eta, \end{aligned} \tag{2.3}$$

and also, there exists $s_2 \in \mathcal{S}_0$ such that

$$\delta(s_2, t_1) = \delta(\mathcal{S}, \mathcal{T}). \tag{2.4}$$

Since $s_1, s_2 \in \mathcal{S}_0$ and $t_0, t_1 \in \mathcal{T}_0$, and the pair $(\mathcal{S}, \mathcal{T})$ has the P-property, from (2.1) and (2.4) we obtain

$$\delta(s_1, s_2) = \delta(t_0, t_1). \tag{2.5}$$

By (2.3) and (2.5) we have

$$\mathcal{F}(\delta(s_1, s_2)) \leq \mathcal{F}(\delta(t_0, t_1)) \leq \mathcal{F}(\Omega(s_0, s_1)) - \eta.$$

Using the triangle inequality of δ, the facts that $t_0 \in \mathcal{K}s_0$ and $t_1 \in \mathcal{R}s_1$, the identities $\delta(s_1, t_0) = \delta(s_2, t_1) = \delta(\mathcal{S}, \mathcal{T})$, from (2.1) and (2.4), and (2.5) we can bound the terms of $\Omega(s_0, s_1)$ as follows:

$$\begin{aligned} \Omega(s_0, s_1) &= \max\{\delta(s_0, s_1), \Delta(s_0, \mathcal{K}s_0) - \delta(\mathcal{S}, \mathcal{T}), \Delta(s_1, \mathcal{R}s_1) - \delta(\mathcal{S}, \mathcal{T}), \frac{\Delta(s_0, \mathcal{R}s_1) + \Delta(s_1, \mathcal{K}s_0)}{2} - \delta(\mathcal{S}, \mathcal{T})\} \\ &\leq \max\{\delta(s_0, s_1), \delta(s_0, t_0) - \delta(\mathcal{S}, \mathcal{T}), \delta(s_1, t_1) - \delta(\mathcal{S}, \mathcal{T}), \frac{\delta(s_0, t_1) + \delta(s_1, t_0)}{2} - \delta(\mathcal{S}, \mathcal{T})\} \\ &\leq \max\{\delta(s_0, s_1), \delta(s_0, s_1) + \delta(s_1, t_0) - \delta(\mathcal{S}, \mathcal{T}), \delta(s_1, t_0) + \delta(t_0, t_1) - \delta(\mathcal{S}, \mathcal{T}), \\ &\quad \frac{\delta(s_0, s_1) + \delta(s_1, s_2) + d(s_2, t_1) + \delta(s_1, t_0)}{2} - \delta(\mathcal{S}, \mathcal{T})\} \\ &\leq \max\{\delta(s_0, s_1), \delta(s_0, s_1) + \delta(\mathcal{S}, \mathcal{T}) - \delta(\mathcal{S}, \mathcal{T}), \delta(\mathcal{S}, \mathcal{T}) + \delta(s_1, s_2) - \delta(\mathcal{S}, \mathcal{T}), \\ &\quad \frac{\delta(s_0, s_1) + \delta(s_1, s_2) + \delta(\mathcal{S}, \mathcal{T}) + \delta(\mathcal{S}, \mathcal{T})}{2} - \delta(\mathcal{S}, \mathcal{T})\} \\ &\leq \max\{\delta(s_0, s_1), \delta(s_1, s_2)\}. \end{aligned}$$

Then $\Omega(s_0, s_1) \leq \max\{\delta(s_0, s_1), \delta(s_1, s_2)\}$. If $\delta(s_0, s_1) < \delta(s_1, s_2)$, then we have

$$\mathcal{F}(\delta(s_1, s_2)) \leq \mathcal{F}(\delta(t_0, t_1)) \leq \mathcal{F}(\delta(s_1, s_2)) - \eta,$$

therefor $\eta \leq 0$, which is a contradiction. Then $\delta(s_1, s_2) \leq \delta(s_0, s_1)$, or $\Omega(s_0, s_1) \leq \delta(s_0, s_1)$, and we have

$$\mathcal{F}(\delta(s_1, s_2)) \leq \mathcal{F}(\delta(t_0, t_1)) \leq \mathcal{F}(\delta(s_0, s_1)) - \eta. \tag{2.6}$$

If $t_1 \in \mathcal{K}s_2 \cap \mathcal{R}s_2$, in the same way as above, we can show that s_2 is the best proximity point of $\mathcal{K}$ and $\mathcal{R}$.

Thus, assume that $t_1 \notin \mathcal{K}s_2 \cap \mathcal{R}s_2$. Let $t_1 \notin \mathcal{K}s_2$. Since $\mathcal{K}s_2$ is compact, Lemma 1.7, there exists $t_2 \in \mathcal{K}s_2 \subset \mathcal{T}_0$ such that

$$0 < \delta(t_1, t_2) \leq \mathrm{H}(\mathcal{K}s_2, \mathcal{R}s_1). \tag{2.7}$$

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Since $\mathcal{F}$ is strictly increasing, by (2.7), we have

$$\begin{aligned}\mathcal{F}(\delta(t_1,t_2)) &\leq \mathcal{F}(\mathrm{H}(\mathcal{K}s_2,\mathcal{R}s_1))\\ &\leq \mathcal{F}(\Omega(s_2,s_1))-\eta,\end{aligned} \tag{2.8}$$

and also, there exists $s_3 \in \mathcal{S}_0$ such that

$$\delta(s_3,t_2)=\delta(\mathcal{S},\mathcal{T}). \tag{2.9}$$

Since $s_2, s_3 \in \mathcal{S}_0$ and $t_1, t_2 \in \mathcal{T}_0$, and the pair $(\mathcal{S},\mathcal{T})$ has the P-property, from (2.4) and (2.9) we obtain

$$\delta(s_2,s_3)=\delta(t_1,t_2). \tag{2.10}$$

By (2.8) and (2.10) we have

$$\mathcal{F}(\delta(s_2,s_3)) \leq \mathcal{F}(\delta(t_1,t_2)) \leq \mathcal{F}(\Omega(s_2,s_1))-\eta,$$

Using the triangle inequality of δ, the facts that $t_2 \in \mathcal{K}s_2$ and $t_1 \in \mathcal{R}s_1$, the identities $\delta(s_1,t_0) = \delta(s_2,t_1) = \delta(s_3,t_2) = \delta(\mathcal{S},\mathcal{T})$, from (2.1) and (2.4) and (2.9), and (2.5) we can bound the terms of $\Omega(s_1,s_2)$ as follows:

$$\begin{aligned}\Omega(s_2,s_1) &= \max\{\delta(s_2,s_1),\Delta(s_2,\mathcal{K}s_2)-\delta(\mathcal{S},\mathcal{T}),\Delta(s_1,\mathcal{R}s_1)-\delta(\mathcal{S},\mathcal{T}),\frac{\Delta(s_2,\mathcal{R}s_1)+\Delta(s_1,\mathcal{K}s_2)}{2}-\delta(\mathcal{S},\mathcal{T})\}\\
&\leq \max\{\delta(s_2,s_1),\delta(s_2,t_2)-\delta(\mathcal{S},\mathcal{T}),\delta(s_1,t_1)-\delta(\mathcal{S},\mathcal{T}),\frac{\delta(s_2,t_1)+\delta(s_1,t_2)}{2}-\delta(\mathcal{S},\mathcal{T})\}\\
&\leq \max\{\delta(s_2,s_1),\delta(s_2,s_3)+\delta(s_3,t_2)-\delta(\mathcal{S},\mathcal{T}),\delta(s_1,t_0)+\delta(t_0,t_1)-\delta(\mathcal{S},\mathcal{T}),\\
&\quad \frac{\delta(s_2,t_1)+\delta(s_1,s_2)+d(s_2,s_3)+\delta(s_3,t_2)}{2}-\delta(\mathcal{S},\mathcal{T})\}\\
&\leq \max\{\delta(s_2,s_1),\delta(s_2,s_3)+\delta(\mathcal{S},\mathcal{T})-\delta(\mathcal{S},\mathcal{T}),\delta(\mathcal{S},\mathcal{T})+\delta(s_1,s_2)-\delta(\mathcal{S},\mathcal{T}),\\
&\quad \frac{\delta(\mathcal{S},\mathcal{T})+\delta(s_1,s_2)+\delta(s_2,s_3)+\delta(\mathcal{S},\mathcal{T})}{2}-\delta(\mathcal{S},\mathcal{T})\}\\
&\leq \max\{\delta(s_1,s_2),\delta(s_2,s_3)\}.\end{aligned}$$

Then $\Omega(s_2,s_1) \leq \max\{\delta(s_1,s_2),\delta(s_2,s_3)\}$. If $\delta(s_1,s_2) < \delta(s_2,s_3)$, then we have

$$\mathcal{F}(\delta(s_2,s_3)) \leq \mathcal{F}(\delta(t_1,t_2)) \leq \mathcal{F}(\delta(s_2,s_3))-\eta,$$

therefor $\eta \leq 0$, which is a contradiction.
Then $\delta(s_2,s_3) \leq \delta(s_1,s_2)$, or $\Omega(s_2,s_1) \leq \delta(s_1,s_2)$, and we have, by (2.6)

$$\mathcal{F}(\delta(s_2,s_3)) \leq \mathcal{F}(\delta(t_1,t_2)) \leq \mathcal{F}(\delta(s_1,s_2))-\eta \leq \mathcal{F}(\delta(s_0,s_1))-2\eta.$$

In general, for all n in $\mathbb{N}$, there exists two sequence $\{S_n\} \subseteq \mathcal{S}_0$ and $\{t_n\} \subseteq \mathcal{T}_0$ such that

- (A_1) $t_{2n} \in \mathcal{K}s_{2n}$ and $t_{2n+1} \in \mathcal{R}s_{2n+1}$;
- (A_2) $\delta(s_{n+1},t_n)=\delta(\mathcal{S},\mathcal{T})$;
- (A_3) $\delta(s_n,s_{n+1})=\delta(t_{n-1},t_n)$;
- (A_4) $\mathcal{F}(\delta(s_n,s_{n+1})) \leq \mathcal{F}(\delta(t_{n-1},t_n)) \leq \mathcal{F}(\delta(s_0,s_1))-n\eta$, for all $n \in \mathbb{N}\cup\{0\}$.

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Put $\beta_n := \delta(s_n, s_{n+1})$ for all $n = 0, 1, 2, \cdots$.
By (A_4), we conclude that

$$\lim_{n\to\infty} \mathcal{F}(\beta_n) = -\infty.$$

From $(\mathcal{F}_2)$ we have

$$\lim_{n\to\infty} \beta_n = 0. \tag{2.11}$$

Also from $(\mathcal{F}_3)$, there exists $\alpha \in (0, 1)$ such that

$$\lim_{n\to\infty} \beta_n^{\alpha} \mathcal{F}(\beta_n) = 0. \tag{2.12}$$

On the other hand, using (A_4)

$$\mathcal{F}(\beta_n) - \mathcal{F}(\beta_0) \leq -n\eta,$$

then we have

$$\beta_n^{\alpha} \mathcal{F}(\beta_n) - \beta_n^{\alpha} \mathcal{F}(\beta_0) \leq -n\beta_n^{\alpha}\eta \leq 0.$$

If $n \to \infty$ in the above inequality by (2.11) and (2.12), we conclude that

$$\lim_{n\to\infty} n\beta_n^{\alpha} = 0.$$

So there exists $\mathcal{N} \in \mathbb{N}$ such that $n\beta_n^{\alpha} \leq 1$ for all $n \geq \mathcal{N}$, i.e, for all $n \geq \mathcal{N}$

$$\beta_n \leq \frac{1}{n^{\frac{1}{\alpha}}}.$$

Therefore, we conclude that the series $\sum_{i=1}^{\infty} \beta_i$ is convergent, so, for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $m, n \in \mathbb{N}$ with $m \geq n \geq N$ we have

$$\sum_{i=n}^{i=m} \beta_i \leq \varepsilon. \tag{2.13}$$

By the triangular inequality and (2.13)

$$\delta(s_m, s_n) \leq \delta(s_{m-1}, s_m) + \ldots + \delta(s_n, s_{n+1}) = \sum_{i=n}^{i=m-1} \beta_i \leq \varepsilon.$$

Then $\{s_n\}$ is a Cauchy sequence in $\mathcal{S}_0 \subseteq \mathcal{S}$. Since $\mathcal{S}$ is a closed subset of the complete metric space $(\mathcal{X}, \delta)$, then there exists some $s \in \mathcal{S}$ such that $\lim_{n\to\infty} s_n = s$. Since $\mathcal{K}$ and $\mathcal{R}$ are continuous with respect to the Hausdorff metric H, therefore

$$\lim_{n\to\infty} \mathrm{H}(\mathcal{K}s_n, \mathcal{K}s) = 0, \tag{2.14}$$

and

$$\lim_{n\to\infty} \mathrm{H}(\mathcal{R}s_n, \mathcal{R}s) = 0. \tag{2.15}$$

Using (A_4), the same as above, we can prove that $\{t_n\}$ is a Cauchy sequence in $\mathcal{T}$. Since $\mathcal{T}$ is a closed subset of the complete metric space $(\mathcal{X}, \delta)$, there exists $t \in \mathcal{T}$ such that $\lim_{n\to\infty} t_n = t$.
Using (A_2) we have

$$\lim_{n\to\infty} \delta(s_{n+1}, t_n) = \delta(s, t) = \delta(\mathcal{S}, \mathcal{T}). \tag{2.16}$$

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Now, we show that $t \in \mathcal{K}s \cap \mathcal{R}s$.
Using $t_{2n} \in \mathcal{K}s_{2n}$ and (2.14), we have

$$\lim_{n\to\infty} \Delta(t_{2n}, \mathcal{K}s) \leq \lim_{n\to\infty} \mathrm{H}(\mathcal{K}s_{2n}, \mathcal{K}s) = 0,$$

then $\Delta(t, \mathcal{K}s) = 0$. Therefore $t \in \mathcal{K}s$, since $\mathcal{K}s$ is closed.
Again, since $t_{2n+1} \in \mathcal{R}s_{2n+1}$ and (2.15) we have

$$\lim_{n\to\infty} \Delta(t_{2n+1}, \mathcal{R}s) \leq \lim_{n\to\infty} \mathrm{H}(\mathcal{R}s_{2n+1}, \mathcal{R}s) = 0,$$

then $\Delta(t, \mathcal{R}s) = 0$. Therefore $t \in \mathcal{R}s$, since $\mathcal{R}s$ is closed.
We conclude that

$$t \in \mathcal{K}s \cap \mathcal{R}s. \tag{2.17}$$

Now, by (2.16) and (2.17), we have

$$\delta(S,T) \leq \Delta(s, \mathcal{K}s) \leq \delta(s,t) \leq \delta(S,T) \Longrightarrow \Delta(s, \mathcal{K}s) = \delta(S,T),$$

and

$$\delta(S,T) \leq \Delta(s, \mathcal{R}s) \leq \delta(s,t) \leq \delta(S,T) \Longrightarrow \Delta(s, \mathcal{R}s) = \delta(S,T),$$

therefore

$$\Delta(s, \mathcal{K}s) = \Delta(s, \mathcal{R}s) = \delta(S,T).$$

Then s is a common best proximity point of the mappings $\mathcal{K}$ and $\mathcal{R}$.

□

The following fixed point result is an immediate consequence of Theorem 2.2 with $\mathcal{S} = \mathcal{T} = \mathcal{X}$.

Corollary 2.3. *Let $(\mathcal{X}, \delta)$ be a complete metric space, and Let $\mathcal{K}, \mathcal{R} : \mathcal{X} \to \mathrm{CB}(\mathcal{X})$ be multivalued mappings satisfy the following conditions:*

i) *There exist $\eta > 0$ and $\mathcal{F} \in \mathbf{F}$ such that for all $s, t \in \mathcal{X}$ with $\mathcal{K}s \neq \mathcal{R}t$.*

$$\eta + \mathcal{F}(\mathrm{H}(\mathcal{K}s, \mathcal{R}t)) \leq \mathcal{F}(\max\{\delta(s,t), \Delta(s, \mathcal{K}s), \Delta(t, \mathcal{R}t), \frac{\Delta(s, \mathcal{R}t) + \Delta(t, \mathcal{K}s)}{2}\}),$$

ii) *$\mathcal{K}s$ and $\mathcal{R}s$ are compact for all $s \in \mathcal{X}$;*

iii) *$\mathcal{K}$ and $\mathcal{R}$ are continuous with respect to the Hausdorff metric* H *on* $\mathrm{CB}(\mathcal{X})$.

Then there exists an element s in $\mathcal{X}$ such that

$$\Delta(s, \mathcal{K}s) = \Delta(s, \mathcal{R}s) = 0.$$

That is, s is a common fixed point of $\mathcal{K}$ and $\mathcal{R}$.

The following result is a special case of Theorem 2.2 when $\mathcal{K}$ and $\mathcal{R}$ are singlevalued non-self-mappings. Also, it is a special result of Theorem 2.1 in [21].

Corollary 2.4. *Let $\mathcal{S}$ and $\mathcal{T}$ be two non-empty closed subsets of a complete metric space $(\mathcal{X}, \delta)$ with $\mathcal{S}_0 \neq \emptyset$ and let $\mathcal{K}, \mathcal{R} : \mathcal{S} \to \mathcal{T}$ be non-self-mappings satisfy the following conditions:*

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i) *There exist* $\eta > 0$ *and* $\mathcal{F} \in \mathbf{F}$ *such that for all* $s, t \in \mathcal{S}$ *with* $\mathcal{K}s \neq \mathcal{R}t$.

$$\eta + \mathcal{F}(\delta(\mathcal{K}s, \mathcal{R}t)) \leq \mathcal{F}(\max\{\delta(s,t), \delta(s,\mathcal{K}s), \delta(t,\mathcal{R}t), \frac{\delta(s,\mathcal{R}t) + \delta(t,\mathcal{K}s)}{2}\}),$$

ii) *the pair* $\{\mathcal{S}, \mathcal{T}\}$ *has the P-property;*

iii) $\mathcal{K}(\mathcal{S}_0) \subseteq \mathcal{T}_0$ *and* $\mathcal{R}(\mathcal{S}_0) \subseteq \mathcal{T}_0$;

iv) $\mathcal{K}$ *and* $\mathcal{R}$ *are continuous.*

Then there exists an element s *in* $\mathcal{S}$ *such that* $\delta(s, \mathcal{K}s) = \delta(s, \mathcal{T}s) = \delta(\mathcal{S}, \mathcal{T})$. *That is,* s *is a common best proximity point of* $\mathcal{K}$ *and* $\mathcal{R}$.

Example 2.5. *Let* $\mathcal{X} = \mathbb{R}^2$ *and* δ *be the Euclidean metric. Let* $\mathcal{S} = \{(-1,0), (-1,3), (-1,6), (-1,9)\}$ *and* $\mathcal{T} = \{(0,-1), (0,2), (0,5), (0,8)\}$. *Then* $\delta(\mathcal{S}, \mathcal{T}) = \sqrt{2}$, $\mathcal{S}_0 = \mathcal{S}$ *and* $\mathcal{T}_0 = \mathcal{T}$. *Clearly,* $(\mathcal{S}, \mathcal{T})$ *satisfies the P-property.*
The multivalued functions $\mathcal{K}, \mathcal{R} : \mathcal{S} \to \mathrm{CB}(\mathcal{T})$ *are defined by*

$$\mathcal{K}s = \begin{cases} \{(0,8), (0,5)\}, & if\, s = (-1,0), \\ (0,8), & otherwise. \end{cases} \qquad and \qquad \mathcal{R}s = \{(0,8)\}.$$

We can easily prove

$$\mathrm{H}(\mathcal{K}s, \mathcal{R}t) < \frac{3}{4}[\Delta(s, \mathcal{K}s) - \delta(\mathcal{S}, \mathcal{T})]$$

for all $s, t \in \mathcal{S}$ *with* $\mathcal{K}s \neq \mathcal{R}t$, *therefore* $\mathrm{H}(\mathcal{K}s, \mathcal{R}t) \leq \frac{3}{4}\Omega(s,t)$ *for all* $s, t \in \mathcal{S}$ *with* $\mathcal{K}s \neq \mathcal{R}t$.
Clearly $\mathcal{K}$ *and* $\mathcal{R}$ *are generalized multivalued* $\mathcal{F}$*-contraction with* $\eta = \ln(\frac{4}{3})$ *and* $\mathcal{F} : (0, \infty) \to (-\infty, \infty)$, $\mathcal{F}(t) = \ln t$. *Therefore, all hypotheses of Theorem 2.2 are satisfied and* $(-1, 9)$ *is common best proximity point of* $\mathcal{K}$ *and* $\mathcal{R}$, *that is* $\Delta((-1,9), \mathcal{K}(-1,9)) = \Delta((-1,9), \mathcal{R}(-1,9)) = \sqrt{2} = \delta(\mathcal{S}, \mathcal{T})$.

In the following, we generalize the concept of multivalued Geraghty-type $\mathcal{F}$-contraction from Debnath [10].

Definition 2.6. Let $\mathcal{S}$ and $\mathcal{T}$ be non-empty closed subsets of a complete metric space $(\mathcal{X}, \delta)$. A pair of mappings $\mathcal{K}, \mathcal{R} : \mathcal{S} \to \mathrm{CB}(\mathcal{T})$ is called a generalized multivalued Geraghty-type $\mathcal{F}$-contraction pair if there exist a Geraghty function γ, a constant $\eta > 0$, and $\mathcal{F} \in \mathbf{F}$ such that

$$\text{for all } s, t \in \mathcal{S} \text{ with } \mathcal{K}s \neq \mathcal{R}t \Longrightarrow \eta + \mathcal{F}(\mathrm{H}(\mathcal{K}s, \mathcal{R}t)) \leq \gamma(\delta(s,t))\mathcal{F}(\Omega(s,t)). \tag{2.18}$$

Theorem 2.7. *Let* $\mathcal{S}$ *and* $\mathcal{T}$ *be non-empty closed subsets of a complete metric space* $(\mathcal{X}, \delta)$ *with* $\mathcal{S}_0 \neq \emptyset$. *Let the multivalued mappings* $\mathcal{K}, \mathcal{R} : \mathcal{S} \to \mathrm{CB}(\mathcal{T})$ *satisfy the following conditions:*

i) *the pair* $\{\mathcal{S}, \mathcal{T}\}$ *has the P-property;*

ii) $\mathcal{K}$ *and* $\mathcal{R}$ *satisfy the generalized multivalued Geraghty-type* $\mathcal{F}$*-contraction;*

iii) $\mathcal{K}s \subseteq \mathcal{T}_0$ *and* $\mathcal{R}s \subseteq \mathcal{T}_0$ *for all* $s \in \mathcal{S}_0$;

iv) $\mathcal{K}s$ *and* $\mathcal{R}s$ *are compact for all* $s \in \mathcal{S}$;

v) $\mathcal{K}$ *and* $\mathcal{R}$ *are continuous with respect to the Hausdorff metric* H *on* $\mathrm{CB}(\mathcal{T})$.

Then $\mathcal{K}$ *and* $\mathcal{R}$ *have a common best proximity point.*

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Proof. We know $\gamma(p) \in [0,1)$ for all $p \in [0,\infty)$, then, using (2.18), we get

$$forall\ s,t \in \mathcal{S} \text{ with } \mathcal{K}s \neq \mathcal{R}t \Longrightarrow \eta + \mathcal{F}(\mathrm{H}(\mathcal{K}s,\mathcal{R}t)) \leq \mathcal{F}(\Omega(s,t)).$$

Therefore, $\mathcal{K}$ and $\mathcal{R}$ is an generalized multivalued $\mathcal{F}$-contraction, and therefore, from Theorem 2.2 it follows that $\mathcal{K}$ and $\mathcal{R}$ have a common best proximity point. □

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