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# FIXED POINTS OF LARGE ENRICHED CONTRACTIONS IN CONVEX PROBABILISTIC $b$-METRIC SPACES AND APPLICATIONS TO FREE MECHANICAL OSCILLATIONS

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Abstract. This study explores a collection of fixed point theorems and the existence of fixed points for large enriched contractions within convex probabilistic $b$-metric spaces. We identify the necessary conditions that guarantee the presence of fixed points for such mappings. The research employs the extended Mann's iteration algorithm in this context. Several intricate examples are presented, and our principal findings are applied to the investigation of free mechanical oscillations, which are described by second-order differential equations.

## 1. Introduction

Nonlinear analysis is a vital and widely utilized branch of mathematics, with fixed point theory serving as one of its central components. This theory is instrumental in establishing the existence of solutions to various equations, including integral, fractional, and differential equations, as well as systems comprising these equations. A significant facet of nonlinear analysis involves developing diverse metric spaces, such as $b$-metric, $G$-metric, $s$-metric, bipolar metric, and probabilistic metric spaces, the latter originally introduced by Menger [32]. The foundational principles of metric spaces were laid out by Fréchet in 1906 [15]. Since then, the concept has undergone numerous modifications and generalizations, including adjustments to the metric function and relaxations of classical axioms, thereby enhancing its scope and applicability. This work proposes a novel framework that extends the structure of probabilistic metric spaces.

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In 1970, Takahashi [39] introduced the concept of convexity in the context of metric spaces, giving rise to what are now termed convex metric spaces. This foundational idea has played a significant role in the advancement of various fixed point theorems. Later, in 1988, Ding [14] investigated the convergence properties of the Ishikawa iteration process for quasi-contractive and quasi-nonexpansive operators within convex metric spaces. Following this, Beg [2] analyzed the convergence of asymptotically nonexpansive mappings using Mann-type iterative methods in uniformly convex metric spaces. In 2016, Fukhar-ud-din and Berinde [16] proposed a modified Noor iteration scheme specifically designed for quasi-contractive-type operators in convex metric spaces. Further extending these developments, Chen et al. [10] in 2020 introduced the framework of convex $b$-metric spaces and adapted the Mann iteration process to accommodate this generalization.

**Definition 1.1** ([39])**.** *Consider a metric space $(\Theta, \eta)$. A mapping $\mathcal{B} : \Theta \times \Theta \times [0, 1] \to \Theta$ is said to exhibit a convex structure if, for all $v \in \Theta$, there exist $\wp, \Re \in \Theta$ and $\lambda \in [0, 1]$ such that*

$$\eta(v, \mathcal{B}(\wp, \Re, \lambda)) \leq \lambda\eta(v, \wp) + (1 - \lambda)\eta(v, \Re).$$

*A metric space $(\Theta, \eta)$ equipped with such a convex structure $\mathcal{B}$ is termed a convex metric space and denoted as $(\Theta, \eta, \mathcal{B})$.*

The notion of $b$-metric spaces was first introduced by Bakhtin [1] and subsequently expanded by Czerwik [13].

**Definition 1.2** ([1, 13])**.** *Let $\Theta$ be a nonempty set, and let the functional $\eta_b : \Theta \times \Theta \to [0, \infty)$ satisfy:*

- $(b_1)$ *$\eta_b(\wp, \Re) = 0$ if and only if $\wp = \Re$,*
- $(b_2)$ *$\eta_b(\wp, \Re) = \eta_b(\Re, \wp)$ for all $\wp, \Re \in \Theta$,*
- $(b_3)$ *there exists a real number $s \geq 1$ such that $\eta_b(\wp, z) \leq s[\eta_b(\wp, \Re) + \eta_b(\Re, z)]$ for all $\wp, \Re, z \in \Theta$.*

*Then $\eta_b$ is called a b-metric on $\Theta$ and a pair $(\Theta, \eta_b)$ is called a b-metric space with coefficient $s$.*

Various investigations have since addressed their characteristics, including convergence criteria, Cauchy sequences, and fixed point results pertinent to nonlinear functional analysis. Comprehensive reviews on these topics can be found in the literature [7–9, 22, 23, 26–29, 34]. In contrast, probabilistic metric spaces were initially formulated by Menger in 1942 [31], followed by significant contributions to fixed point theory from Sehgal and Bharucha-Reid [4].

**Definition 1.3** ([31])**.** *A triangular norm (abbreviated, $t$-norm) is a binary operation $\mathcal{Y}$ on $[0, 1]$, which satisfies the following conditions:*

- (a) *$\mathcal{Y}$ is associative and commutative;*
- (b) *$\mathcal{Y}$ is continuous;*
- (c) *$\mathcal{Y}(a, 1) = a$ for all $a \in [0, 1]$;*
- (d) *$\mathcal{Y}(a, b) \leq \mathcal{Y}(c, d)$ whenever $a \leq c$ and $b \leq a$, for each $a, b, c, d \in [0, 1]$.*

**Definition 1.4** ([31])**.** *A Menger probabilistic metric space (briefly, Menger PM-space) is a triple $(\Theta, \urcorner, \mathcal{Y})$, where $\Theta$ is a nonempty set, $\mathcal{Y}$ is a continuous $t$-norm and $\urcorner$ is a mapping from $\Theta \times \Theta$ into $D^+$ such that if $\urcorner_{\wp,\Re}$ denotes the value of $\urcorner$ at the pair $(\wp, \Re)$, then the following conditions hold:*

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(PM1) $\daleth_{\wp,\Re}(\aleph) = 1$ *for all* $\aleph > 0 \Leftrightarrow \wp = \Re$,
(PM2) $\daleth_{\wp,\Re}(\aleph) = \daleth_{\Re,\wp}(\aleph)$ *for all* $\wp, \Re \in \Theta$ *and* $\aleph > 0$,
(PM3) $\daleth_{\wp,z}(\aleph_1 + \aleph_2) \geq \mathcal{Y}(\daleth_{\wp,\Re}(\aleph_1), \daleth_{\Re,z}(\aleph_2))$ *for all* $\wp, \Re, z \in \Theta$ *and* $\aleph_1, \aleph_2 \geq 0$.

Schweizer and Sklar [37] conducted thorough studies on Menger probabilistic metric spaces ($MPM$-spaces), addressing both single-valued and multi-valued mappings as documented in several works [11, 12, 18, 19, 21, 24, 25, 30, 33, 35, 36, 38]. In 2015, Hasanvand and Khanehgir [20] established the framework of Menger probabilistic $b$-metric spaces ($MPbM$-spaces) and proved a fixed point theorem for single-valued mappings.

**Definition 1.5** ( [31]). *A Menger probabilistic $b$-metric space (briefly, Menger $PbM$-space) with coefficient $\alpha \in (0, 1]$ is a triple $(\Theta, \daleth, \mathcal{Y})$, where $\Theta$ is a nonempty set, $\mathcal{Y}$ is a continuous $t$-norm and $\daleth$ is a mapping from $\Theta \times \Theta$ into $D^+$ such that if $\daleth_{\wp,\Re}$ denotes the value of $\daleth$ at the pair $(\wp, \Re)$, then the following conditions hold:*
(PM1) $\daleth_{\wp,\Re}(\aleph) = 1$ *for all* $\aleph > 0 \Leftrightarrow \wp = \Re$;
(PM2) $\daleth_{\wp,\Re}(\aleph) = \daleth_{\Re,\wp}(\aleph)$ *for all* $\wp, \Re \in \Theta$ *and* $\aleph > 0$;
(PM3) $\daleth_{\wp,z}(\aleph_1 + \aleph_2) \geq \mathcal{Y}(\daleth_{\wp,\Re}(\alpha\aleph_1), \daleth_{\Re,z}(\alpha\aleph_2))$ *for all* $\wp, \Re, z \in \Theta$ *and* $\aleph_1, \aleph_2 \geq 0$.

A notable advancement in this area was made by Branciari in 2000 with the introduction of the quadrilateral inequality, which played a crucial role in extending the Banach contraction principle and led to the emergence of rectangular metric spaces [5]. Additionally, Burton [6] introduced the concept of large contraction mappings in 1996 to address integral equations. These mappings generalize Banach contractions; while every Banach contraction qualifies as a large contraction, the converse does not hold.

**Definition 1.6** ([6]). *Let $(\Theta, \eta)$ be a metric space. A mapping $\mathcal{R} : \Theta \to \Theta$ is termed a large contraction if for any $\wp, \Re \in \Theta$ with $\wp \neq \Re$, it holds that $\eta(\mathcal{R}\wp, \mathcal{R}\Re) < \eta(\wp, \Re)$, and for every $\epsilon > 0$, there exists $0 < \delta < 1$ such that*

$$\eta(\wp, \Re) \geq \epsilon \Rightarrow \eta(\mathcal{R}\wp, \mathcal{R}\Re) \leq \delta\eta(\wp, \Re).$$

More recently, Berinde and Pacurar [3] introduced the concept of enriched contractions within convex metric spaces.

**Definition 1.7** ( [3]). *Let $(\Theta, \eta, \mathcal{B})$ be a convex metric space. A mapping $\mathcal{R} : \Theta \to \Theta$ is called an enriched contraction if there exist $r, \lambda \in [0, 1)$ such that*

$$\eta(\mathcal{B}(\wp, \mathcal{R}\wp, \lambda), \mathcal{B}(\Re, \mathcal{R}\Re, \lambda)) \leq r\eta(\wp, \Re)$$

*for all distinct $\wp, \Re \in \Theta$.*

For a comprehensive understanding of foundational topics such as $b$-metric spaces, $MPM$-spaces, distribution functions, and $H$-type (Hadzić type) $t$-norms, readers are encouraged to consult references [1, 5, 11, 13, 17, 19] and the works cited therein. The primary objective of this paper is to extend the concept of enriched contractions and investigate the existence of fixed points within the context of

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convex probabilistic $b$-metric spaces ($CPbMs$). The manuscript is organized into four main sections. Section 1 provides an introduction. In Section 2, we formally define the $CPbM$ space by incorporating convex structures and discuss the notion of large enriched contractions in this setting. Additionally, we generalize Mann's iteration process to $CPbM$ spaces. Section 3 further develops the theory of large enriched contractions in $CPbM$ spaces and presents new fixed point theorems for single-valued mappings in this framework. Section 4 applies the theoretical results to demonstrate the existence of solutions for initial value problems associated with free mechanical oscillations described by second-order differential equations.

## 2. **Enriched contraction in $CPbM$-spaces**

This section addresses the existence and uniqueness of fixed points for large enriched contractions in $CPbM$-spaces.

**Definition 2.1.** *Let $(\Theta, \daleth, \mathcal{Y})$ be a probabilistic $b$-metric space with coefficient $\alpha \in (0,1]$ and continuous $t$-norm $\mathcal{Y}$. A mapping $\mathcal{S} : \Theta \times \Theta \times [0,1] \to \Theta$ is said to be a convex structure on $\Theta$, if for every $(\wp, \Re) \in \Theta \times \Theta$ holds $\mathcal{S}(\wp, \Re, 0) = \Re$, $\mathcal{S}(\wp, \Re, 1) = \wp$ and for all $\wp, \Re, v \in \Theta$, $\lambda \in [0,1]$ and $\epsilon > 0$*

$$\daleth_{\mathcal{S}(\wp,\Re,\lambda),v}(2\epsilon) \geq \mathcal{Y}(\daleth_{\wp,v}(\frac{\epsilon}{\lambda}), \daleth_{\Re,v}(\frac{\epsilon}{1-\lambda})). \tag{2.1}$$

A probabilistic $b$-metric space $(\Theta, \daleth, \mathcal{Y})$ with coefficient $\alpha \in (0,1]$ and a convex structure $\mathcal{S}$ is called a convex probabilistic $b$-metric space.

Example 2.2. *Let $\Theta = \mathbb{R}$ and $\mathcal{S} : \Theta \times \Theta \times [0,1] \to \Theta$ defined by*

$$\mathcal{S}(\wp, \Re, \lambda) = \frac{\wp + \Re}{2},$$

*for all $\wp, \Re \in \mathbb{R}$ and $\lambda \in (0,1)$ is a convex structure on probabilistic $b$-metric space $(\mathbb{R}, \daleth, \mathcal{Y})$ induced by a $b$-metric $\eta(\wp, \Re) = |\wp - \Re|^2$ whit with coefficient $s = 2$ on $\Theta$, where $\mathcal{Y}(a,b) = min\{a,b\}$ is continuous $t$-norm for $a, b \in [0,1]$, and*

$$\daleth_{\wp,\Re}(\aleph) = \begin{cases} 0, & \aleph \leq \eta(\wp, \Re) \\ 1, & \eta(\wp, \Re) < \aleph. \end{cases}$$

*for all $\wp, \Re \in \Theta$ and $\aleph \geq 0$. Let us prove this assertion. Firstly, we have that $\mathcal{S}(\wp, \Re, 0) = \Re$, and $\mathcal{S}(\wp, \Re, 1) = \wp$ for all $\wp, \Re \in \Theta$. Now, let us prove that inequality (2.1) is satisfied. If we assume that*

$$\min\{\daleth_{\wp,v}(\frac{\aleph}{\lambda}), \daleth_{v,\Re}(\frac{\aleph}{1-\lambda})\} = 0,$$

*then inequality (2.1) is a trivially satisfied because we get $\daleth_{\mathcal{S}(\wp,\Re,\lambda),v}(2\aleph) \geq 0$. Now we will assume that $\daleth_{\wp,v}(\frac{\aleph}{\lambda}) = 1$ and $\daleth_{v,\Re}(\frac{\aleph}{1-\lambda}) = 1$. Then we have that $\frac{\aleph}{\lambda} > \eta(\wp, v)$ and $\frac{\aleph}{1-\lambda} > \eta(\Re, v)$, i.e. $\aleph > \lambda\eta(\wp, v)$ and $\aleph > (1-\lambda)\eta(\Re, v)$. Utilizing the inequality $(n+m)^l \leq 2^{l-1}(n^l + m^l)$ for any $n, m \in [0, +\infty)$ and*

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*$l \geq 1$, we consider the following setting,*

$$\begin{aligned}
\eta(\mathcal{S}(\wp,\Re,\lambda),v) &= \eta(\frac{\wp+\Re}{2},v) \\
&= |\frac{\wp+\Re}{2}-v|^2 \\
&\leq 2(2^{-2}|\wp-v|^2+2^{-2}|\Re-v|^2) \\
&= \frac{1}{2}(|\wp-v|^2+|\Re-v|^2) \\
&= \lambda|\wp-v|^2+(1-\lambda)|\Re-v|^2=\lambda\eta(\wp,v)+(1-\lambda)\eta(\Re,v) \\
&< 2\aleph.
\end{aligned}$$

*that is, $2\aleph-\eta(\mathcal{S}(\wp,\Re,\lambda),v)>0$ and so $\daleth_{\mathcal{S}(\wp,\Re,\lambda),v}(2\aleph)=1$. Therefore, inequality 2.1 holds for all $\wp,\Re,v\in\Theta$. and $\aleph>0$.*

**Lemma 2.3.** *Let $(\Theta,\daleth,\mathcal{Y},\mathcal{S})$ be a CPbM-space. For all $\wp,\Re\in\Theta$ and any $\lambda\in[0,1]$,*

$$\daleth_{\wp,\Re}(\aleph)=\mathcal{Y}(\daleth_{\wp,\mathcal{S}(\wp,\Re,\lambda)}(\frac{\aleph}{2}),\daleth_{\mathcal{S}(\wp,\Re,\lambda),\Re}(\frac{\aleph}{2}))$$

*Proof.* By the triangle inequality, we get

$$\begin{aligned}
\daleth_{\wp,\Re}(\aleph) &\geq \mathcal{Y}(\daleth_{\wp,\mathcal{S}(\wp,\Re,\lambda)}(\frac{\aleph}{2}),\daleth_{\mathcal{S}(\wp,\Re,\lambda),\Re}(\frac{\aleph}{2})) \\
&\geq \mathcal{Y}[\mathcal{Y}(\daleth_{\wp,\wp}(\frac{\aleph}{4\lambda}),\daleth_{\wp,\Re}(\frac{\aleph}{4(1-\lambda)})),\mathcal{Y}(\daleth_{\wp,\Re}(\frac{\aleph}{4\lambda}),\daleth_{\Re,\Re}(\frac{\aleph}{4(1-\lambda)}))] \\
&= \mathcal{Y}[\daleth_{\wp,\Re}(\frac{\aleph}{4(1-\lambda)}),\daleth_{\wp,\Re}(\frac{\aleph}{4\lambda})] \\
&\geq \daleth_{\wp,\Re}(\aleph).
\end{aligned}$$

□

**Lemma 2.4.** *Let $(\Theta,\daleth,\mathcal{Y},\mathcal{S})$ be a CPbM-space. For all $\wp,\Re\in\Theta$, and any $\lambda\in[0,1]$, we have $\daleth_{\wp,\mathcal{S}(\wp,\Re,\lambda)}(\aleph)=\daleth_{\wp,\Re}(\frac{\aleph}{2(1-\lambda)})$, and $\daleth_{\mathcal{S}(\wp,\Re,\lambda),\Re}(\aleph)=\daleth_{\wp,\Re}(\frac{\aleph}{2\lambda})$*

*Proof.* From convex structure on $\Theta$, we get

$$\daleth_{\wp,\mathcal{S}(\wp,\Re,\lambda)}(\aleph)\geq\mathcal{Y}(\daleth_{\wp,\wp}(\frac{\aleph}{2\lambda}),\daleth_{\wp,\Re}(\frac{\aleph}{2(1-\lambda)}))=\daleth_{\wp,\Re}(\frac{\aleph}{2(1-\lambda)})$$

and

$$\daleth_{\mathcal{S}(\wp,\Re,\lambda),\Re}(\aleph)\geq\mathcal{Y}(\daleth_{\wp,\Re}(\frac{\aleph}{2\lambda}),\daleth_{\Re,\Re}(\frac{\aleph}{2(1-\lambda)}))=\daleth_{\wp,\Re}(\frac{\aleph}{2\lambda}).$$

If we had strict inequality in either of the above two inequalities, then by Lemma 2.3, we would reach the contradiction

$$\daleth_{\wp,\Re}(\aleph)=\mathcal{Y}(\daleth_{\wp,\mathcal{S}(\wp,\Re,\lambda)}(\frac{\aleph}{2}),\daleth_{\mathcal{S}(\wp,\Re,\lambda),\Re}(\frac{\aleph}{2}))\geq\daleth_{\wp,\Re}(\aleph).$$

□

Let $(\Theta,\daleth,\mathcal{Y},\mathcal{S})$ be a $CPbM$-space and $\mathcal{R}:\Theta\to\Theta$ be a self mapping. Denote by $Fix(\mathcal{R})$ the set of all fixed points of $\mathcal{R}$, that is,

$$Fix(\mathcal{R})=\wp\in\Theta:\mathcal{R}\wp=\wp.$$

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**Lemma 2.5.** *Let $(\Theta, \daleth, \mathcal{Y}, \mathcal{S})$ be a CPbM-space and $\mathcal{R} : \Theta \to \Theta$ be a self mapping. Define the mapping $\mathcal{R}_\lambda : \Theta \to \Theta$ by*

$$\mathcal{R}_\lambda \wp = \mathcal{S}(\wp, \mathcal{R}\wp, \lambda), \wp \in \Theta.$$

*Then, for any $\lambda \in [0, 1)$,*

$$Fix(\mathcal{R}) = Fix(\mathcal{R}_\lambda).$$

*Proof.* For $\lambda = 0$, $\mathcal{R}_\lambda = \mathcal{R}$ and the assertion is trivial. Assume $\lambda \in (0, 1)$ and let $\wp \in Fix(\mathcal{R})$. This means $\wp = \mathcal{R}\wp$ and therefore

$$\daleth_{\wp,\mathcal{R}\wp}(\aleph) = \daleth_{\wp,\mathcal{S}(\wp,\mathcal{R}\wp,\lambda)}(\aleph) \geq \mathcal{Y}(\daleth_{\wp,\wp}(\frac{\aleph}{2\lambda}), \daleth_{\wp,\mathcal{R}\wp}(\frac{\aleph}{2(1-\lambda)})) = 1$$

i.e., $\wp \in Fix(\mathcal{R}_\lambda)$.

Conversely, assume $\wp \in Fix(\mathcal{R}_\lambda)$. This means that $\daleth_{\wp,\mathcal{R}_\lambda\wp}(\aleph) = 1$, which implies $\daleth_{\wp,\mathcal{S}(\wp,\mathcal{R}\wp,\lambda)}(\aleph) = 1$. By Lemma 2.4,

$$\daleth_{\wp,\mathcal{S}(\wp,\mathcal{R}\wp,\lambda)}(\aleph) = \daleth_{\wp,\mathcal{R}\wp}(\frac{\aleph}{2(1-\lambda)}),$$

so it follows that

$$\daleth_{\wp,\mathcal{R}\wp}(\frac{\aleph}{2(1-\lambda)}) = 1$$

which, implies $\daleth_{\wp,\mathcal{R}\wp}(\aleph) = 1$. □

**Definition 2.6.** *Let $(\Theta, \daleth, \mathcal{Y})$ denote a CPbM-space, and let $\Phi$ represent the collection of all functions $\phi : [0, +\infty) \to [0, +\infty)$ such that $\phi(r) < r$ and $\lim_{n\to\infty} \phi^n(r) = 0$ for every $r > 0$. A mapping $\mathcal{R} : \Theta \to \Theta$ is defined as a large enriched contraction if, for any distinct $\wp, \Re \in \Theta$ and for $\lambda \in [0, 1)$, the following holds:*

$$\daleth_{\mathcal{R}\lambda\wp,\mathcal{R}\lambda\Re}(\aleph) > \daleth_{\wp,\Re}(\aleph),$$

*where $\mathcal{R}\lambda\wp = \mathcal{S}(\wp, \mathcal{R}\wp, \lambda)$. Moreover, for every $\epsilon \in (0, 1)$, there exists $\phi \in \Phi$ such that*

$$\daleth\wp, \Re(\aleph) \leq 1 - \epsilon \Rightarrow \daleth_{\mathcal{R}\lambda\wp,\mathcal{R}\lambda\Re}(\phi\aleph) \geq \daleth_{\wp,\Re}(\aleph).$$

**Definition 2.7.** *Let $(\Theta, \daleth, \mathcal{Y})$ be a CPbM-space and let $\mathcal{R} : \Theta \to \Theta$ be a mapping. We generalize Mann's iteration scheme within the context of CPbM-spaces as follows:*

$$\wp_{n+1} = \mathcal{S}(\wp_n, \mathcal{R}\wp_n, \lambda_n), \quad n \in \mathbb{N},$$

*where each $\wp_n \in \Theta$ and $\lambda_n \in [0, 1]$. The sequence $\wp_n$ is thus referred to as Mann's iteration sequence for the mapping $\mathcal{R}$.*

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## 3. Fixed point theorems in $CPbM$-space

In this section, we present results regarding the existence and uniqueness of fixed points for large enriched contractions within the context of a $CPbM$-space.

**Theorem 3.1.** *Let $(\Theta, \daleth, \mathcal{Y}, \mathcal{S})$ with coefficient $\alpha \in (0,1]$ denote a complete CPbM-space, and let $\mathcal{R} : \Theta \to \Theta$ be a large enriched contraction mapping. Suppose there exists $u_0 \in \Theta$ and a constant $L \in (0,1)$ such that $\daleth_{\mathcal{R}\lambda^n u0, u_0}(\aleph) \geq 1 - L$ for all $n \geq 1$. Under these conditions, $\mathcal{R}$ possesses a unique fixed point in $\Theta$.*

*Proof.* Initially, we shall prove that $\lim_{n\to\infty} \daleth_{\mathcal{R}_\lambda^{n+1}u_0, \mathcal{R}_\lambda^n u_0}(\aleph) = 1$. Using the definition of a large enriched contraction, we have

$$\daleth_{\mathcal{R}_\lambda u_1, \mathcal{R}_\lambda u_2}(\aleph) > \daleth_{u_1,u_2}(\aleph). \tag{3.1}$$

Setting $u_1 = \mathcal{R}_\lambda^n u_0$ and $u_2 = \mathcal{R}_\lambda^{n-1} u_0$ in 3.1, we obtain

$$\daleth_{\mathcal{R}_\lambda^{n+1}u_0, \mathcal{R}_\lambda^n u_0}(\aleph) > \daleth_{\mathcal{R}_\lambda^n u_0, \mathcal{R}_\lambda^{n-1}u_0}(\aleph) > ... > \daleth_{\mathcal{R}_\lambda u_0, u_0}(\aleph). \tag{3.2}$$

Thus, $\daleth_{\mathcal{R}_\lambda^{n+1}u_0, \mathcal{R}_\lambda^n u_0}(\aleph) > \daleth_{\mathcal{R}_\lambda u_0, u_0}(\aleph)$ for all $\aleph > 0$. Since $\lim_{\aleph\to\infty} \daleth_{\mathcal{R}_\lambda u_0, u_0}(\aleph) = 1$, there exists an$\aleph_0$ such that $\daleth_{\mathcal{R}_\lambda u_0, u_0}(\aleph_0) > 1 - \epsilon$ for $\epsilon \in (0,1)$. Thus, we obtain

$$\daleth_{\mathcal{R}_\lambda^{n+1}u_0, \mathcal{R}_\lambda^n u_0}(\aleph_0) > \daleth_{\mathcal{R}_\lambda u_0, u_0}(\aleph_0) > 1 - \epsilon,$$

which this means $\lim_{n\to\infty} \daleth_{\mathcal{R}_\lambda^{n+1}u_0, \mathcal{R}_\lambda^n u_0}(\aleph) = 1$.

The subsequent aim is to demonstrate that the sequence $u_n$ defined by $\mathcal{R}\lambda^n u0 = u_{n+1}$ forms a Cauchy sequence. Assume for contradiction that $u_n$ fails to be Cauchy. Then there exist $\epsilon \in (0,1)$ and sequences of integers $m_k, n_k$, and $N_k$ such that certain properties hold,

$$\daleth_{\mathcal{R}_\lambda^{m_k}u_0, \mathcal{R}_\lambda^{n_k}u_0}(\aleph) \leq 1 - \epsilon,$$

for some $m_k > n_k > N_k$.

As $\mathcal{R}$ is a large enriched contraction, there exists a $\phi \in \Phi$ such that

$$\daleth_{\mathcal{R}_\lambda^{m_k}u_0, \mathcal{R}_\lambda^{n_k}u_0}(\phi\aleph) \geq \daleth_{\mathcal{R}_\lambda^{m_k-1}u_0, \mathcal{R}_\lambda^{n_k-1}u_0}(\aleph),$$

for all $\aleph > 0$. Hence, we obtain

$$\daleth_{\mathcal{R}_\lambda^{m_k}u_0, \mathcal{R}_\lambda^{n_k}u_0}(\phi^2\aleph) \geq \daleth_{\mathcal{R}_\lambda^{m_k-1}u_0, \mathcal{R}_\lambda^{n_k-1}u_0}(\phi\aleph) \geq \daleth_{\mathcal{R}_\lambda^{m_k-2}u_0, \mathcal{R}_\lambda^{n_k-2}u_0}(\aleph),$$

for all $\aleph > 0$. Thus,

$$\daleth_{\mathcal{R}_\lambda^{m_k}u_0, \mathcal{R}_\lambda^{n_k}u_0}(\phi^n\aleph) \geq \daleth_{\mathcal{R}_\lambda u_0, u_0}(\aleph)$$

for all $\aleph > 0$. Since $lim_{\aleph\to\infty} \daleth_{\mathcal{R}_\lambda u_0, u_0}(\aleph) = 1$, there exists a $\aleph^*$ such that $\daleth_{\mathcal{R}_\lambda u_0, u_0}(\aleph^*) > 1 - \epsilon$ for $\epsilon \in (0,1)$. Also, by $\lim_{n\to\infty} \phi^n \aleph = 0$, there exists a $N^* \in \mathbb{N}$ such that $\phi^n(\aleph^*) < \delta$ for $\delta > 0,\ n > N^*$. Thus, we obtain

$$\daleth_{\mathcal{R}_\lambda^{m_k}u_0, \mathcal{R}_\lambda^{n_k}u_0}(\delta) \geq \daleth_{\mathcal{R}_\lambda^{m_k}u_0, \mathcal{R}_\lambda^{n_k}u_0}(\phi^n\aleph^*) \geq \daleth_{\mathcal{R}_\lambda u_0, u_0}(\aleph^*) > 1 - \epsilon,$$

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for $n > N^*$, which this means $\lim_{n\to\infty} \daleth_{\mathcal{R}_\lambda^{m_k}u_0,\mathcal{R}_\lambda^{n_k}u_0}(\aleph) = 1$, which is a contradiction. So, the sequence $\{u_n\}$ given by $\mathcal{R}_\lambda^n u_0 = u_{n+1}$ is a Cauchy sequence.

In the final step, we confirm both existence and uniqueness of this fixed point. Since $(\Theta, \daleth, \mathcal{Y}, \mathcal{S})$ is complete, there exists some $u \in \Theta$ such that $\lim_{n\to\infty} \mathcal{R}\lambda^n u0 = u$. The continuity of $\mathcal{R}\lambda$ ensures that $\mathcal{R}\lambda u = u$. If another fixed point $u'$ of $\mathcal{R}_\lambda$ exists with $u' \neq u$, then for some $\epsilon \in (0,1)$ it holds that $\daleth(u, u', \aleph) \leq 1-\epsilon$. Utilizing the contractive property yields a contradiction.

$$\daleth_{u,u'}(\phi\aleph) = \daleth_{\mathcal{R}\lambda u,\mathcal{R}\lambda u'}(\phi\aleph) \geq \daleth_{u,u'}(\aleph),$$

which contradicts the assumption that $u \neq u'$. Thus, $\mathcal{R}_\lambda$ admits a unique fixed point. By invoking Lemma 2.5, it follows that $\mathcal{R}u = u$. □

**Corollary 3.2.** *Let $(\Theta, \daleth, \mathcal{Y})$ with coefficient $\alpha \in (0,1]$ be a complete PbM-space, and let $\mathcal{R} : \Theta \to \Theta$ be a large enriched contraction. Suppose there exists some $u_0 \in \Theta$ and a constant $L \in (0,1)$ such that $\daleth_{\mathcal{R}\lambda^n u0,u_0}(\aleph) \geq 1 - L$ for all $n \geq 1$. Then, $\mathcal{R}$ possesses a unique fixed point in $\Theta$.*

Example 3.3. *Consider now the mapping $\mathcal{R} : [0,1] \to [0,1]$ defined by $\mathcal{R}u = u - u^4$ for all $u \in [0,1]$. Assume further that $\daleth_{u_1,u_2}(\aleph) = \frac{\aleph}{\aleph+|u_1-u_2|^2}$, $\phi(r) = \frac{r}{2}$, and $\mathcal{S}(u_1, u_2, \lambda) = \lambda u_1 + (1-\lambda)u_2$. Under these definitions, the mapping $\mathcal{R}$ qualifies as a large enriched contraction.*

$$\begin{aligned}
\daleth_{\mathcal{R}_\lambda u_1,\mathcal{R}_\lambda u_2}(\aleph)5 &= \frac{\aleph}{\aleph + |\lambda u_1 + (1-\lambda)(u_1 - u_1^4) - \lambda u_2 - (1-\lambda)(u_2 u_2^4)|^2} \\
&= \frac{\aleph}{\aleph + |u_1u_2|^2|1 - (1-\lambda)(u_1^2 + u_2^2)(u_1 + u_2)|^2}.
\end{aligned}$$

*Using the inequalities $|u_1 - u_2|^2 \leq 2(u_1^2 + u_2^2)$,$\phi(r) \leq r$ and $|u_1 - u_2| \leq |u_1 + u_2|$, we have*

$$\begin{aligned}
\daleth_{\mathcal{R}_\lambda u_1,\mathcal{R}_\lambda u_2}(\phi\aleph) &= \frac{\frac{\aleph}{2}}{\frac{\aleph}{2} + |\lambda u_1 + (1-\lambda)(u_1 - u_1^4) - \lambda u_2 - (1-\lambda)(u_2 u_2^4)|^2} \\
&= \frac{\frac{\aleph}{2}}{\frac{\aleph}{2} + |u_1u_2|^2|1 - (1-\lambda)(u_1^2 + u_2^2)(u_1 + u_2)|^2} \\
&\geq \frac{\frac{\aleph}{2}}{\frac{\aleph}{2} + |u_1u_2|^2|1 - (1-\lambda)|u_1 - u_2|\frac{|u_1-u_2|^2}{2}|^2} \\
&= \frac{\frac{\aleph}{2}}{\frac{\aleph}{2} + |u_1u_2|^2|1 - (1-\lambda)\frac{|u_1-u_2|^3}{2}|^2} \\
&= \frac{\aleph}{\aleph + 2.|u_1u_2|^2|1 - (1-\lambda)\frac{|u_1-u_2|^3}{2}|^2} \\
&\geq \frac{\aleph}{\aleph + |u_1u_2|^2} \\
&= \daleth_{u_1,u_2}(\aleph).
\end{aligned}$$

*Thus, $\mathcal{R}$ is a large enriched contraction. Therefore, by Theorem 3.1 $\mathcal{R}$ has a unique fixed point.*

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## 4. Modelling free mechanical oscillations

Oscillation refers to the repeated or periodic fluctuation, typically over time, of a quantity around a central value (often an equilibrium point) or between different states. Common instances include a swinging pendulum or alternating electrical current. In physics, oscillatory models are frequently employed to approximate complex phenomena, such as atomic interactions. Oscillatory behavior is not limited to mechanical systems; it is prevalent across diverse scientific disciplines, including cardiac cycles in physiology, economic cycles in finance, predator–prey dynamics in ecology, geyser eruptions in geology, vibrations in musical instruments, neural firing patterns in neuroscience, and the pulsation of variable stars in astronomy. The term “vibration” specifically denotes mechanical oscillations. In control engineering, rapid oscillations—referred to as chattering or flapping—can be undesirable, as they may hinder system stability, for example, in sliding mode control or valve operation. The most elementary mechanical oscillator consists of a mass attached to a linear spring subjected only to gravitational and elastic forces. This system can be modeled on surfaces with minimal friction, such as air tables or ice. The equilibrium position is defined by the static state of the spring. When displaced from equilibrium, a restoring force acts on the mass to return it to its original position; however, because of inertia, the mass overshoots, generating a new restoring force in the opposite direction. Introducing a constant force like gravity shifts the equilibrium point. Duration of one complete oscillation is known as the period. Systems where the restoring force is directly proportional to displacement—such as the spring-mass oscillator—are governed by simple harmonic motion, with their dynamics described by the simple harmonic oscillator equation. In such systems, energy alternates between kinetic and potential forms as the mass moves through its path. Key characteristics of oscillatory systems include an equilibrium state and a restoring force that intensifies with increased displacement from equilibrium. For a spring-mass system, Hooke’s law quantifies this restoring force:

$$\vec{F} = -k\vec{x}$$

Let $k$ be a positive constant, and let $F$ denote the restoring force, which is directly proportional to the displacement $x$. The motion of the system can be described using the vertical displacement $\wp$ of the mass. Suppose $m$ represents the mass, $k\frac{d\wp}{dt}$ denotes the damping force due to the dashpot, and $n\wp$ is the restoring force generated by the spring, where both $k$ and $n$ are constants. The resulting equation of motion is given by

$$m\frac{d^2\wp}{dt^2} = -k\frac{d\wp}{dt} - n\wp.$$

Assume that initially, the mass is displaced by a distance $\wp_0$ and released from rest. Therefore, at time $t = 0$, $\wp = \wp_0$ and $\frac{d\wp}{dt} = 0$. By expressing the differential equation in its standard form,

$$m\frac{d^2\wp}{dt} + k\frac{d\wp}{dt} + n\wp = 0 \tag{4.1}$$

or

$$\frac{d^2\wp}{dt} + \bar{\omega}\frac{d\wp}{dt} + v\wp = 0,$$

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we introduce $\bar{\omega} = \frac{k}{m}$ and $v = \frac{n}{m}$. The behavior of the oscillations governed by this differential equation fundamentally depends on the relative magnitudes of the mechanical parameters $m$, $k$, and $n$.

The Green's function corresponding to equation (4.1) for critically damped motion, under the initial conditions $\wp(0) = 0$ and $\dot{\wp}(0) = a$, is determined

$$E(\beta, \nu) = \begin{cases} -\nu e^{\rho(\nu-\beta)}, & 0 \le \nu \le \beta \le 1 \\ -\beta e^{\rho(\nu-\beta)}, & 0 \le \beta \le \nu \le 1. \end{cases} \tag{4.2}$$

where $\rho > 0$ is a constant defined in terms of $v$ and $\bar{\omega}$.

Let us define $\Theta = C([0,1], \mathbb{R}^+)$ as the set of all continuous functions mapping $[0,1]$ into the non-negative real numbers. Consider a function $\eta : \Theta \times \Theta \to [0, \infty)$ defined appropriately;

$$\eta(\wp, \Re) = max_{\beta \in [0,1]} |\wp(\beta) - \Re(\beta)|^2 e^{-2L\beta}, \qquad \wp, \Re \in \Theta, L > 0.$$

it can be verified that $\eta$ constitutes a complete $b$-metric with parameter $s = 2$. Next, we introduce a mapping $\daleth : \Theta \times \Theta \to D^+$ as follows,

$$\daleth_{\wp,\Re}(\aleph) = \chi(\aleph - \eta(\wp, \Re))$$

for $\aleph > 0$, where

$$\chi(\aleph) = \begin{cases} 0 & \text{if } \aleph \le 0 \\ 1 & \text{if } \aleph > 0. \end{cases}$$

We know that $(\Theta, \daleth, min, \mathcal{S})$ where $\mathcal{S}(\wp, \Re, \lambda) = \lambda\wp + (1-\lambda)\Re$ is a complete $CPbM$-space with coefficient $\varrho = \frac{1}{2}$ .

In the subsequent theorem, we examine oscillator equations in which damping causes the system to return to equilibrium as rapidly as possible without overshooting or oscillating about this position.

**Theorem 4.1.** *Let $(\Theta, \daleth, \min, \mathcal{S})$, where $\mathcal{S}(\wp, \Re, \lambda) = \lambda\wp + (1-\lambda)\Re$, be a complete $CPbM$-space with parameter $\varrho = \frac{1}{2}$, and let $\mathcal{R} : \Theta \to \Theta$ be a self-mapping such that:*

$$\mathcal{R}\omega(\beta) = \int_0^\beta E(\beta, \nu)\mho(\nu, \omega(\nu))d\nu, \tag{4.3}$$

*where $\mho : [0,1] \times \mathbb{R} \to \mathbb{R}$ is an increasing function satisfying certain conditions.*

$$|\mho(\beta, \mathcal{R}\omega(\beta)) - \mho(\beta, \mathcal{R}\vartheta(\beta))| \le \frac{1}{\sqrt{2}} |\omega(\beta)) - \vartheta(\beta)|.$$

*Under these circumstances, the differential equation (4.1) possesses a unique solution.*

*Proof.* Solving equation (4.1) is equivalent to finding a solution to an associated integral equation.

$$\omega(\beta) = \int_0^\beta E(\beta, \nu)\mho(\nu, \omega(\nu))d\nu.$$

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Thus, a function $z$ solves equation (4.1) if and only if it is a fixed point of $\mathcal{R}$. For any $\omega, \vartheta \in \Theta$, it follows that certain contractive properties hold,

$$\begin{aligned}
\eta(\mathcal{R}_\lambda\omega, \mathcal{R}_\lambda\vartheta) &= \eta(\mathcal{S}(\omega, \mathcal{R}\omega, \lambda), \mathcal{S}(\vartheta, \mathcal{R}\vartheta, \lambda)) \\
&= max_{\beta\in[0,1]}(|\lambda\omega + (1-\lambda)\mathcal{R}\omega - \lambda\vartheta - (1-\lambda)\mathcal{R}\vartheta|^2 e^{-2L\beta}) \\
&= max_{\beta\in[0,1]}(|\lambda(\omega-\vartheta) + (1-\lambda)(\mathcal{R}\omega - \mathcal{R}\vartheta)|^2 e^{-2L\beta}) \\
&= max_{\beta\in[0,1]}(|\lambda(\omega-\vartheta) + (1-\lambda)(\int_0^\beta E(\beta,\nu)\mho(\nu,\omega(\nu))d\nu - \\
&\quad \int_0^\beta E(\beta,\nu)\mho(\nu,\vartheta(\nu))d\nu)|^2 e^{-2L\beta}) \\
&\le max_{\beta\in[0,1]}(2|\lambda(\omega-\vartheta)|^2 e^{-2L\beta} + 2|(1-\lambda)(\int_0^\beta E(\beta,\nu)\mho(\nu,\omega(\nu))d\nu - \\
&\quad \int_0^\beta E(\beta,\nu)\mho(\nu,\vartheta(\nu))d\nu)|^2 e^{-2L\beta}) \\
&= max_{\beta\in[0,1]}(2\lambda^2|(\omega-\vartheta)|^2 e^{-2L\beta} + 2(1-\lambda)^2|(\int_0^\beta E(\beta,\nu)\mho(\nu,\omega(\nu))d\nu - \\
&\quad \int_0^\beta E(\beta,\nu)\mho(\nu,\vartheta(\nu))d\nu)|^2 e^{-2L\beta}) \\
&\le max_{\beta\in[0,1]}(2\lambda^2|(\omega-\vartheta)|^2 e^{-2L\beta} + 2(1-\lambda)^2((\int_0^\beta E(\beta,\nu)|\mho(\nu,\omega(\nu)) - \\
&\quad \mho(\nu,\vartheta(\nu))|)^2 e^{-2L\beta} d\nu)) \\
&\le max_{\beta\in[0,1]}(2\lambda^2|(\omega-\vartheta)|^2 e^{-2L\beta} + 2(1-\lambda)^2((\int_0^\beta E(\beta,\nu)|\omega(\beta)-\vartheta(\beta)|)^2 e^{-2L\beta} d\nu)) \\
&\le 2\lambda^2\eta(\omega,\vartheta) + 2(1-\lambda)^2\frac{1}{2}\eta(\omega,\vartheta) max_{\beta\in[0,1]}(\int_0^\beta E(\beta,\nu)d\nu)^2
\end{aligned}$$

on the other hand, $max_{\beta\in[0,1]}\int_0^\beta E(\beta,\nu)d\nu < 1$ so we have

$$\eta(\mathcal{R}_\lambda\omega, \mathcal{R}_\lambda\vartheta) \le 2\lambda^2\eta(\omega,\vartheta) + (1-\lambda)^2\eta(\omega,\vartheta) = (\lambda-1)^2\eta(\omega,\vartheta) \le \eta(\omega,\vartheta).$$

Putting $\phi\aleph = \aleph$, we derive

$$\begin{aligned}
\daleth_{\mathcal{R}_\lambda\omega, \mathcal{R}_\lambda\vartheta}(\aleph) &= \chi(\aleph - \eta(\mathcal{R}_\lambda\omega, \mathcal{R}_\lambda\vartheta)) \\
&\ge \chi(\aleph - \eta(\omega,\vartheta)) \\
&= \daleth_{\omega,\vartheta}(\aleph).
\end{aligned}$$

Consequently, by invoking Theorem 3.1, $\mathcal{R}$ admits a fixed point, which corresponds to a solution of equation(4.1). □

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## 5. Conclusion

Given that $CPbM$-spaces represent a relatively recent development in mathematical literature, this work aims to advance their study by introducing a new framework that synthesizes concepts from both $MPbM$-spaces and $CM$-spaces. Utilizing these foundational ideas, we establish novel fixed point results. Section 2 presents new fixed point theorems for single-valued operators in $CPbM$-spaces and explores their application to free mechanical oscillations modeled by second-order differential equations.

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