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CENTRALIZER OF SOME SPECIAL TYPE OF ADDITIVE MAPS IN PRIME RINGS

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Abstract. Let $\mathscr{R}$ be a prime ring with char $(\mathscr{R}) \neq 2, 3$ and $f(\zeta_1, \ldots, \zeta_n)$ be a non-central multilinear polynomial over $\mathscr{C}(= \mathscr{Z}(\mathscr{U}))$, where $\mathscr{U}$ be the Utumi ring of quotients of $\mathscr{R}$. Let $d \neq 0$ be a derivation of $\mathscr{R}$. Suppose that $\mathscr{F}, \mathscr{G}$ are two generalized derivations of $\mathscr{R}$. Let $\mathscr{A} = \{[(\mathscr{F}^2 + \mathscr{G})(f(\zeta_1, \ldots, \zeta_n)), f(\zeta_1, \ldots, \zeta_n)] : \zeta_1, \ldots, \zeta_n \in \mathscr{R}\}$. In the present article, we determine the nature of the maps $\mathscr{F}$ and $\mathscr{G}$, when $d(\mathscr{A}) = 0$. Moreover, it is proved that if $\mathscr{A} \neq 0$, then $C_{\mathscr{R}}(\mathscr{A}) = \mathscr{Z}(\mathscr{R})$, where $C_{\mathscr{R}}(\mathscr{A}) = \{a \in \mathscr{R} : [a, x] = 0 \ \forall x \in \mathscr{A}\}$ is the centralizer of $\mathscr{A}$.

1. Introduction

In this paper, $\mathscr{R}$ always be a prime ring with center $\mathscr{Z}(\mathscr{R})$, $\mathscr{U}$ be its Utumi ring of quotients. The center of $\mathscr{U}$ is denoted by $\mathscr{C}$ which is called extended centroid of $\mathscr{R}$. By a derivation d on $\mathscr{R}$, one usually means an additive mapping $d : \mathscr{R} \to \mathscr{R}$ such that for any $x, y \in \mathscr{R}$, $d(xy) = d(x)y + xd(y)$. By a generalized derivation g on $\mathscr{R}$, one usually means an additive mapping $g : \mathscr{R} \to \mathscr{R}$ such that for any $x, y \in \mathscr{R}$, $g(xy) = g(x)y + xd(y)$ for some derivation d in $\mathscr{R}$. Every derivation is a generalized derivation. Thus generalized derivation map is the generalization of the map derivation. For any $a, b \in \mathscr{R}$, we denote $[a, b] = ab - ba$, the commutator of a and b.

The multilinear polynomial $f(\zeta_1, \ldots, \zeta_n)$ over the field $\mathscr{C}$ is written as

$$f(\zeta_1, \ldots, \zeta_n) = \zeta_1\zeta_2 \cdots \zeta_n + \sum_{\sigma \in S_n, \sigma \neq id} \alpha_\sigma \zeta_{\sigma(1)}\zeta_{\sigma(2)} \cdots \zeta_{\sigma(n)}$$

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where $\alpha_\sigma \in \mathscr{C}$ and S_n is the symmetric group of degree n. Let $\mathscr{A}$ be a nonempty subset of $\mathscr{R}$. We define the set $f(\mathscr{A}) = \{f(\zeta_1, \ldots, \zeta_n) | \zeta_1, \ldots, \zeta_n \in \mathscr{A}\}$. A mapping χ is said to be commuting on $\mathscr{A}$ if $[\chi(a), a] = 0$ for all $a \in \mathscr{A}$ (resp. centralizing on $\mathscr{A}$ if $[\chi(a), a] \in \mathscr{Z}(\mathscr{R})$ for all $a \in \mathscr{A}$). The study of commuting and centralizing maps was initiated by Posner in 1957. We know that the global structure of a prime ring $\mathscr{R}$ is often tightly connected to the behaviour of additive mappings defined on $\mathscr{R}$, which act on some suitable subsets of the whole ring. For example, Posner [22] says that existence of a nonzero centralizing derivation in a prime ring implies the ring to be commutative. Since then many authors investigated commuting and centralizing maps in different directions (see [6–8, 10–12, 16, 25]). For instance, Lee and Lee [16] proved that if $[d(u), u] \in \mathscr{Z}(\mathscr{R})$ for all $u \in f(\mathscr{I})$, where $\mathscr{I}$ is a nonzero ideal of $\mathscr{R}$, then either $f(x_1, \ldots, x_n)$ is central-valued on R or char $(\mathscr{R}) = 2$ and $\mathscr{R}$ satisfies s_4.

In [2, Theorem 2], Argaç and De Filippis studied the above result replacing derivation d by a generalized derivation $\mathscr{F}$. They proved the following:

Let K be a commutative ring with unity and $\mathscr{R}$ be a noncommutative prime K-algebra with center $\mathscr{Z}(\mathscr{R})$, $\mathscr{C} = \mathscr{Z}(\mathscr{U})$ the extended centroid of $\mathscr{R}$, where $\mathscr{U}$ is the Utumi quotient ring of $\mathscr{R}$. Suppose that $\mathscr{I}$ is a nonzero ideal of $\mathscr{R}$ and $f(\zeta_1, \ldots, \zeta_n)$ a noncentral multilinear polynomial over K. If $\mathscr{F}$ is a nonzero generalized derivation of $\mathscr{R}$ such that $[\mathscr{F}(u), u] = 0$ for all $u \in f(\mathscr{I})$, then one of the following holds:

(1) *there exists $\lambda' \in \mathscr{C}$ such that $\mathscr{F}(x) = \lambda' x$ for all $x \in \mathscr{R}$;*
(2) *$f(x_1, \ldots, x_n)^2$ is central valued on $\mathscr{R}$ and there exist $a' \in \mathscr{U}$ and $\lambda' \in \mathscr{C}$ such that $\mathscr{F}(x) = a'x + xa' + \lambda' x$ for all $x \in \mathscr{R}$;*
(3) *char $(\mathscr{R}) = 2$ and $\mathscr{R}$ satisfies s_4.*

In [11] Dhara and De Filippis studied the case $\mathscr{F}^2(u)u - u\mathscr{G}^2(u) = 0$ for all $u \in \{f(\zeta_1, \ldots, \zeta_n) | \zeta_1, \ldots, \zeta_n \in \mathscr{I}\}$, where $\mathscr{I}$ is an ideal of prime ring $\mathscr{R}$ and $\mathscr{F}$, $\mathscr{G}$ two nonzero generalized derivations of $\mathscr{R}$. In case $\mathscr{F} = \mathscr{G}$, the identity becomes $[\mathscr{G}^2(u), u] = 0$ for all $u \in f(\mathscr{I})$.

Combining the two situations $[\mathscr{F}(u), u] = 0$ and $[\mathscr{G}^2(u), u] = 0$, we get $[(\mathscr{G}^2 + \mathscr{F})(u), u] = 0$ for all $u \in f(\mathscr{I})$ which was studied by De Filippis et al. [5]. Note that $(\mathscr{G}^2 + \mathscr{F})$ is a special type of additive map which was introduced and studied by Lee et al. in [17, Theorem 2.1] when $\mathscr{G}$ and $\mathscr{F}$ are derivations. Further this special type of additive map was studied by Rehman and De Filippis in [23] when $\mathscr{G}$ and $\mathscr{F}$ are generalized derivations.

On the other hand, De Filippis and Di Vincenzo [9] studied $d([\mathscr{F}(u), u]) = 0$ for all $u \in f(\mathscr{R})$, where d is a nonzero derivation of $\mathscr{R}$ and $\mathscr{F}$ is a nonzero generalized derivation of $\mathscr{R}$ and then obtained forms of the maps.

In [24], Tiwari et al. studied the case $d([\mathscr{G}^2(u), u]) = 0$ for all $u \in f(\mathscr{R})$, where d is a nonzero derivation of $\mathscr{R}$ and $\mathscr{G}$ is a generalized derivation of $\mathscr{R}$.

In the present paper, we will examine the situation $d([(\mathscr{G}^2 + \mathscr{F})(u), u]) = 0$ for all $u \in f(\mathscr{R})$.

More precisely, we will prove the following:

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Theorem 1.1. *Let $\mathscr{R}$ be a prime ring with char $(\mathscr{R}) \neq 2,3$ and $f(\zeta_1,\ldots,\zeta_n)$ be a non-central multilinear polynomial over $\mathscr{C}(=\mathscr{Z}(\mathscr{U}))$, where $\mathscr{U}$ be the Utumi ring of quotients of $\mathscr{R}$. Let $d \neq 0$ be a derivation of $\mathscr{R}$. Suppose that $\mathscr{F},\mathscr{G}$ are two generalized derivations of $\mathscr{R}$.*

If

$$d\Bigg(\Bigg[(\mathscr{F}^2+\mathscr{G})(f(\zeta_1,\ldots,\zeta_n)), f(\zeta_1,\ldots,\zeta_n)\Bigg]\Bigg) = 0$$

for all $\zeta_1,\ldots,\zeta_n \in \mathscr{R}$, then one of the following holds:

(1) *there exist $b,p \in \mathscr{U}$ such that $\mathscr{F}(x) = xb$ and $\mathscr{G}(x) = xp$ for all $x \in \mathscr{R}$ with $b^2+p \in \mathscr{C}$;*
(2) *there exist $b,p \in \mathscr{U}$ such that $\mathscr{F}(x) = bx$ and $\mathscr{G}(x) = px$ for all $x \in \mathscr{R}$ with $b^2+p \in \mathscr{C}$;*
(3) *$f(\zeta_1,\ldots,\zeta_n)^2$ is central valued on $\mathscr{R}$ and one of the following holds:*
   (a) *there exist $b,p,q \in \mathscr{U}$ such that $\mathscr{F}(x) = xb$ and $\mathscr{G}(x) = px+xq$ for all $x \in \mathscr{R}$ with $b^2+q-p \in \mathscr{C}$;*
   (b) *there exist $b,p,q \in \mathscr{U}$ such that $\mathscr{F}(x) = bx$ and $\mathscr{G}(x) = px+xq$ for all $x \in \mathscr{R}$ with $b^2+p-q \in \mathscr{C}$.*

Let $\mathscr{S}$ be a nonempty subset of $\mathscr{R}$. The centralizer of $\mathscr{S}$ is denoted by $C_{\mathscr{R}}(\mathscr{S})$ and defined by $C_{\mathscr{R}}(\mathscr{S}) = \{a \in \mathscr{R} : [a,x] = 0 \ \forall x \in \mathscr{S}\}$. In [1], Albas et al. proved that if the set $\mathscr{S} = \{[\mathscr{F}(f(\zeta_1,\ldots,\zeta_n)), f(\zeta_1,\ldots,\zeta_n)] : \zeta_1,\ldots,\zeta_n \in \mathscr{R}\}$ is nonzero, then $C_{\mathscr{R}}(\mathscr{S}) = \mathscr{Z}(\mathscr{R})$, where $\mathscr{F}$ is a nonzero generalized skew derivation of $\mathscr{R}$.

Let $\mathscr{A} = \{[(\mathscr{F}^2+\mathscr{G})(f(\zeta_1,\ldots,\zeta_n)), f(\zeta_1,\ldots,\zeta_n)] : \zeta_1,\ldots,\zeta_n \in \mathscr{R}\}$. In the above theorem, the nature of the maps $\mathscr{F}$ and $\mathscr{G}$ are determined, when $d(\mathscr{A}) = 0$. From above theorem it is very clear that if $d \neq 0$, then all the conclusions give $\mathscr{A} = \{0\}$. Thus we conclude that if $\mathscr{A} \neq \{0\}$, then d must be zero. Assuming $d(x) = [a,x]$ for all $x \in R$, if $\mathscr{A} \neq 0$, then $[a,\mathscr{A}] = 0$ implies $a \in Z(R)$, i.e., $C_{\mathscr{R}}(\mathscr{A}) = \mathscr{Z}(\mathscr{R})$. Therefore, the following Corollary is straight forward.

Corollary 1.2. *Let $\mathscr{R}$ be a prime ring with char $(\mathscr{R}) \neq 2,3$ and $f(\zeta_1,\ldots,\zeta_n)$ be a non-central multilinear polynomial over $\mathscr{C}(=\mathscr{Z}(\mathscr{U}))$, where $\mathscr{U}$ be the Utumi ring of quotients of $\mathscr{R}$. Let $\mathscr{A} = \{[(\mathscr{F}^2+\mathscr{G})(f(\zeta_1,\ldots,\zeta_n)), f(\zeta_1,\ldots,\zeta_n)] : \zeta_1,\ldots,\zeta_n \in \mathscr{R}\}$. If $\mathscr{A} \neq \{0\}$, then $C_{\mathscr{R}}(\mathscr{A}) = \mathscr{Z}(\mathscr{R})$.*

Moreover, we have the following corollaries from the Theorem 1.1.

Corollary 1.3. *Let $\mathscr{R}$ be a noncommutative prime ring with char $(\mathscr{R}) \neq 2,3$, $\mathscr{U}$ be the Utumi ring of quotients of $\mathscr{R}$ and let $d \neq 0$ be a derivation of $\mathscr{R}$. Suppose that $\mathscr{F},\mathscr{G}$ are two generalized derivations of $\mathscr{R}$. If*

$$d\Bigg(\Bigg[(\mathscr{F}^2+\mathscr{G})(x), x\Bigg]\Bigg) = 0$$

for all $x \in \mathscr{R}$, then one of the following holds:

(1) *there exist $b,p \in \mathscr{U}$ such that $\mathscr{F}(x) = xb$ and $\mathscr{G}(x) = xp$ for all $x \in \mathscr{R}$ with $b^2+p \in \mathscr{C}$;*
(2) *there exist $b,p \in \mathscr{U}$ such that $\mathscr{F}(x) = bx$ and $\mathscr{G}(x) = px$ for all $x \in \mathscr{R}$ with $b^2+p \in \mathscr{C}$.*

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Corollary 1.4. *Let $\mathscr{R}$ be a prime ring with char $(\mathscr{R}) \neq 2, 3$ and let d, g, h be three nonzero derivations of $\mathscr{R}$. If*

$$d\left(\left[(g^2 + h)(x), x\right]\right) = 0$$

for all $x \in \mathscr{R}$, then $\mathscr{R}$ must be commutative.

2. Some results

Through out this section, $\mathscr{R}$ always be a prime ring, $\mathscr{I}$ a two sided ideal of $\mathscr{R}$ and $f(\zeta_1, \ldots, \zeta_n)$ a multilinear polynomial over $\mathscr{C}$, which is not central valued on $\mathscr{R}$. The $\mathscr{C}$ denotes the extended centroid of $\mathscr{R}$ which is the center of Utumi ring of quotients $\mathscr{U}$.

The following facts are frequently used to prove our results.

Fact 2.1. *By* [3], *$\mathscr{I}$, $\mathscr{R}$ and $\mathscr{U}$ satisfy the same generalized polynomial identities (GPIs) with coefficients in $\mathscr{U}$.*

Fact 2.2. *By* [18], *$\mathscr{I}$, $\mathscr{R}$ and $\mathscr{U}$ satisfy the same differential identities.*

Fact 2.3. (*See* [15,18]) *Let $Der(\mathscr{U})$ be the set of all derivations on $\mathscr{U}$ and D_{int} be the set of all inner derivations on $\mathscr{U}$.*

A derivation word is an additive map Δ of the form $\Delta = d_1^{s_1} d_2^{s_2} \cdots d_m^{s_m}$, with each $d_i \in \mathrm{Der}(\mathscr{U})$ and $s_i \geq 1$. By a differential polynomial, we mean a generalized polynomial of the form $\Phi(x_i^{\Delta_j})$ with coefficients in $\mathscr{U}$, where x_i are noncommutative indeterminates on which the derivations words Δ_j act as unary operations. Thus the identity $\Phi(x_i^{\Delta_j}) = 0$ for all $x_i \in T$ is the differential identity satisfied by T.

By [15, Theorem 2] (*see also* [18, Theorem 1]), *we have the following result:*

Let $d_1, \ldots, d_m \in \mathrm{Der}(\mathscr{U})$ and derivations words Δ_j are in the form

$$\Delta_j = d_1^{s_{1,j}} d_2^{s_{2,j}} \cdots d_m^{s_{m,j}} \quad j = 1, \ldots, n$$

where

$$s = max\{s_{i,j},\ i = 1, \ldots, m\ j = 1, \ldots, n\}.$$

If $d_1, \ldots, d_m$ are $\mathscr{C}-$linearly independent modulo D_{int}, $s < p$, with char$(\mathscr{R}) = p \neq 0$, and $\Phi(x_i^{\Delta_j})$ be a differential identity on $\mathscr{R}$, then $\Phi(y_{ji})$ is a GPI on $\mathscr{R}$, where y_{ji} are distinct indeterminates.

In particular, if char $(\mathscr{R}) \neq 2$ and d is a nonzero derivation on $\mathscr{R}$ such that

$$\Phi(\zeta_1, \ldots, \zeta_n, \zeta_1^d, \ldots, \zeta_n^d, \zeta_1^{d^2}, \ldots, \zeta_n^{d^2})$$

is a differential identity on a prime ring $\mathscr{R}$, then either $d \in D_{int}$ or $\mathscr{R}$ satisfies

$$\Phi(\zeta_1, \ldots, \zeta_n, \xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n) = 0.$$

Fact 2.4. ([9, Lemma 1]) *Let $\mathscr{C}$ be an infinite field, t a positive integer with $t \geq 2$ and $\mathscr{R} = M_t(\mathscr{C})$, the algebra of all $t \times t$ matrices over $\mathscr{C}$. Let $B_1, \ldots, B_k$ be not scalar matrices in $\mathscr{R}$. Then there must exists at least one invertible matrix $Q \in \mathscr{R}$ such that all the entries of the matrices $QB_1Q^{-1}, \ldots, QB_kQ^{-1}$ have non-zero values.*

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Fact 2.5. *We write the multilinear polynomial over the field $\mathscr{C}$ as*

$$f(\zeta_1, \ldots, \zeta_n) = \zeta_1\zeta_2\cdots\zeta_n + \sum_{\sigma\in S_n, \sigma\neq id} \alpha_\sigma \zeta_{\sigma(1)}\zeta_{\sigma(2)}\cdots\zeta_{\sigma(n)}$$

where $\alpha_\sigma \in \mathscr{C}$ and S_n is the symmetric group of degree n.

Let d be a derivations on $\mathscr{R}$. Then action of d on $f(\zeta_1, \ldots, \zeta_n)$ is as follows:

$$d(f(\zeta_1, \ldots, \zeta_n)) = f^d(\zeta_1, \ldots, \zeta_n) + \sum_i f(\zeta_1, \ldots, d(\zeta_i), \ldots, \zeta_n),$$

where $f^d(\zeta_1, \ldots, \zeta_n)$ be the polynomials obtained from $f(\zeta_1, \ldots, \zeta_n)$ replacing each coefficients α_σ with $\delta(\alpha_\sigma)$. Similarly, we have

$$\begin{aligned} d^2(f(\zeta_1, \ldots, \zeta_n)) &= f^{d^2}(\zeta_1, \ldots, \zeta_n) + 2\sum_i f^d(\zeta_1, \ldots, d(\zeta_i), \ldots, \zeta_n) \\ &\quad + \sum_i f(\zeta_1, \ldots, d^2(\zeta_i), \ldots, \zeta_n) \\ &\quad + \sum_i f(\zeta_1, \ldots, d(\zeta_i), \ldots, d(\zeta_j), \ldots, \zeta_n). \end{aligned}$$

Fact 2.6. *If $\mathscr{R}$ satisfies a nontrivial generalized polynomial identity (GPI) $\chi(\zeta_1, \ldots, \zeta_n) = 0$, then it is also satisfied by $\mathscr{U}$ by [3]. Let $\mathscr{E}$ be the algebraic closure of $\mathscr{C}$. We know that if $\mathscr{C}$ is infinite, then $\chi(\zeta_1, \ldots, \zeta_n) = 0$ for all $\zeta_1, \ldots, \zeta_n \in \mathscr{U} \otimes_{\mathscr{C}} \mathscr{E}$. Since both of $\mathscr{U}$ and $\mathscr{U} \otimes_{\mathscr{C}} \mathscr{E}$ are prime and centrally closed (see [13,* Theorems 2.5 and 3.5]*), we may replace $\mathscr{R}$ by $\mathscr{U}$ or $\mathscr{R}$ by $\mathscr{U} \otimes_{\mathscr{C}} \mathscr{E}$ according to $\mathscr{C}$ finite or infinite and hence we may assume that $\mathscr{R}$ is centrally closed over $\mathscr{C}$. Then by [21], $\mathscr{R}$ is a primitive ring having a nonzero socle soc($\mathscr{R}$) and $\mathscr{C}$ is its associated division ring. By Jacobson's theorem [14,* p.75]*, $\mathscr{R}$ is isomorphic to a dense ring of linear transformations of a vector space V over $\mathscr{C}$.*

Fact 2.7. *Every generalized derivation $\mathscr{F}$ on $\mathscr{R}$ can be uniquely extended to a generalized derivation of $\mathscr{U}$ such that the form of $\mathscr{F}$ will be $\mathscr{F}(x) = ax + d(x)$ for all $x \in \mathscr{U}$, where $a \in \mathscr{U}$ and d is a derivation on $\mathscr{U}$ (see [19,* Theorem 3]*).*

Fact 2.8. *Let char ($\mathscr{R}$) $\neq 2$ and $f(\zeta_1, \ldots, \zeta_n)$ a multilinear polynomial over $\mathscr{C}$. If for any $i = 1, \ldots, n$,*

$$[f(\zeta_1, \ldots, z_i, \ldots, \zeta_n), f(\zeta_1, \ldots, \zeta_n)] = 0$$

for all $\zeta_1, \ldots, \zeta_n, z_i \in \mathscr{R}$, then the polynomial $f(\zeta_1, \ldots, \zeta_n)$ is central-valued on $\mathscr{R}$.

Proof. Let a be a noncentral element of $\mathscr{R}$. Then replacing z_i with $[a, \zeta_i]$ we have that for any $i = 1, \ldots, n$

$$[f(\zeta_1, \ldots, [a, \zeta_i], \ldots, \zeta_n), f(\zeta_1, \ldots, \zeta_n)] = 0$$

and so

$$[\sum_{i=0}^{n} f(\zeta_1, \ldots, [a, \zeta_i], \ldots, \zeta_n), f(\zeta_1, \ldots, \zeta_n)] = 0$$

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which implies, $[a, f(\zeta_1,\dots,\zeta_n)]_2 = 0$ for all $\zeta_1,\dots,\zeta_n \in \mathscr{R}$. By [16, Theorem], $f(\zeta_1,\dots,\zeta_n)$ is central-valued on $\mathscr{R}$. $\square$

3. The case: maps are inner.

We dedicate this section, when all generalized derivations are inner and the derivation is inner. Let $\mathscr{F}(x) = bx + xc$, $\mathscr{G}(x) = px + xq$ and $d(x) = [a, x]$ for all $x \in \mathscr{R}$, and suitable fixed $a, b, c, p, q \in \mathscr{U}$. Let $f(\zeta_1,\dots,\zeta_n)$ be a non-central multilinear polynomial over $\mathscr{C}$ with n noncommuting variables. Then situation

$$d([(\mathscr{F}^2 + \mathscr{G})(x), x]) = 0$$

gives

$$[a, [(b^2+p)x + 2bxc + x(c^2+q), x]] = 0$$

for all $x \in f(\mathscr{R}) = \{f(\zeta_1,\dots,\zeta_n) | \zeta_1,\dots,\zeta_n \in \mathscr{R}\}$. This gives the generalized polynomial identity

$$\begin{gathered}\Phi(\zeta_1,\dots,\zeta_n) = \\ a(b^2+p)f(\zeta_1,\dots,\zeta_n)^2 + 2abf(\zeta_1,\dots,\zeta_n)cf(\zeta_1,\dots,\zeta_n) \\ +af(\zeta_1,\dots,\zeta_n)(c^2+q)f(\zeta_1,\dots,\zeta_n) - af(\zeta_1,\dots,\zeta_n)(b^2+p)f(\zeta_1,\dots,\zeta_n) \\ -2af(\zeta_1,\dots,\zeta_n)bf(\zeta_1,\dots,\zeta_n)c - af(\zeta_1,\dots,\zeta_n)^2(c^2+q) \\ -(b^2+p)f(\zeta_1,\dots,\zeta_n)^2a - 2bf(\zeta_1,\dots,\zeta_n)cf(\zeta_1,\dots,\zeta_n)a \\ -f(\zeta_1,\dots,\zeta_n)(c^2+q)f(\zeta_1,\dots,\zeta_n)a + f(\zeta_1,\dots,\zeta_n)(b^2+p)f(\zeta_1,\dots,\zeta_n)a \\ +2f(\zeta_1,\dots,\zeta_n)bf(\zeta_1,\dots,\zeta_n)ca + f(\zeta_1,\dots,\zeta_n)^2(c^2+q)a = 0\end{gathered}$$

for all $\zeta_1,\dots,\zeta_n \in \mathscr{R}$.

Proposition 3.1. *Let $\mathscr{C}$ be an infinite field, $\mathscr{R} = M_m(\mathscr{C})$, $m \geq 2$ be the ring of all $m \times m$ matrices over $\mathscr{C}$, $f(\zeta_1,\dots,\zeta_n)$ a non-central multilinear polynomial over $\mathscr{C}$ and $a, b, c, p, q \in \mathscr{R}$. If*

$$\Phi(\zeta_1,\dots,\zeta_n) = 0$$

for all $\zeta_1,\dots,\zeta_n \in \mathscr{R}$, then either $a \in \mathscr{C}.I_m$ or $b \in \mathscr{C}.I_m$ or $c \in \mathscr{C}.I_m$.

Proof. On contrary, we assume that $a \notin \mathscr{Z}(\mathscr{R})$, $b \notin \mathscr{Z}(\mathscr{R})$ and $c \notin \mathscr{Z}(\mathscr{R})$. Now we show that this case leads to a contradiction.

By Fact 2.4, there exists a $\mathscr{C}$-automorphism ϕ of $M_m(\mathscr{C})$ such that $\phi(a)$, $\phi(b)$ and $\phi(c)$ have all non-zero entries. Evidently, R satisfies the condition

$$\begin{gathered}\phi(a(b^2+p))x^2 + 2\phi(ab)x\phi(c)x + \phi(a)x\phi(c^2+q)x - \phi(a)x\phi(b^2+p)x \\ -2\phi(a)x\phi(b)x\phi(c) - \phi(a)x^2\phi(c^2+q) - \phi(b^2+p)x^2\phi(a) - 2\phi(b)x\phi(c)x\phi(a) \\ -x\phi(c^2+q)x\phi(a) + x\phi(b^2+p)x\phi(a) + 2x\phi(b)x\phi(ca) + x^2\phi((c^2+q)a) = 0\end{gathered} \tag{3.1}$$

for all $x \in f(\mathscr{R})$.

Let e_{ij} denotes the usual matrix unit with 1 in (i,j)-entry and zero elsewhere. Since $f(\zeta_1,\dots,\zeta_n)$ is not central, by [18] (see also [20]), there exist matrices $\zeta_1,\dots,\zeta_n \in M_m(\mathscr{C})$ such that $f(\zeta_1,\dots,\zeta_n) = \gamma e_{ij}$ with $i \neq j$, where $0 \neq \gamma \in \mathscr{C}$. Thus we can substitute the value of $f(\zeta_1,\dots,\zeta_n)$ with γe_{ij} in (3.1) and then left and right multiplying by e_{ij}, we have

$$-2e_{ij}\phi(a)e_{ij}\phi(b)e_{ij}\phi(c)e_{ij} - 2e_{ij}\phi(b)e_{ij}\phi(c)e_{ij}\phi(a)e_{ij} = 0 \tag{3.2}$$

which implies $\phi(a)_{ji}\phi(b)_{ji}\phi(c)_{ji} = 0$. This is a contradiction, since $\phi(a)$, $\phi(b)$ and $\phi(c)$ have all non-zero entries. Thus we conclude that either $a \in \mathscr{C}.I_m$ or $b \in \mathscr{C}.I_m$ or $c \in \mathscr{C}.I_m$. □

Proposition 3.2. *Let $\mathscr{C}$ be any field, $\mathscr{R} = M_m(\mathscr{C})$, $m \geq 2$ be the ring of all $m \times m$ matrices over C with char $\mathscr{R} \neq 2$, $f(x_1,\dots,x_n)$ a non-central multilinear polynomial over $\mathscr{C}$ and $a,b,c,p,q \in \mathscr{R}$. If*

$$\Phi(\zeta_1,\dots,\zeta_n) = 0$$

for all $\zeta_1,\dots,\zeta_n \in \mathscr{R}$, then either $a \in \mathscr{C}.I_m$ or $b \in \mathscr{C}.I_m$ or $c \in \mathscr{C}.I_m$.

Proof. If $\mathscr{C}$ is infinite, then the conclusions follow by Proposition 3.1.
So we assume that $\mathscr{C}$ is finite. Let K be an infinite field which is an extension of the field $\mathscr{C}$. Let $\overline{\mathscr{R}} = M_m(K) \cong \mathscr{R} \otimes_{\mathscr{C}} K$. Note that the multilinear polynomial $f(x_1,\dots,x_n)$ is central-valued on $\mathscr{R}$ if and only if it is central-valued on $\overline{\mathscr{R}}$.

Given the generalized polynomial identity for $\mathscr{R}$

$$\Phi(\zeta_1,\dots,\zeta_n) = 0.$$

Moreover, it is a multi-homogeneous of multi-degree $(2,\dots,2)$ in the indeterminates $\zeta_1,\dots,\zeta_n$. Now the complete linearization of $\Phi(\zeta_1,\dots,\zeta_n)$ gives a multilinear generalized polynomial $\Theta(\zeta_1,\dots,\zeta_n,\eta_1, \dots,\eta_n)$ in $2n$ indeterminates. Notice that

$$\Theta(\zeta_1,\dots,\zeta_n,\zeta_1,\dots,\zeta_n) = 2^n\Phi(\zeta_1,\dots,\zeta_n).$$

Clearly the multilinear polynomial $\Theta(\zeta_1,\dots,\zeta_n,\eta_1,\dots,\eta_n)$ is a generalized polynomial identity for R and $\overline{R}$ too. Since $char(\mathscr{C}) \neq 2$ we obtain $\Phi(\zeta_1,\dots,\zeta_n) = 0$ for all $\zeta_1,\dots,\zeta_n \in \overline{\mathscr{R}}$ and then conclusion follows by Proposition 3.1. □

Thus in the same manner, we have the following Corollary.

Corollary 3.3. *Let $\mathscr{R} = M_m(\mathscr{C})$, $m \geq 2$ be the ring of all matrices over the field $\mathscr{C}$ with char $(\mathscr{R}) \neq 2$ and $f(x_1,\dots,x_n)$ a non-central multilinear polynomial over $\mathscr{C}$ and $a,b,c,p,q,a',a'',b',c',p',q' \in \mathscr{R}$. If*

$$a'x^2 + 2a''xcx + axb'x - axc'x - 2axbxc - ax^2b'$$
$$-p'x^2a - 2bxcxa - xb'xa + xp'xa + 2xbxca + x^2q' = 0$$

for all $x \in f(\mathscr{R})$, then either $a \in \mathscr{C}.I_m$ or $b \in \mathscr{C}.I_m$ or $c \in \mathscr{C}.I_m$.

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Corollary 3.4. *Let $\mathscr{R} = M_m(\mathscr{C})$, $m \geq 2$ be the ring of all matrices over the field $\mathscr{C}$ with char $(\mathscr{R}) \neq 2$ and $a, b, c, p, q, a', a'', b', c', p', q' \in \mathscr{R}$. If*

$$a'x^2 + 2a''xcx + axb'x - axc'x - 2axbxc - ax^2b'$$
$$-p'x^2a - 2bxcxa - xb'xa + xp'xa + 2xbxca + x^2q' = 0$$

for all $x \in \mathscr{R}$, then either $a \in \mathscr{C}.I_m$ or $b \in \mathscr{C}.I_m$ or $c \in \mathscr{C}.I_m$.

Lemma 3.5. *Let $\mathscr{R}$ be a noncommutative prime ring with char $(\mathscr{R}) \neq 2$, $\mathscr{C} = \mathscr{Z}(\mathscr{U})$ be its extended centroid, where $\mathscr{U}$ is the Utumi quotient ring of $\mathscr{R}$ and $f(\zeta_1, \ldots, \zeta_n)$ a multilinear polynomial over $\mathscr{C}$, which is not central valued on $\mathscr{R}$. Suppose that for some $a, b, c, p, q \in \mathscr{U}$, $d(x) = [a, x]$, $\mathscr{F}(x) = bx + xc$ and $\mathscr{G}(x) = px + xq$ for all $x \in \mathscr{R}$ such that*

$$d([(\mathscr{F}^2 + \mathscr{G})(f(\zeta_1, \ldots, \zeta_n)), f(\zeta_1, \ldots, \zeta_n)]) = 0$$

for all $\zeta_1, \ldots, \zeta_n \in \mathscr{R}$. Then one of the following holds:

(1) $\mathscr{F}(x) = x(b + c)$ and $\mathscr{G}(x) = x(p + q)$ for all $x \in \mathscr{R}$ with $(b + c)^2 + p + q \in \mathscr{C}$;
(2) $\mathscr{F}(x) = (b + c)x$ and $\mathscr{G}(x) = (p + q)x$ for all $x \in \mathscr{R}$ with $(b + c)^2 + p + q \in \mathscr{C}$;
(3) $f(x_1, \ldots, x_n)^2$ is central valued on $\mathscr{R}$ and one of the following holds:
  (a) $\mathscr{F}(x) = x(b + c)$ and $\mathscr{G}(x) = px + xq$ for all $x \in \mathscr{R}$ with $(b + c)^2 + q - p \in \mathscr{C}$;
  (b) $\mathscr{F}(x) = (b + c)x$ and $\mathscr{G}(x) = px + xq$ for all $x \in \mathscr{R}$ with $(b + c)^2 + p - q \in \mathscr{C}$.

Proof. By hypothesis, we have

$$\Phi(\zeta_1, \ldots, \zeta_n) = 0$$

for all $\zeta_1, \ldots, \zeta_n \in \mathscr{R}$. This is a GPI for $\mathscr{U}$ also (see Fact 2.1).

Let $a \notin \mathscr{C}$, $b \notin \mathscr{C}$ and $c \notin \mathscr{C}$.

Suppose that $\Phi(\zeta_1, \ldots, \zeta_n) = 0$ is a trivial GPI for $\mathscr{U}$. Let $T = \mathscr{U} *_{\mathscr{C}} \mathscr{C}\{\zeta_1, \zeta_2, \ldots, \zeta_n\}$, the free product of $\mathscr{U}$ and $\mathscr{C}\{\zeta_1, \ldots, \zeta_n\}$, the free $\mathscr{C}$-algebra in noncommuting indeterminates $\zeta_1, \zeta_2, \ldots, \zeta_n$. Then, $\Phi(\zeta_1, \ldots, \zeta_n)$ is zero element in $T = \mathscr{U} *_{\mathscr{C}} \mathscr{C}\{\zeta_1, \ldots, \zeta_n\}$. Since $a \notin \mathscr{C}$, $b \notin \mathscr{C}$ and $c \notin \mathscr{C}$, the term

$$-2af(\zeta_1, \ldots, \zeta_n)bf(\zeta_1, \ldots, \zeta_n)c - 2bf(\zeta_1, \ldots, \zeta_n)cf(\zeta_1, \ldots, \zeta_n)a$$

appears nontrivially in (3.3) and hence

$$-2af(\zeta_1, \ldots, \zeta_n)bf(\zeta_1, \ldots, \zeta_n)c - 2bf(\zeta_1, \ldots, \zeta_n)cf(\zeta_1, \ldots, \zeta_n)a = 0 \in T.$$

This implies that a and b are $\mathscr{C}$-dependent. Let $a = \alpha b$ for some $\alpha \in \mathscr{C}$. Then from above

$$-2\alpha bf(\zeta_1, \ldots, \zeta_n)bf(\zeta_1, \ldots, \zeta_n)c - 2bf(\zeta_1, \ldots, \zeta_n)cf(\zeta_1, \ldots, \zeta_n)\alpha b = 0 \in T.$$

Again this implies that b and c are $\mathscr{C}$-dependent, that is, $b = \beta c$ for some $\beta \in \mathscr{C}$. Thus from above relation

$$-4\alpha\beta^2 cf(\zeta_1, \ldots, \zeta_n)cf(\zeta_1, \ldots, \zeta_n)c = 0 \in T.$$

Since $c \notin \mathscr{C}$, we must have $\alpha = 0$ or $\beta = 0$. This implies that either $a = 0$ or $b = 0$, a contradiction.

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Next suppose that $\Phi(\zeta_1,\ldots,\zeta_n)=0$ is a non-trivial GPI for $\mathscr{U}$. Then by Fact 2.6, $\mathscr{R}$ is isomorphic to a dense ring of linear transformations of a vector space V over $\mathscr{C}$.

Let $\dim_{\mathscr{C}}V=m$. By density of $\mathscr{R}$, $\mathscr{R}\cong M_m(\mathscr{C})$. Given that $f(r_1,\ldots,r_n)$ is not central valued on $\mathscr{R}$ and therefore, $\mathscr{R}$ must be noncommutative. Hence $m\geq 2$. In this case, by Proposition 3.2, we have either a or b or c are in $\mathscr{C}$, a contradiction. Let $\dim_{\mathscr{C}}V=\infty$. Since the set $f(\mathscr{R})$ is dense on $\mathscr{R}$ (Lemma 2 in [26]), from above, $\mathscr{R}$ satisfies

$$a(b^2+p)r^2+2abrcr+ar(c^2+q)r-ar(b^2+p)r$$
$$-2arbrc-ar^2(c^2+q)-(b^2+p)r^2a-2brcra$$
$$-r(c^2+q)ra+r(b^2+p)ra+2rbrca+r^2(c^2+q)a=0.$$

For any $e^2=e\in soc(\mathscr{R})$ we have $e\mathscr{R}e\cong M_t(\mathscr{C})$ with $t=\dim_{\mathscr{C}}Ve$. Since $a\notin\mathscr{C}$, $b\notin\mathscr{C}$ and $c\notin\mathscr{C}$, then there exist $h_1,h_2,h_3,h_4\in soc(\mathscr{R})$ such that either $[a,h_1]\neq 0$, $[b,h_2]\neq 0$ and $[c,h_3]\neq 0$. By Litoff's theorem [4], there exists idempotent $e\in soc(\mathscr{R})$ such that $ah_1,h_1a,bh_2,h_2b,ch_3,h_3c,h_1,h_2,h_3\in e\mathscr{R}e$. We have $e\mathscr{R}e\cong M_k(\mathscr{C})$ with $k=\dim_{\mathscr{C}}Ve$. Since $\mathscr{R}$ satisfies generalized identity

$$e\{a(b^2+p)(ere)^2+2aberecere+aere(c^2+q)ere-aere(b^2+p)ere$$
$$-2aereberec-a(ere)^2(c^2+q)-(b^2+p)(ere)^2a-2berecerea$$
$$-ere(c^2+q)erea+ere(b^2+p)erea+2erebereca+(ere)^2(c^2+q)a\}e=0.$$

the subring $e\mathscr{R}e$ satisfies

$$ea(b^2+p)er^2+2eaberecer+eaere(c^2+q)er-eaere(b^2+p)er$$
$$-2eaereberece-eaer^2e(c^2+q)e-e(b^2+p)er^2eae-2eberecereae$$
$$-re(c^2+q)ereae+re(b^2+p)ereae+2reberecae+r^2e(c^2+q)ae=0.$$

Then by Corollary 3.4 either eae or ebe or ece are central elements of $e\mathscr{R}e$. Thus $ah_1=(eae)h_1=h_1eae=h_1a$ or $bh_2=(ebe)h_2=h_2(ebe)=h_2b$ or $ch_3=(ece)h_3=h_3(ece)=h_3c$, a contradiction.

Thus up to now, we have proved that either a or b or c are in $\mathscr{C}$. Therefore, we have the following four cases:

<u>Case-I:</u> $b\in C$.

In this case, $\mathscr{F}(x)=x(b+c)$ for all $x\in\mathscr{R}$ and hence by hypothesis $\mathscr{R}$ satisfies

$$[a,[pf(\zeta_1,\ldots,\zeta_n)+f(\zeta_1,\ldots,\zeta_n)(q+(b+c)^2),f(\zeta_1,\ldots,\zeta_n)]]=0.$$

By [9], one of the following holds:

(i) $p,q+(b+c)^2\in\mathscr{C}$. Thus $\mathscr{F}(x)=x(b+c)$ and $\mathscr{G}(x)=x(p+q)$ for all $x\in\mathscr{R}$ with $(b+c)^2+p+q\in\mathscr{C}$;

(ii) $p-q-(b+c)^2\in\mathscr{C}$ and $f(x_1,\ldots,x_n)^2$ is central valued on $\mathscr{R}$. Thus $\mathscr{F}(x)=x(b+c)$ and $\mathscr{G}(x)=px+xq$ for all $x\in\mathscr{R}$ with $(b+c)^2+q-p\in\mathscr{C}$.

<u>Case-II:</u> $c\in\mathscr{C}$.

In this case, $\mathscr{F}(x)=(b+c)x$ for all $x\in\mathscr{R}$ and hence by hypothesis $\mathscr{R}$ satisfies

$$[a,[(p+(b+c)^2)f(\zeta_1,\ldots,\zeta_n)+f(\zeta_1,\ldots,\zeta_n)q,f(\zeta_1,\ldots,\zeta_n)]]=0.$$

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By [9], one of the following holds:

(i) $q, p+(b+c)^2 \in \mathscr{C}$. Thus $\mathscr{F}(x) = (b+c)x$ and $\mathscr{G}(x) = (p+q)x$ for all $x \in \mathscr{R}$ with $(b+c)^2+p+q \in \mathscr{C}$;

(ii) $p + (b+c)^2 - q \in \mathscr{C}$ and $f(x_1, \ldots, x_n)^2$ is central valued on $\mathscr{R}$. Thus $\mathscr{F}(x) = (b+c)x$ and $\mathscr{G}(x) = px + xq$ for all $x \in \mathscr{R}$ with $(b+c)^2 + p - q \in \mathscr{C}$.

□

4. Proof of Theorem 1.1

Lemma 4.1. *Let R be a noncommutative prime ring with char $(\mathscr{R}) \neq 2$, $\mathscr{C} = \mathscr{Z}(\mathscr{U})$ be its extended centroid, where $\mathscr{U}$ is the Utumi quotient ring of $\mathscr{R}$ and $f(\zeta_1, \ldots, \zeta_n)$ a multilinear polynomial over $\mathscr{C}$, which is not central valued on $\mathscr{R}$. Suppose that $0 \neq a \in \mathscr{R}$ and $\mathscr{F}$, $\mathscr{G}$ two generalized derivations of $\mathscr{R}$. If*

$$\Big[a, [(\mathscr{F}^2 + \mathscr{G})(f(\zeta_1, \ldots, \zeta_n)), f(\zeta_1, \ldots, \zeta_n)]\Big] = 0$$

for all $\zeta_1, \ldots, \zeta_n \in \mathscr{R}$, then one of the following holds:

(1) *there exist $b, p \in \mathscr{U}$ such that $\mathscr{F}(x) = xb$ and $\mathscr{G}(x) = xp$ for all $x \in \mathscr{R}$ with $b^2 + p \in \mathscr{C}$;*

(2) *there exist $b, p \in \mathscr{U}$ such that $\mathscr{F}(x) = bx$ and $\mathscr{G}(x) = px$ for all $x \in \mathscr{R}$ with $b^2 + p \in \mathscr{C}$;*

(3) *$f(\zeta_1, \ldots, \zeta_n)^2$ is central valued on R and one of the following holds:*

 (a) *there exist $b, p, q \in \mathscr{U}$ such that $\mathscr{F}(x) = xb$ and $\mathscr{G}(x) = px + xq$ for all $x \in \mathscr{R}$ with $b^2 + q - p \in \mathscr{C}$;*

 (b) *there exist $b, p, q \in \mathscr{U}$ such that $\mathscr{F}(x) = bx$ and $\mathscr{G}(x) = px + xq$ for all $x \in \mathscr{R}$ with $b^2 + p - q \in \mathscr{C}$.*

Proof. In view of Fact 2.7, there exist $b, p \in \mathscr{U}$ and derivations d', δ of $\mathscr{U}$ such that $\mathscr{F}(x) = bx + \delta(x)$ and $\mathscr{G}(x) = px + g(x)$. In view of Fact 2.1 and Fact 2.2, we have

$$[a, [(\mathscr{F}(b) + p)f(\zeta) + 2b\delta(f(\zeta)) + \delta^2(f(\zeta)) + g(f(\zeta)), f(\zeta)]] = 0 \tag{4.1}$$

for all $\zeta = (\zeta_1, \ldots, \zeta_n) \in \mathscr{U}^n$, where δ and g are two derivations on $\mathscr{U}$.

If δ and g both are inner derivations of $\mathscr{R}$, then by Lemma 3.5 we obtain our conclusions. Thus we assume that not both of δ and g are inner. Now we consider the following two cases:

Case-I: *δ and g are linearly C-dependent modulo inner derivations of $\mathscr{U}$.*

Let $\alpha_1\delta + \alpha_2 g = ad_q$, i.e., $\alpha_1\delta(x) + \alpha_2 g(x) = [q, x]$ for some $\alpha_1, \alpha_2 \in \mathscr{C}$ and $q \in \mathscr{U}$. If $\alpha_2 = 0$, then $\delta(x) = [q', x]$ inner derivation with $q' = \alpha_1^{-1}q$. Then g must be outer and hence by using Fact 2.3 we can replace $g(x_i)$ with t_i in (4.1) and then $\mathscr{U}$ satisfies blended component

$$\Big[a, [\sum_i f(\zeta_1, \ldots, \eta_i, \ldots, \zeta_n), f(\zeta_1, \ldots, \zeta_n)]\Big] = 0. \tag{4.2}$$

Replacing η_i with $[p', \zeta_i]$ for some $p' \notin \mathscr{C}$, we have

$$\Big[a, [[p', f(\zeta_1, \ldots, \zeta_n)], f(\zeta_1, \ldots, \zeta_n)]\Big] = 0. \tag{4.3}$$

By [9], $a \in C$ or $p' \in \mathscr{C}$, a contradiction.

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Thus we assume that $\alpha_2 \neq 0$. Then we can write $g(x) = \lambda\delta(x) + [q', x]$, where $q' = \alpha_2^{-1}q$ and $\lambda = -\alpha_2^{-1}\alpha_1$. By hypothesis, $\mathscr{U}$ satisfies

$$[a, [(\mathscr{F}(b) + p)f(\zeta) + 2b\delta(f(\zeta)) + \delta^2(f(\zeta)) + \lambda\delta(f(\zeta)) + [q', f(\zeta)], f(\zeta)]] = 0 \tag{4.4}$$

for all $\zeta = (\zeta_1, \dots, \zeta_n) \in \mathscr{U}^n$. Since δ is outer, by Fact 2.3 we can replace $\delta(\zeta_i)$ with η_i and $\delta^2(\zeta_i)$ with s_i in (4.4) and then $\mathscr{U}$ satisfies blended component

$$\Big[a, [\sum_i f(\zeta_1, \dots, s_i, \dots, \zeta_n), f(\zeta_1, \dots, \zeta_n)]\Big] = 0. \tag{4.5}$$

This is same as (4.8). By same argument, it leads to a contradiction.

Case-II: *δ and g are linearly $\mathscr{C}$-independent modulo inner derivations of $\mathscr{U}$.*

By Fact 2.3 we can replace $g(\zeta_i)$ with y_i, $\delta(\zeta_i)$ with t_i and $\delta^2(\zeta_i)$ with s_i in (4.1) and then $\mathscr{U}$ satisfies blended component

$$\Big[a, [\sum_i f(\zeta_1, \dots, s_i, \dots, \zeta_n), f(\zeta_1, \dots, \zeta_n)]\Big] = 0. \tag{4.6}$$

This is same as (4.8). By same argument, it leads to a contradiction. □

Lemma 4.2. *Let $\mathscr{R}$ be a noncommutative prime ring with char $(\mathscr{R}) \neq 2$, $\mathscr{C} = \mathscr{Z}(\mathscr{U})$ be its extended centroid, where $\mathscr{U}$ is the Utumi quotient ring of $\mathscr{R}$ and $f(\zeta_1, \dots, \zeta_n)$ a multilinear polynomial over $\mathscr{C}$, which is not central valued on $\mathscr{R}$. Assume that d is a nonzero derivation of $\mathscr{R}$, $\mathscr{F}$ a generalized derivation of $\mathscr{R}$ and $\mathscr{G}(x) = px + xq$ for some $p, q \in \mathscr{U}$ such that*

$$d([\mathscr{F}^2(f(\zeta_1, \dots, \zeta_n)) + \mathscr{G}(f(\zeta_1, \dots, \zeta_n)), f(\zeta_1, \dots, \zeta_n)]) = 0$$

for all $\zeta_1, \dots, \zeta_n \in \mathscr{R}$, then one of the following holds:

(1) *there exist $b, p \in \mathscr{U}$ such that $\mathscr{F}(x) = xb$ and $\mathscr{G}(x) = xp$ for all $x \in \mathscr{R}$ with $b^2 + p \in \mathscr{C}$;*
(2) *there exist $b, p \in \mathscr{U}$ such that $\mathscr{F}(x) = bx$ and $\mathscr{G}(x) = px$ for all $x \in \mathscr{R}$ with $b^2 + p \in \mathscr{C}$;*
(3) *$f(\zeta_1, \dots, \zeta_n)^2$ is central valued on $\mathscr{R}$ and one of the following holds:*
    (a) *there exist $b, p, q \in \mathscr{U}$ such that $\mathscr{F}(x) = xb$ and $\mathscr{G}(x) = px + xq$ for all $x \in \mathscr{R}$ with $b^2 + q - p \in \mathscr{C}$;*
    (b) *there exist $b, p, q \in \mathscr{U}$ such that $\mathscr{F}(x) = bx$ and $\mathscr{G}(x) = px + xq$ for all $x \in \mathscr{R}$ with $b^2 + p - q \in \mathscr{C}$.*

Proof. In view of Fact 2.7, there exist $b \in U$ and derivations δ of $\mathscr{U}$ such that $\mathscr{F}(x) = bx + \delta(x)$. In view of Fact 2.1 and 2.2, we have

$$d([(\mathscr{F}(b) + p)f(\zeta) + 2b\delta(f(\zeta)) + \delta^2(f(\zeta)) + f(\zeta)q, f(\zeta)]] = 0 \tag{4.7}$$

for all $\zeta = (\zeta_1, \dots, \zeta_n) \in \mathscr{U}^n$. Now we consider the following cases:

Case-I: *δ and d are linearly $\mathscr{C}$-dependent modulo inner derivations of $\mathscr{U}$.*

Let $\alpha_1\delta + \alpha_2 d = ad_q$, i.e., $\alpha_1\delta(x) + \alpha_2 d(x) = [q, x]$ for some $\alpha_1, \alpha_2 \in \mathscr{C}$ and $q \in \mathscr{U}$. If $\alpha_1 = 0$, then $d(x) = [q', x]$ inner derivation with $q' = \alpha_2^{-1}q$. Then δ must be outer and hence by using Fact 2.3 we

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can replace $\delta(\zeta_i)$ with t_i and $\delta^2(\zeta_i)$ with s_i in (4.16) and then U satisfies blended component

$$\Big[q', [\sum_i f(\zeta_1, \ldots, s_i, \ldots, \zeta_n), f(\zeta_1, \ldots, \zeta_n)]\Big] = 0. \tag{4.8}$$

Replacing s_i with $[p', \zeta_i]$ for some $p' \notin \mathscr{C}$, we have

$$\Big[q', [[p', f(\zeta_1, \ldots, \zeta_n)], f(\zeta_1, \ldots, \zeta_n)]\Big] = 0. \tag{4.9}$$

By [9], $p' \in \mathscr{C}$ or $q' \in \mathscr{C}$. Since $p' \notin \mathscr{C}$, we have $q' = \alpha_2^{-1} q \in \mathscr{C}$, i.e., $d = 0$, a contradiction.

If $\alpha_1 \neq 0$, then $\delta(x) = \beta d(x) + [q'', x]$, where $\beta = -\alpha_1^{-1}\alpha_2$ and $q'' = \alpha_1^{-1} q$. Thus replacing this value in (4.16) and then by Fact 2.3 we can replace $d^3(\zeta_i)$ with t_i and hence $\mathscr{U}$ satisfies blended components

$$\beta^2 \Big[\sum_i f(\zeta_1, \ldots, t_i, \ldots, \zeta_n), f(\zeta_1, \ldots, \zeta_n)\Big] = 0. \tag{4.10}$$

This implies $\beta = 0$, and hence $\delta(x) = [q'', x]$ inner derivation. By (4.16)

$$d([(\mathscr{F}(b) + p) f(\zeta) + 2b[q'', f(\zeta)] + [q'', [q'', f(\zeta)]] + f(\zeta) q, f(\zeta)]] = 0 \tag{4.11}$$

for all $\zeta = (\zeta_1, \ldots, \zeta_n) \in \mathscr{U}^n$. Since d is outer, by Fact 2.3 we can replace $d(\zeta_i)$ with t_i and hence $\mathscr{U}$ satisfies

$$\begin{aligned} &\Big[(\mathscr{F}(b) + p) \sum_i f(\zeta_1, \ldots, t_i, \ldots, \zeta_n) + 2b[q'', \sum_i f(\zeta_1, \ldots, t_i, \ldots, \zeta_n)] \\ &+ [q'', [q'', \sum_i f(\zeta_1, \ldots, t_i, \ldots, \zeta_n)]] + \sum_i f(\zeta_1, \ldots, t_i, \ldots, \zeta_n) q, f(\zeta_1, \ldots, \zeta_n)]\Big] \\ &+ \Big[(\mathscr{F}(b) + p) f(\zeta_1, \ldots, \zeta_n) + 2b[q'', f(\zeta_1, \ldots, \zeta_n)] + [q'', [q'', f(\zeta_1, \ldots, \zeta_n)]] \\ &+ f(\zeta_1, \ldots, \zeta_n) q, \sum_i f(\zeta_1, \ldots, t_i, \ldots, \zeta_n)\Big] = 0. \end{aligned} \tag{4.12}$$

In particular, for $t_1 = \zeta_1$ and $t_2 = \cdots = t_n = 0$, we have from above that

$$2[(\mathscr{F}(b) + p) f(\zeta) + 2b[q'', f(\zeta)] + [q'', [q'', f(\zeta)]] + f(\zeta) q, f(\zeta)] = 0 \tag{4.13}$$

which gives by using char $(R) \neq 2$ that

$$[(\mathscr{F}^2 + \mathscr{G})(f(\zeta)), f(\zeta)] = 0 \tag{4.14}$$

for all $\zeta = (\zeta_1, \ldots, \zeta_n) \in \mathscr{U}^n$. Then by [5], we have our conclusions.

Case-II: *δ and d are linearly $\mathscr{C}$-independent modulo inner derivations of $\mathscr{U}$.*

In this case by Fact 2.3 we can replace $d\delta^2(\zeta_i)$ with t_i and hence $\mathscr{U}$ satisfies blended components

$$\Big[\sum_i f(\zeta_1, \ldots, t_i, \ldots, \zeta_n), f(\zeta_1, \ldots, \zeta_n)\Big] = 0. \tag{4.15}$$

By Fact 2.8, this leads to a contradiction. □

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Lemma 4.3. *Let $\mathscr{R}$ be a noncommutative prime ring with char $(\mathscr{R}) \neq 2$, $\mathscr{C} = \mathscr{Z}(\mathscr{U})$ be its extended centroid, where $\mathscr{U}$ is the Utumi quotient ring of $\mathscr{R}$ and $f(\zeta_1, \dots, \zeta_n)$ a multilinear polynomial over $\mathscr{C}$, which is not central valued on $\mathscr{R}$. Assume that d is a nonzero derivation of $\mathscr{R}$, $\mathscr{G}$ a generalized derivations of $\mathscr{R}$ and $\mathscr{F}(x) = bx + xc$ for some $b, c \in \mathscr{U}$ such that*

$$d([\mathscr{F}^2(f(\zeta_1, \dots, \zeta_n)) + \mathscr{G}(f(\zeta_1, \dots, \zeta_n)), f(\zeta_1, \dots, \zeta_n)]) = 0$$

for all $\zeta_1, \dots, \zeta_n \in \mathscr{R}$, then one of the following holds:

(1) *there exist $b, p \in \mathscr{U}$ such that $\mathscr{F}(x) = xb$ and $\mathscr{G}(x) = xp$ for all $x \in \mathscr{R}$ with $b^2 + p \in \mathscr{C}$;*
(2) *there exist $b, p \in \mathscr{U}$ such that $\mathscr{F}(x) = bx$ and $\mathscr{G}(x) = px$ for all $x \in \mathscr{R}$ with $b^2 + p \in \mathscr{C}$;*
(3) *$f(\zeta_1, \dots, \zeta_n)^2$ is central valued on R and one of the following holds:*
   (a) *there exist $b, p, q \in \mathscr{U}$ such that $\mathscr{F}(x) = xb$ and $\mathscr{G}(x) = px + xq$ for all $x \in \mathscr{R}$ with $b^2 + q - p \in \mathscr{C}$;*
   (b) *there exist $b, p, q \in \mathscr{U}$ such that $\mathscr{F}(x) = bx$ and $\mathscr{G}(x) = px + xq$ for all $x \in \mathscr{R}$ with $b^2 + p - q \in \mathscr{C}$.*

Proof. In view of Fact 2.7, there exist $p \in \mathscr{U}$ and derivations g of $\mathscr{U}$ such that $\mathscr{G}(x) = px + g(x)$. In view of Fact 2.1 and Fact 2.2, we have

$$d([(b^2 + p)f(\zeta) + 2bf(\zeta)c + f(\zeta)c^2 + g(f(\zeta)), f(\zeta)]) = 0 \tag{4.16}$$

for all $\zeta = (\zeta_1, \dots, \zeta_n) \in \mathscr{U}^n$. Now we consider the following cases:

Case-I: *d and g are linearly $\mathscr{C}$-dependent modulo inner derivations of $\mathscr{U}$.*

Let $\alpha_1 d + \alpha_2 g = ad_q$, i.e., $\alpha_1 d(x) + \alpha_2 g(x) = [q, x]$ for some $\alpha_1, \alpha_2 \in C$ and $q \in U$. If $\alpha_1 = 0$, then $g(x) = [q', x]$ inner derivation with $q' = \alpha_2^{-1} q$. Then d must be outer and hence by using Fact 2.3 we can replace $d(\zeta_i)$ with t_i in (4.16) and then U satisfies blended component

$$\begin{aligned}&\Big[(b^2 + p + q')\sum_i f(\zeta_1, \dots, t_i, \dots, \zeta_n) + 2b\sum_i f(\zeta_1, \dots, t_i, \dots, \zeta_n)c \\ &\quad + \sum_i f(\zeta_1, \dots, t_i, \dots, \zeta_n)(c^2 - q'), f(\zeta_1, \dots, \zeta_n)\Big] \\ &+ \Big[(b^2 + p + q')f(\zeta_1, \dots, \zeta_n) + 2bf(\zeta_1, \dots, \zeta_n)c + f(\zeta_1, \dots, \zeta_n)(c^2 - q'), \\ &\quad \sum_i f(\zeta_1, \dots, t_i, \dots, \zeta_n)\Big] = 0.\end{aligned}$$

In particular, for $t_1 = \zeta_1$ and $\zeta_2 = \dots = \zeta_n = 0$, we have from above that

$$2[(b^2 + p + q')f(\zeta) + 2bf(\zeta)c + f(\zeta)(c^2 - q'), f(\zeta)] = 0 \tag{4.17}$$

which gives by using char $(R) \neq 2$ that

$$[(\mathscr{F}^2 + \mathscr{G})(f(\zeta)), f(\zeta)] = 0 \tag{4.18}$$

for all $\zeta = (\zeta_1, \dots, \zeta_n) \in \mathscr{U}^n$. Then by [5], we have our conclusions.

Case-II: *d and g are linearly $\mathscr{C}$-independent modulo inner derivations of $\mathscr{U}$.*

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By Fact 2.3 we can replace $dg(\zeta_i)$ with t_i in (4.16) and then $\mathscr{U}$ satisfies blended component

$$\left[\sum_i f(\zeta_1,\ldots,t_i,\ldots,\zeta_n), f(\zeta_1,\ldots,\zeta_n)\right]=0. \tag{4.19}$$

By Fact 2.8, this leads to a contradiction. □

Proof of Theorem 1.1. If any one of d, $\mathscr{F}$ and $\mathscr{G}$ are inner, then we have our conclusions by Lemma 4.1, Lemma 4.2 and Lemma 4.3. Thus we are to consider the case when all of them are outer.

In view of Fact 2.7, there exist $b,p\in\mathscr{U}$ and derivations d',δ of $\mathscr{U}$ such that $\mathscr{F}(x)=bx+\delta(x)$ and $\mathscr{G}(x)=px+g(x)$. In view of Fact 2.1 and Fact 2.2, we have

$$d([(\mathscr{F}(b)+p)f(\zeta)+2b\delta(f(\zeta))+\delta^2(f(\zeta))+g(f(\zeta)),f(\zeta)])=0 \tag{4.20}$$

for all $\zeta=(\zeta_1,\ldots,\zeta_n)\in\mathscr{U}^n$, where δ and g are two derivations on $\mathscr{U}$.

Let d, d' and δ be $\mathscr{C}$-independent modulo inner derivations of $\mathscr{U}$.
Applying Fact 2.3, since char $(\mathscr{R})\neq 2,3$, replacing in (4.20) each $d\delta^2(\zeta_i)$ with y_i, we note that $\mathscr{U}$ satisfies the blended component

$$[\sum_i f(\zeta_1,\ldots,y_i,\ldots,\zeta_n),f(\zeta_1,\ldots,\zeta_n)]=0.$$

By Fact 2.8, this leads to a contradiction.
In all that follows we then assume that d, g and δ are $\mathscr{C}$-dependent modulo inner derivations of $\mathscr{U}$, i.e. there exist $\lambda,\mu,\nu\in\mathscr{C}$ and $q\in\mathscr{U}$ such that

$$\lambda d(x)+\mu g(x)+\nu\delta(x)=[q,x]\quad\forall x\in\mathscr{R}. \tag{4.21}$$

Notice that, since d is not inner, then $(\mu,\nu)\neq(0,0)$.
Without loss of generality, we assume that $\mu\neq 0$. By (4.21) we have that

$$g(x)=\lambda' d(x)+\nu'\delta(x)+[q',x]\quad\forall x\in\mathscr{R} \tag{4.22}$$

where $\lambda'=-\mu^{-1}\lambda$, $\nu'=-\mu^{-1}\nu$ and $q'=\mu^{-1}q$. Then (4.20) reduces to

$$\begin{aligned} d\Big([(\mathscr{F}(b)+p)f(\zeta)+2b\delta(f(\zeta))+\delta^2(f(\zeta))\\ +\lambda' d(f(\zeta))+\nu'\delta(f(\zeta))+[q',f(\zeta)],f(f(\zeta))]\Big)=0\end{aligned} \tag{4.23}$$

for all $\zeta=(\zeta_1,\ldots,\zeta_n)\in\mathscr{U}^n$. This is a differential identity in which the derivation words that appear are of the type

$$d,\delta,d\delta,\delta^2,d^2,d\delta^2.$$

If we assume first that d,δ are linearly $\mathscr{C}$-independent modulo inner derivations, then by using again Fact 2.3, we may replace each $d\delta^2(\zeta_i)$ with y_i. By computations, U satisfies blended component

$$[\sum_i f(\zeta_1,\ldots,y_i,\ldots,\zeta_n),f(\zeta_1,\ldots,\zeta_n)]=0$$

which gives contradiction by Fact 2.8.

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Next we assume that d, δ are linearly $\mathscr{C}$-dependent modulo inner derivations, that is, $d(x) = \alpha\delta(x) + [p', x]$ for all $x \in \mathscr{R}$. By (4.23),

$$\alpha\delta\Big([(\mathscr{F}(b)+p)f(\zeta) + 2b\delta(f(\zeta)) + \delta^2(f(\zeta)) + (\lambda'\alpha+\nu')\delta(f(\zeta)) + [\lambda'p'+q', f(\zeta)], f(f(\zeta))]\Big) + \Big[p', [(F(b)+p)f(\zeta) + 2b\delta(f(\zeta)) + \delta^2(f(\zeta)) + (\lambda'\alpha+\nu')\delta(f(\zeta)) + [\lambda'p'+q', f(\zeta)], f(f(\zeta))]\Big] = 0$$

for all $\zeta = (\zeta_1, \dots, \zeta_n) \in \mathscr{U}^n$.

This is a differential identity in which the derivation words that appear are of the type

$$\delta, \delta^2, \delta^3.$$

By using again Fact 2.3, we may replace each $\delta^3(\zeta_i)$ with y_i. By computations, $\mathscr{U}$ satisfies blended component

$$\alpha[\sum_i f(\zeta_1, \dots, y_i, \dots, \zeta_n), f(\zeta_1, \dots, \zeta_n)] = 0.$$

Now $\alpha = 0$ implies d is inner, a contradiction. Thus $\alpha \neq 0$ and hence $\mathscr{U}$ satisfies

$$[\sum_i f(\zeta_1, \dots, y_i, \dots, \zeta_n), f(\zeta_1, \dots, \zeta_n)] = 0$$

which gives contradiction by Fact 2.8. This completes the proof of the theorem.

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