http://jims.iranjournals.ir

Journal of the Iranian Mathematical Society
ISSN (print): ????-????, ISSN (on-line): ????-????
Vol. x No. x (201x), pp. xx-xx.
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http://en.ims.ir/

WEYL-ZAK TRANSFORMS AND TWISTED TRANSLATIONS: A REPRESENTATION VIEWPOINT ON LCA GROUPS

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Communicated by

ABSTRACT. This paper presents a unified approach to the Weyl-Zak transform on a locally compact abelian (LCA) group G, grounded in projective representation theory of the time-frequency plane $G \times \widehat{G}$. We define a projective representation and a twisted translation on this plane and prove their central intertwining relation with the Weyl transform. Using a uniform lattice $L \subset G$ and its annihilator $L^{\perp} \subset \widehat{G}$, we derive a generalized Weyl-Zak transform and show it is unitary from $L^2(G \times \widehat{G})$ to $L^2(S_L \times S_{L^{\perp}} \times G)$, where S_L and $S_{L^{\perp}}$ are fundamental domains for L and $L^{\perp}$. This provides a cohesive, representation-theoretic foundation beyond the Euclidean setting. Finally, we introduce a bracket product based on the Weyl-Zak transform and demonstrate its essential properties regarding twisted translations.

1. Introduction

The Zak transform and the Weyl transform are fundamental tools in harmonic analysis, with deep connections to time-frequency analysis [12, 5], quantum mechanics [2], and the theory of pseudo-differential operators [24]. Their interplay is particularly relevant to the study of shift-invariant spaces, Gabor systems, and time-frequency representations. In particular, Gabor analysis on locally compact abelian (LCA) groups, including finite abelian groups, provides a natural setting for time-frequency methods and twisted translation structures [3, 13]. Related constructions occur in more general forms in the theory of Gabor transforms, wave-packet transforms, and Zak transforms on semidirect product groups; see [1, 7, 8]. Moreover, related projective representations arise naturally in the frameworks of abstract wave-packet groups and generalized Weyl–Heisenberg groups; see [9, 10]. These connections place the present work within a broader representation-theoretic approach to time-frequency analysis.

MSC(2010): Primary: 43A65; Secondary: 43A15, 43A25.

Keywords: projective representation, twisted translates, Weyl transforms, Weyl-Zak transforms.

Received: dd mmmm yyyy, Accepted: dd mmmm yyyy.

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On the Euclidean space $\mathbb{R}^{2n}$, the interaction between the Weyl and Zak transforms was studied by Radha and co-authors [23], who introduced the *Weyl–Zak transform* in connection with twisted translations. Furthermore, convolution-type transforms associated with twisted translations and projective representations have been investigated in the literature; see, for example, [17, 18, 19]. The multiplier approach has also been considered in the setting of covariant functions, where it provides a unified perspective on convolution-type transforms associated with twisted translations and projective representations; see [6, 11]. The extension of these ideas to LCA groups provides a unified framework encompassing both continuous and discrete settings. The Zak transform on LCA groups has been studied since the work of Kutyniok [16], while Weyl transforms in related settings have been considered in [20, 22]. Nevertheless, a formulation of the Weyl–Zak transform on general LCA groups based on the specific multiplier and twisted translation structure considered in this paper has not, to the best of our knowledge, been developed in the present form.

Motivated by the Euclidean construction in [23], we develop a Weyl–Zak transform for arbitrary LCA groups. Our approach is representation-theoretic. The use of a multiplier α and the associated projective representation π (equivalently, the α-representation) is not merely a formal modification of the classical Weyl–Heisenberg picture; rather, it provides a natural framework for encoding the twisted translation structure that underlies the Weyl–Zak transform on $G \times \widehat{G}$. In particular, the projective representation makes the cocycle interaction explicit and allows the Weyl transform to be formulated in a way that is compatible with the twisted action on $L^2(G \times \widehat{G})$. This viewpoint is especially useful on general LCA groups, where the representation-theoretic structure clarifies the relation between the transform, the multiplier, and the induced translation operators. We then introduce a carefully chosen multiplier α on $G\times\widehat{G}$ and the associated projective representation. This representation induces a twisted translation operator on $L^2(G \times \widehat{G})$. Our definitions are tailored to establish a natural intertwining relation between the resulting Weyl transform and the corresponding twisted translations. This intertwining relation is a central ingredient in the development of our framework.

The paper is organized as follows. In Section 2, we recall the required background and notation, including multipliers, projective representations, and Fourier analysis on LCA groups. In Section 3, we construct a multiplier on $G \times \widehat{G}$, define the corresponding projective representation, and develop the Weyl and Weyl–Zak transforms. We establish their main properties, including their isometric behavior and their interaction with twisted translations. Finally, we define a bracket map via the Weyl–Zak transform and investigate its algebraic and functional properties.

2. **Preliminaries and notation**

Let G be a locally compact group and $\mathbb{T}$ denote the unit circle group. A function $\alpha : G \times G \to \mathbb{T}$ is called a multiplier (or $2-$cocycle) if it satisfies the following conditions for all $g_1, g_2, g_3, g \in G$.

$$(i) \quad \alpha(g_1, g_2g_3)\alpha(g_2, g_3) = \alpha(g_1g_2, g_3)\alpha(g_1, g_2),$$

$$(ii) \quad \alpha(g, e_G) = \alpha(e_G, g) = 1,$$

where e_G denotes the identity element of G. A projective representation π of the locally compact group G is a mapping $g \mapsto \pi(g)$ from G into the group $U(\mathbb{H})$, of all unitary operators on the Hilbert space $\mathbb{H}$, such that for all $g, h \in G$, we have

$$\pi(g)\pi(h) = \alpha(g,h)\pi(gh),$$

where α is a multiplier defined on $G \times G$. Throughout this paper, we use the standard terminology "projective representation" for a representation determined by a multiplier (2-cocycle) α. (See more detailes in [3, 14, 15]).
Let G be a locally compact abelian group, and let $L \subset G$ be a discrete subgroup. The annihilator of L, denoted by $L^\perp$, is the subset of $\widehat{G}$ defined as

$$L^\perp = \{\chi \in \widehat{G} : \chi(\ell) = 1, \ \forall \ell \in L\},$$

which is a closed subgroup of $\widehat{G}$. According to Pontryagin duality, $L^\perp$ is isomorphic to the dual of the quotient $\frac{G}{L}$, and thus, $L^\perp \cong \widehat{(\frac{G}{L})}$.(See [4], Theorem 4.30). A fundamental domain $S_L \subset G$ for L is a Borel measurable section such that every $x \in G$ can be written uniquely as $x = \ell s$, where $\ell \in L, s \in S_L$. The existence of such a domain is guaranteed.(See [16], Lemma 2).

The Fourier transform $\hat{f}$ of any function $f \in L^1(G)$ is defined by

$$\hat{f}(\xi) = \int_G f(x)\overline{\xi(x)}dx,$$

where dx is the Haar measure on G. As usual we extend the Fourier transform to an isometry from $L^2(G)$ into $L^2(\hat{G})$, the so-called Plancherel isomorphism. By the Fourier inversion formula, if $f \in L^1(G)$ and $\hat{f} \in L^1(\hat{G})$, then

$$f(x) = \int_{\hat{G}} \hat{f}(\xi)\xi(x)d\xi,$$

where $d\xi$ is the dual Haar measure on $\widehat{G}$ normalized with respect to dx.

Let L be a closed subgroup of a locally compact abelian group G. The normalized Haar measure $d\mu_{\frac{G}{L}}$ of $\frac{G}{L}$ and the Haar measure $d\ell$ on L can be chosen such that for $f \in L^1(G)$,

$$\int_{\frac{G}{L}} \int_L f(x\ell)d\ell d\mu_{\frac{G}{L}}(xL) = \int_G f(x)dx, \tag{2.1}$$

and for $f \in L^1(G) \bigcap L^2(G)$, then

$$\int_{\frac{G}{L}} |\int_L f(x\ell)d\ell|^2 d\mu_{\frac{G}{L}}(xL) = \int_{L^\perp} |\hat{f}(\omega)|^2 d\omega, \tag{2.2}$$

where $d\omega$ is the dual Haar measure on $L^\perp$ normalized with respect to $d\ell$. (For more details see [4]).

3. **Main results**

Throughout this section, let G be a locally compact abelian group with dual group $\widehat{G}$. Our goal is to introduce a specific multiplier on $G \times \widehat{G}$, define a corresponding projective representation, and establish foundational results regarding Weyl and Weyl-Zak transforms and their interactions with twisted translations.

Let G be a locally compact abelian group. Define the mapping α from $(G \times \hat{G}) \times (G \times \hat{G})$ into $\mathbb{T}$ such that

$$\alpha((x_1, \xi_1), (x_2, \xi_2)) = \xi_2(x_1)\overline{\xi_1(x_2)}.$$

This mapping is a multiplier. Indeed, for $(x_1, \xi_1), (x_2, \xi_2), (x_3, \xi_3) \in G \times \hat{G}$ we have,

$$\begin{aligned}
\alpha((x_1, \xi_1), (x_2, \xi_2))\alpha((x_1x_2, \xi_1\xi_2), (x_3, \xi_3)) &= \overline{\xi_1(x_2)}\xi_2(x_1)\xi_3(x_1x_2)\overline{\xi_1\xi_2(x_3)} \\
&= \overline{\xi_1(x_2)}\xi_2(x_1)\xi_3(x_1)\xi_3(x_2)\overline{\xi_1(x_3)\xi_2(x_3)} \\
&= \alpha((x_1, \xi_1), (x_2x_3, \xi_2\xi_3))\alpha((x_2, \xi_2), (x_3, \xi_3)).
\end{aligned}$$

Also, for $(x, \xi) \in G \times \hat{G}$ and for unit element $(e_G, 1_{\hat{G}})$ of the group $G \times \hat{G}$ we have,

$$\alpha((x, \xi), (e_G, 1_{\hat{G}})) = \alpha((e_G, 1_{\hat{G}}), (x, \xi)) = 1, \qquad ((x, \xi) \in G \times \hat{G}).$$

This structure provides the groundwork for defining a projective representation of $G \times \hat{G}$ on $U(L^2(G))$ as follows:
For a locally compact abelian (LCA) group G with the dual group $\widehat{G}$, we define,

$$\pi : G \times \hat{G} \to U(L^2(G)), \quad [\pi(x, \xi)f](y) = \xi(y)^2\xi(x)f(xy),$$

for $(x, \xi) \in G \times \hat{G}, y \in G$ and $f \in L^2(G)$. π is a projective representation of $G \times \hat{G}$ on $U(L^2(G))$. In fact,

$$\begin{aligned}
[\pi((x_1, \xi_1)(x_2, \xi_2))f](y) &= [\pi(x_1x_2, \xi_1\xi_2)f](y) \\
&= \xi_1\xi_2(y)^2\xi_1\xi_2(x_1x_2)f(x_1x_2y) \\
&= \xi_1(y)^2\xi_2(y)^2\xi_1(x_1)\xi_2(x_2)\xi_1(x_2)\xi_2(x_1)f(x_1x_2y),
\end{aligned}$$

on the other hand,

$$\begin{aligned}
[\pi(x_1, \xi_1)\pi(x_2, \xi_2)f](y) &= \xi_1(y)^2\xi_1(x_1)[\pi(x_2, \xi_2)f](x_1y) \\
&= \xi_1(y)^2\xi_1(x_1)\xi_2(x_1y)^2\xi_2(x_2)f(x_2x_1y) \\
&= \xi_1(y)^2\xi_1(x_1)\xi_2(x_1)^2\xi_2(y)^2\xi_2(x_2)f(x_2x_1y),
\end{aligned}$$

for $(x_1, \xi_1), (x_2, \xi_2) \in G \times \hat{G}$. So,

$$\pi(x_1, \xi_1)\pi(x_2, \xi_2) = \alpha((x_1, \xi_1)(x_2, \xi_2))\pi((x_1, \xi_1)(x_2, \xi_2)).$$

Thus, π is projective representation. Consider $G_\alpha = G \times \hat{G} \times \mathbb{T}$ by the multiplication

$$(x_1, \xi_1, t_1).(x_2, \xi_2, t_2) = (x_1x_2, \xi_1\xi_2, t_1t_2\overline{\xi_1(x_2)}\xi_2(x_1)),$$

is a locally compact group. Then $\tilde{\pi}$ of G_α on $L^2(G)$ by $[\tilde{\pi}(x, \xi, t)f](y) = t\xi(y)^2\xi(x)f(xy)$, is a continuous unitary representation.

In the following we define the Weyl transform on $G \times \hat{G}$ as follows:

$$[W(\phi)f](y) = \int_G \int_{\hat{G}} \phi(x, \xi)[\pi(x, \xi)f](y)dxd\xi,$$

where, $\phi \in C_c(G \times \hat{G})$ and $f \in L^2(G)$. So we get,

$$\begin{aligned} [W(\phi)f](y) &= \int_G \int_{\hat{G}} \phi(x, \xi)\xi(y)^2\xi(x)f(xy)dxd\xi \\ &= \int_G \int_{\hat{G}} \phi(xy^{-1}, \xi)\xi(y)^2\xi(xy^{-1})f(x)dxd\xi \\ &= \int_G \int_{\hat{G}} \phi(xy^{-1}, \xi)\xi(y)\xi(x)f(x)dxd\xi. \end{aligned}$$

The Weyl transform is an integral operator with the kernel

$$K_\phi(x, y) = \int_{\hat{G}} \phi(xy^{-1}, \xi)\xi(xy)d\xi.$$

If $\phi \in L^1 \bigcap L^2(G \times \hat{G})$, then $K_\phi \in L^2(G \times G)$, which implies that $W(\phi)$ is a Hilbert-Schmidt operator whose norm is given by $\|W(\phi)\|_{\mathbb{HS}} = \|K_\phi\|_{L^2(G\times G)}$, where $\mathbb{HS}$ is the Hilbert space of Hilbert Schmidt operators on $L^2(G \times G)$ with inner product $\prec T, S \succ = tr(TS^*)$.(See [25]). Now we define twisted translate.

Definition 3.1. *Let ϕ be in $L^2(G \times \hat{G})$. For $(y, \chi) \in G \times \hat{G}$ we define twisted translate of ϕ, denoted by $T^t_{(y,\chi)}$ as:*

$$T^t_{(y,\chi)}\phi(x, \xi) = \phi(xy^{-1}, \xi\bar{\chi})\chi(x)\overline{\xi(y)}.$$

Remark 3.2. *The twisted translate introduced in Definition 3.1 is adapted to the phase-space group $G \times \widehat{G}$ and is chosen so that it interacts naturally with the Weyl transform studied in this paper. In particular, it involves a simultaneous translation on both variables together with the modulation factor $\chi(x)\xi(y)$, which is essential for the covariance relations established later. This normalization is therefore different from the twisted translation convention used in* [22].

In the following proposition we show that T^t is projective representation.

Proposition 3.3. *The mapping $T^t : (y, \chi) \mapsto T^t_{(y,\chi)}$ defines a projective representation of $G \times \widehat{G}$ on $L^2(G \times \widehat{G})$.*

Proof. It is clear that for $\phi \in L^2(G\times\hat{G})$, the operator $T^t_{(y,\chi)}$ is unitary on $L^2(G\times\hat{G})$, for $(y,\chi)\in G\times\hat{G}$. Also, for $(y_1,\chi_1),(y_2,\chi_2)\in G\times\hat{G}$ we have,

$$\begin{aligned}
T^t_{(y_1,\chi_1)}T^t_{(y_2,\chi_2)}\phi(x,\xi) &= T^t_{(y_2,\chi_2)}\phi(xy_1^{-1},\xi\bar{\chi_1})\chi_1(x)\overline{\xi(y_1)}\\
&= \phi(xy_1^{-1}y_2^{-1},\xi\bar{\chi_1}\bar{\chi_2})\chi_2(xy_1^{-1})\overline{\xi\bar{\chi_1}(y_2)}\chi_1(x)\overline{\xi(y_1)}\\
&= \phi(xy_1^{-1}y_2^{-1},\xi\bar{\chi_1}\bar{\chi_2})\chi_2(x)\overline{\chi_2(y_1)\xi(y_2)}\chi_1(y_2)\chi_1(x)\overline{\xi(y_1)}\\
&= \phi(xy_1^{-1}y_2^{-1},\xi\bar{\chi_1}\bar{\chi_2})\chi_1\chi_2(x)\overline{\xi(y_1y_2)\chi_2(y_1)}\chi_1(y_2)\\
&= \overline{\chi_2(y_1)}\chi_1(y_2)T^t_{(y_1,\chi_1)(y_2,\chi_2)}\phi(x,\xi).
\end{aligned}$$

Thus,

$$T^t_{(y_1,\chi_1)}T^t_{(y_2,\chi_2)} = \overline{\chi_2(y_1)}\chi_1(y_2)T^t_{(y_1,\chi_1)(y_2,\chi_2)}.$$

□

The following proposition shows that how the kernel of the Weyl transform shifts under twisted translation.

Proposition 3.4. *For $\phi\in L^2(G\times\hat{G})$ and $(z,\chi)\in G\times\hat{G}$ we have,*

$$K_{T^t_{(z,\chi)}\phi}(x,y) = \chi(x)^2\overline{\chi(z)}K_\phi(xz^{-1},y).$$

Proof. By the definition of kernel and twisted translate we have,

$$\begin{aligned}
K_{T^t_{(z,\chi)}\phi}(x,y) &= \int_{\hat{G}} T^t_{(z,\chi)}\phi(xy^{-1},\xi)\xi(xy)d\xi\\
&= \int_{\hat{G}} \phi(xy^{-1}z^{-1},\xi\bar{\chi})\overline{\xi(z)}\chi(xy^{-1})\xi(xy)d\xi\\
&= \int_{\hat{G}} \phi(xy^{-1}z^{-1},\xi)\overline{\xi\chi(z)}\chi(x)\overline{\chi(y)}\xi\chi(xy)d\xi\\
&= \chi(x)^2\overline{\chi(z)}\int_{\hat{G}} \phi(xy^{-1}z^{-1},\xi)\xi(xyz^{-1})d\xi\\
&= \chi(x)^2\overline{\chi(z)}K_\phi(xz^{-1},y).
\end{aligned}$$

□

The relation between the Weyl transform and twisted translate is as follows:

Proposition 3.5. *For $\phi\in L^2(G\times\hat{G})$ and $(z,\chi)\in G\times\hat{G}$ we have,*

$$W(T^t_{(z,\chi)}\phi) = W(\phi)\pi(z,\chi).$$

Proof. For $f \in L^2(G)$ and $(z,\chi) \in G \times \hat{G}$ we get,

$$\begin{aligned}
[W(T^t_{(z,\chi)}\phi)f](y) &= \int_G \int_{\hat{G}} T^t_{(z,\chi)}\phi(x,\xi)[\pi(x,\xi)f](y)dxd\xi \\
&= \int_G \int_{\hat{G}} \phi(xz^{-1},\xi\bar{\chi})\chi(x)\overline{\xi(z)}\xi(y)^2\xi(x)f(xy)dxd\xi \\
&= \int_G \int_{\hat{G}} \phi(x,\xi\bar{\chi})\chi(xz)\overline{\xi(z)}\xi(y)^2\xi(xz)f(xzy)dxd\xi \\
&= \int_G \int_{\hat{G}} \phi(x,\xi)\chi(x)\chi(z)\overline{\xi\chi(z)}\xi\chi(y)^2\xi\chi(xz)f(xyz)dxd\xi \\
&= \int_G \int_{\hat{G}} \phi(x,\xi)\chi(x)\chi(z)\xi(y)^2\chi(y)^2\xi(x)\chi(x)f(xyz)dxd\xi,
\end{aligned}$$

on the other hand we have,

$$\begin{aligned}
[W(\phi)\pi(z,\chi)f](y) &= \int_G \int_{\hat{G}} \phi(x,\xi)[\pi(x,\xi)\pi(z,\chi)f](y)dxd\xi \\
&= \int_G \int_{\hat{G}} \phi(x,\xi)\chi(x)\overline{\xi(z)}[\pi(xz,\xi\chi)f](y)dxd\xi \\
&= \int_G \int_{\hat{G}} \phi(x,\xi)\chi(x)\overline{\xi(z)}\xi\chi(y)^2\xi\chi(xz)f(xzy)dxd\xi \\
&= \int_G \int_{\hat{G}} \phi(x,\xi)\chi(x)\xi(y)^2\chi(y)^2\xi(x)\chi(x)\chi(z)f(xyz)dxd\xi.
\end{aligned}$$

So the proof is completed. $\square$

We now synthesize the Weyl and Zak transforms to define the central object of this paper.

Definition 3.6. *Let $L \subset G$ be a uniform lattice with annihilator $L^\perp \subset \widehat{G}$, and let $S_L \subset G$ and $S_{L^\perp} \subset \widehat{G}$ be fixed fundamental domains for L and $L^\perp$, respectively. The* Weyl–Zak transform *Z_w of a function $\phi \in L^2(G \times \widehat{G})$ is defined by*

$$Z_w\phi(x,\xi,y) = \sum_{l \in L} K_\phi(lx,y)\overline{\xi(l)},$$

where $(x,\xi,y) \in S_L \times S_{L^\perp} \times G$ and $K_\phi(x,y) = \int_{\hat{G}} \phi(xy^{-1},\xi)\xi(xy)d\xi$.

We now state the main result of this section, which shows that the Weyl-Zak transform is a unitary operator between the relevant L^2-spaces.

Theorem 3.7. *The Weyl-Zak transform Z_w is an isometric isomorphism between $L^2(G \times \hat{G})$ and $L^2(S_L \times S_{L^\perp} \times G)$.*

Proof. For $\phi \in L^2(G \times \hat{G})$ by the formulas (2.2), (2.1) and plancharel theorem we get,

$$\begin{aligned}
\|Z_w\phi\|^2_{L^2(S_L \times S_{L^\perp} \times G)} &= \int_{S_L}\int_{S_{L^\perp}}\int_G |Z_w\phi(x,\xi,y)|^2 dyd\xi dx \\
&= \int_{S_L^\perp}\int_G\int_{S_L} |\sum_{l\in L} K_\phi(lx,y)\overline{\xi(l)}|^2 dxdyd\xi \\
&= \int_{S_L^\perp}\int_G \sum_{\gamma\in L^\perp} |\widehat{K_\phi(\gamma\xi,y)}|^2 dyd\xi \\
&= \int_G\int_{S_L^\perp} \sum_{\gamma\in L^\perp} |\widehat{K_\phi(\gamma\xi,y)}|^2 dyd\xi \\
&= \int_G\int_{\hat{G}} |\widehat{K_\phi(\omega,y)}|^2 d\omega dy \\
&= \int_G\int_G |K_\phi(x,y)|^2 dxdy \\
&= \|K_\phi\|^2_{L^2(G\times\hat{G})} \\
&= \|\phi\|^2_{L^2(G\times\hat{G})}
\end{aligned}$$

Moreover, Z_w is surjective. To prove this, let $F \in L^2(S_L \times S_{L^\perp} \times G)$. For almost every $x \in S_L$ and $y \in G$, we have $F(x,\cdot,y) \in L^2(S_{L^\perp})$. Its Fourier expansion on the quotient group $\widehat{L} \cong \widehat{G}/L^\perp$ (which we identify with $S_{L^\perp}$) is given by

$$F(x,\xi,y) = \sum_{\ell\in L} \widehat{F}(x,\ell,y)\overline{\xi(\ell)}, \tag{3.1}$$

where $\widehat{F}(x,\cdot,y)$ denotes the Fourier coefficients. Now, define a function K_F on $G \times G$ by

$$K_F(x,y) := \widehat{F}(s,\ell,y), \quad \text{for } x = \ell s \text{ with } \ell \in L,\ s \in S_L.$$

This definition is well-defined since the decomposition $x = \ell s$ is unique. Using the Plancherel formula on the quotient $\widehat{G}/L^\perp$, we compute:

$$\begin{aligned}
\int_G\int_G |K_F(x,y)|^2\,dx\,dy &= \int_G\int_{S_L}\sum_{\ell\in L} |K_F(\ell s,y)|^2\,ds\,dy \\
&= \int_G\int_{S_L}\sum_{\ell\in L} |\widehat{F}(s,\ell,y)|^2\,ds\,dy \\
&= \int_G\int_{S_L}\int_{\widehat{L}} |F(s,\eta,y)|^2\,d\eta\,ds\,dy \\
&= \int_G\int_{S_L}\int_{S_{L^\perp}} |F(s,\eta,y)|^2\,d\eta\,ds\,dy < \infty,
\end{aligned}$$

where we used the identification $\widehat{L} \cong S_{L^\perp}$. Hence, $K_F \in L^2(G \times G)$.

Since the Weyl transform W is a unitary (and hence surjective) map from $L^2(G \times \widehat{G})$ onto the space of Hilbert–Schmidt operators on $L^2(G)$, which is isomorphic to $L^2(G\times G)$, there exists a unique $\phi \in L^2(G \times \widehat{G})$ such that its Weyl kernel satisfies

$$K_\phi(x, y) = K_F(x, y) \quad \text{for a.e. } x, y \in G.$$

Here, K_ϕ is the kernel associated to the Weyl transform W_ϕ, defined by $(W_\phi f)(y) = \int_G K_\phi(x, y) f(x)\, dx$ for $f \in L^2(G)$. Finally, we verify that $Z_w\phi = F$. Using the definition of the Weyl–Zak transform and the relation $x = \ell' s$ with $\ell' \in L,\ s \in S_L$, we obtain for $(x, \xi, y) \in S_L \times S_{L^\perp} \times G$:

$$\begin{aligned}
(Z_w\phi)(x, \xi, y) &= \sum_{\ell\in L} K_\phi(\ell x, y)\,\overline{\xi(\ell)} \\
&= \sum_{\ell\in L} K_F(\ell\ell' s, y)\,\overline{\xi(\ell)} \quad (\text{since } K_\phi = K_F) \\
&= \sum_{\ell\in L} \widehat{F}(s, \ell\ell', y)\,\overline{\xi(\ell)} \quad (\text{by definition of } K_F) \\
&= \overline{\xi(\ell')} \sum_{\ell\in L} \widehat{F}(s, \ell\ell', y)\,\overline{\xi(\ell\ell')} \quad (\text{multiplying and dividing by } \overline{\xi(\ell')}) \\
&= \overline{\xi(\ell')} \sum_{\lambda\in L} \widehat{F}(s, \lambda, y)\,\overline{\xi(\lambda)} \quad (\text{setting } \lambda = \ell\ell') \\
&= \overline{\xi(\ell')} F(s, \xi, y) \quad (\text{by the Fourier expansion (3.1)}).
\end{aligned}$$

To conclude, we recall that any function H in the range of the Weyl–Zak transform must satisfy the quasi-periodicity condition $H(\ell' s, \xi, y) = \overline{\xi(\ell')} H(s, \xi, y)$ for $\ell' \in L, s \in S_L$. The function F was constructed precisely via its Fourier coefficients $\widehat{F}(s, \ell, y)$, and this construction guarantees that F itself satisfies the same condition: $F(x, \xi, y) = F(\ell' s, \xi, y) = \overline{\xi(\ell')} F(s, \xi, y)$. Therefore,

$$(Z_w\phi)(x, \xi, y) = \overline{\xi(\ell')} F(s, \xi, y) = F(\ell' s, \xi, y) = F(x, \xi, y),$$

which completes the proof of surjectivity. $\square$

The following proposition shows the relation between image of twisted translates of ϕ with image of ϕ under Weyl-Zak transform.

Proposition 3.8. *Let $\phi \in L^2(G \times \hat{G})$. The Weyl-Zak transform of twisted transform of ϕ is given by,*

$$Z_w T^t_{(s,\eta)}\phi(x, \xi, y) = \eta(x)^2 \overline{\xi(s)} Z_w\phi(x, \xi, y),$$

for, $(s, \eta) \in L \times L^\perp, x \in S_L, \xi \in S_{L^\perp}, y \in G$.

Proof. For $\phi \in L^2(G \times \hat{G})$, we get

$$\begin{aligned} Z_w T^t_{(s,\eta)}\phi(x,\xi,y) &= \sum_{l\in L} K_{T^t_{(s,\eta)}\phi}(xl,y)\overline{\xi(l)} \\ &= \eta(x)^2\overline{\eta(s)}\sum_{l\in L} K_\phi(xs^{-1}l,y)\overline{\xi(l)} \\ &= \eta(x)^2\overline{\xi(s)}\sum_{l\in L} K_\phi(xl,y)\overline{\xi(l)} \\ &= \eta(x)^2\overline{\xi(s)} Z_w\phi(x,\xi,y). \end{aligned}$$

□

Using the Weyl-Zak transform, we now introduce a bracket product $[\cdot,\cdot]$ on $L^2(G \times \widehat{G})$.

Definition 3.9. *For two functions $\phi, \psi \in L^2(G \times \hat{G})$, we define the bracket $[\phi, \psi]$ via their Weyl-Zak transform as follows:*

$$[\phi,\psi](x,\xi) = \int_G Z_w\phi(x,\xi,y)\overline{Z_w\psi(x,\xi,y)}dy = \prec Z_w\phi(x,\xi,.), Z_w\psi(x,\xi,.) \succ_{L^2(G)} .$$

This definition is well-defined. In fact, by the Cauchy- Schwartz inequality we have,

$$\begin{aligned} |\int_G Z_w\phi(x,\xi,y)\overline{Z_w\psi(x,\xi,y)}dy| &\leq \int_G |Z_w\phi(x,\xi,y)||\overline{Z_w\psi(x,\xi,y)}|dy \\ &\leq \left(\int_G |Z_w\phi(x,\xi,y)|^2 dy\right)^{1/2} \left(\int_G |Z_w\psi(x,\xi,y)|^2 dy\right)^{1/2} < \infty, \end{aligned}$$

since $Z_w\phi(x,\xi,.) \in L^2(G)$. Also,

$$[\phi,\phi](x,\xi) = \int_G |Z_w\phi(x,\xi,y)|^2 dy \geq 0,$$

for $(x,\xi) \in S_L \times S_{L^\perp}$. Furthermore,

$$|[\phi,\psi](x,\xi)| \leq ([\phi,\phi](x,\xi))^{1/2}\,([\psi,\psi](x,\xi))^{1/2} .$$

We claim that the bracket product $[\phi,\psi]$ is integrable on $S_L\times S_{L^\perp}$. Indeed,

$$\begin{aligned}
&\|[\phi,\psi]\|_{L^1(S_L\times S_{L^\perp})}=\int_{S_L}\int_{S_{L^\perp}}[\phi,\psi](x,\xi)dxd\xi\\
&\leq\int_{S_L}\int_{S_{L^\perp}}\left([\phi,\phi](x,\xi)\right)^{1/2}\left([\psi,\psi](x,\xi)\right)^{1/2}dxd\xi\\
&\leq\left(\int_{S_L}\int_{S_{L^\perp}}[\phi,\phi](x,\xi)dxd\xi\right)^{1/2}\left(\int_{S_L}\int_{S_{L^\perp}}[\psi,\psi](x,\xi)dxd\xi\right)^{1/2}\\
&=\left(\int_{S_L}\int_{S_{L^\perp}}\int_G|Z_w\phi(x,\xi,y)|^2dxd\xi dy\right)^{1/2}\left(\int_{S_L}\int_{S_{L^\perp}}\int_G|Z_w\psi(x,\xi,y)|^2dxd\xi dy\right)^{1/2}\\
&=\|Z_w\phi\|^2\|Z_w\psi\|^2\\
&=\|\phi\|^2\|\psi\|^2<\infty
\end{aligned}$$

In the following proposition, we investigate some interesting properties about twisted translates and brackets.

Proposition 3.10. *For $(s,\eta)\in L\times L^\perp$ and $\phi,\psi\in L^2(G\times\hat{G})$, we have the following properties:*

(i) $\prec\phi,T^t_{(s,\eta)}\psi\succ_{L^2(G\times\hat{G})}=\int_{S_L}\int_{S_{L^\perp}}[\phi,\psi](x,\xi)\overline{\eta(x)^2}\xi(s)dxd\xi.$

(ii) $[T^t_{(s,\eta)}\phi,\psi](x,\xi)=\eta(x)^2\overline{\xi(s)}[\phi,\psi](x,\xi).$

(iii) $[\phi,T^t_{(s,\eta)}\psi](x,\xi)=\overline{\eta(x)^2}\xi(s)[\phi,\psi](x,\xi).$

(iv) $\phi\perp V^t(\psi) if \quad and \quad only \quad if [\phi,\psi]=0 \qquad a.e (x,\xi)\in S_L\times S_{L^\perp},$
where, $V^t(\psi)=\overline{span\{T^t_{(s,\eta)}\psi;(s,\eta)\in L\times L^\perp\}}.$

Proof. Let $(s,\eta)\in L\times L^\perp$ and $\phi,\psi\in L^2(G\times\hat{G})$. Then we get,

$$\begin{aligned}
\prec\phi,T^t_{(s,\eta)}\psi\succ_{L^2(G\times\hat{G})}&=\prec Z_w\phi,Z_wT^t_{(s,\eta)}\psi\succ_{L^2(S_L\times S_{L^\perp}\times G)}\\
&=\int_{S_L}\int_{S_{L^\perp}}\int_G Z_w\phi(x,\xi,y)\overline{Z_w\psi(x,\xi,y)\eta(x)^2\xi(s)}dxd\xi dy\\
&=\int_{S_L}\int_{S_{L^\perp}}\overline{\eta(x)^2}\xi(s)\left(\int_G Z_w\phi(x,\xi,y)\overline{Z_w\psi(x,\xi,y)}dy\right)dxd\xi\\
&=\int_{S_L}\int_{S_{L^\perp}}[\phi,\psi](x,\xi)\overline{\eta(x)^2}\xi(s)dxd\xi.
\end{aligned}$$

For part (ii) we have,

$$\begin{aligned}
\prec T^t_{(s,\eta)}\phi, T^t_{(s',\eta')}\psi \succ_{L^2(G\times\hat{G})} &= \prec \phi, T^t_{(s^{-1},\bar{\eta})}T^t_{(s',\eta')}\psi \succ_{L^2(G\times\hat{G})} \\
&= \prec \phi, \overline{\eta'(s^{-1})\eta(s)}T^t_{(s^{-1}s',\bar{\eta}\eta')}\psi \succ_{L^2(G\times\hat{G})} \\
&= \overline{\eta'(s)}\eta(s) \prec \phi, T^t_{(s^{-1}s',\bar{\eta}\eta')}\psi \succ_{L^2(G\times\hat{G})} \\
&= \int_{S_L}\int_{S_{L^\perp}} [\phi,\psi](x,\xi)\overline{\bar{\eta}\eta'}^2(x)\xi(s^{-1}s')dxd\xi \\
&= \int_{S_L}\int_{S_{L^\perp}} [\phi,\psi](x,\xi)\eta(x)^2\overline{\eta'(x)}^2\overline{\xi(s)}\xi(s')dxd\xi.
\end{aligned}$$

On the other hands,

$$\prec T^t_{(s,\eta)}\phi, T^t_{(s',\eta')}\psi \succ_{L^2(G\times\hat{G})} = \int_{S_L}\int_{S_{L^\perp}} [T^t_{(s,\eta)}\phi,\psi](x,\xi)\overline{\eta'(x)}^2\xi(s')dxd\xi,$$

so we get,

$$[T^t_{(s,\eta)}\phi,\psi](x,\xi) = \eta(x)^2\overline{\xi(s)}[\phi,\psi](x,\xi), \quad a.e.(x,\xi)\in S_L\times S_{L^\perp}.$$

part(iii):

$$[\phi, T^t_{(s,\eta)}\psi](x,\xi) = [T^t_{(s^{-1},\bar{\eta})}\phi,\psi](x,\xi) = \overline{\eta(x)^2}\xi(s)[\phi,\psi](x,\xi).$$

And for part (iv), suppose that $\phi \perp V^t(\psi)$. Then $\prec \phi, T^t_{(s,\eta)}\psi \succ = 0$, for $(s,\eta)\in L\times L^\perp$. By part (i) it is equivalent to

$$\int_{S_L}\int_{S_{L^\perp}} [\phi,\psi](x,\xi)\overline{\eta(x)^2}\xi(s)dxd\xi = 0,$$

for $(s,\eta)\in L\times L^\perp$, which is equivalent to $[\phi,\psi](x,\xi)=0$, a.e. $(s,\eta)\in L\times L^\perp$. $\square$

Remark 3.11 (Relation with the Euclidean case)**.** *The present setting on LCA groups contains the classical Euclidean situation as a special case. When $G=\mathbb{R}^n$, the dual group $\widehat{G}$ is again identified with $\mathbb{R}^n$, so that $G\times\widehat{G}\simeq\mathbb{R}^{2n}$. Under this identification, the twisted translate in Definition 3.1 coincides with the operator defined in* [21, Definition 2.4]. *Likewise, the Weyl–Zak transform in Definition 3.6 and the bracket product in Definition 3.9 agree with the corresponding Euclidean constructions discussed in* [23, Section 3]. *Hence our results provide a unified extension of the Euclidean theory to arbitrary LCA groups, including the framework of* [23].

Remark 3.12. *Consider the representation $\tilde{\pi}: G\times\hat{G}\to U(L^2(\hat{G}))$ defined by*

$$[\tilde{\pi}(x,\xi)f](\eta) = \xi(x)\eta(x)^2 f(\xi\eta),$$

where $f\in L^2(\hat{G})$ and $(x,\xi)\in G\times\hat{G}$. This mapping $\tilde{\pi}$ constitutes a projective representation, with the multiplier $\alpha:(G\times\hat{G})\times(G\times\hat{G})\to\mathbb{C}$ given by

$$\alpha((x_1,\xi_1),(x_2,\xi_2)) = \xi_1(x_2)\overline{\xi_2(x_1)}.$$

Moreover, for $\phi \in C_c(G \times \hat{G})$, the Weyl transform W_ϕ on $L^2(\hat{G})$ can be defined by

$$[W_\phi(f)](\eta) = \int_G \int_{\hat{G}} \phi(x,\xi)\xi(x)\eta(x)^2 f(\xi\eta)dxd\xi,$$

for all $f \in L^2(\hat{G})$. The kernel of this Weyl transform, denoted by $K_\phi : \hat{G} \times \hat{G} \to \mathbb{C}$, is given by

$$K_\phi(\xi,\eta) = \int_G \phi(x,\xi\bar{\eta})\eta\xi(x)dx.$$

In addition, under twisted translation by (z,x), the kernel transforms are as follows:

$$K_{T^t_{(z,\chi)}\phi}(\xi,\eta) = \eta(z)^2\chi(z)K_\phi(\xi,\eta\chi).$$

Remark 3.13. *The constructions of this paper admit a natural dual formulation. Instead of working on $L^2(G)$, one may consider the representation $\tilde{\pi} : G \times \widehat{G} \to U(L^2(\widehat{G}))$ defined by*

$$[\tilde{\pi}(x,\xi)f](\eta) = \xi(x)\eta(x)^2 f(\xi\eta), \qquad f \in L^2(\widehat{G}),\ (x,\xi) \in G \times \widehat{G}.$$

This mapping $\tilde{\pi}$ is a projective representation with the same multiplier α given in (3.12)*. For $\phi \in C_c(G \times \widehat{G})$, the corresponding Weyl transform $\widetilde{W}_\phi$ on $L^2(\widehat{G})$ is then*

$$[\widetilde{W}_\phi(f)](\eta) = \int_G \int_{\widehat{G}} \phi(x,\xi)\,\xi(x)\eta(x)^2 f(\xi\eta)\,dx\,d\xi, \quad f \in L^2(\widehat{G}),$$

and its kernel $\widetilde{K}_\phi : \widehat{G} \times \widehat{G} \to \mathbb{C}$ satisfies

$$\widetilde{K}_\phi(\xi,\eta) = \int_G \phi(x,\xi\bar{\eta})\,\eta\xi(x)\,dx.$$

Under twisted translation by $(z,\chi) \in G \times \widehat{G}$, this kernel transforms as

$$\widetilde{K}_{T^t_{(z,\chi)}\phi}(\xi,\eta) = \eta(z)^2\chi(z)\,\widetilde{K}_\phi(\xi,\eta\chi).$$

This dual picture provides an equivalent framework for defining the Weyl–Zak transform. Indeed, one may set

$$\widetilde{Z}_w\phi(\xi,x,\eta) = \sum_{\nu \in L^\perp} \widetilde{K}_\phi(\xi\nu,\eta)\,\overline{x(\nu)},$$

where $(\xi,x,\eta) \in S_{L^\perp} \times S_L \times \widehat{G}$ and

$$\widetilde{K}_\phi(\xi,\eta) = \int_G \phi(\xi\eta^{-1},x)\,x(\xi\eta)\,dx.$$

for $(x,\xi) \in S_L \times S_{L^\perp}$ and $\eta \in \widehat{G}$. The results of this paper, including the unitarity and bracket product theory, carry over to this setting with appropriate modifications.

Data availability. No data sets were generated or analysed during the current study.

Conflicts of Interest. The authors confirm the above information which is true and correct and indicate their agreement.

Acknowledgments

We would like to extend our sincere gratitude to the anonymous reviewers for their helpful comments and suggestions, which have greatly contributed to enhancing the quality of this manuscript.

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