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ON NATURALLY REDUCTIVE (α_1, α_2)-METRICS

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Abstract. In this article, we show that if $(M = G/H, g)$ is a naturally reductive homogeneous Riemannian manifold and $\mathfrak{g}' \subset \mathfrak{h} + \mathfrak{m}_1$, then the (α_1, α_2)-metric F induced by g and the decomposition $\mathfrak{m} = \mathfrak{m}_1 + \mathfrak{m}_2$ is also naturally reductive. To establish this, we provide an explicit formula for the Cartan tensor of homogeneous (α_1, α_2)-metrics. We further examine the relationship between geodesic vector fields on homogeneous Riemannian spaces and their counterparts on homogeneous (α_1, α_2)-spaces. Finally, we construct left-invariant (α_1, α_2)-metrics on the tangent bundle of Lie groups from left-invariant Randers metrics on the base Lie group and study their geometric relations.

1. Introduction

Shen and Chern were the first to identify, while investigating product Finsler metrics [6], that there is no natural method for defining a product Finsler metric on the product manifold formed by a pair of Finsler manifolds. However, when the Finsler metrics are Riemannian, it becomes possible to construct various product Finsler metrics. In their book, the authors introduced a norm known as the f-product, which essentially marks the emergence of Finsler metrics referred to as (α_1, α_2)-metrics, although this term itself was not explicitly stated. Notably, the f-product is a specific instance of the Berwald-type (α_1, α_2)-metric.

In 2016, Deng and Xu first introduced the concept of Finsler metrics of the (α_1, α_2) type while researching Clifford-Wolf transformations of homogeneous Finsler spaces (see [10] and [11]). They particularly focused on Clifford-Wolf homogeneous Randers spaces and left-invariant (α, β)-metrics of Clifford-Wolf type [19, 20]. Their investigate these concepts on f-products and their generalizations.

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They stated that (α_1, α_2)-metrics, in a sense, can be viewed as a generalization of (α, β)-metrics. Additionally, they defined a notion of suitable normalized data for homogeneous (α_1, α_2)-metrics, which they used to explore geometric properties. The authors also characterized the conditional Clifford-Wolf homogeneity of left-invariant (α_1, α_2)-metrics on simply connected compact Lie groups, demonstrating that in special cases, these metrics must be Riemannian.

In 2022, Zhang and Xu introduced the concept of a standard homogeneous (α_1, α_2)-metric, generalizing normal homogeneity in Finsler geometry while maintaining relatively good computability. They explored the geodesic orbit (g.o.) property of this metric and established a criterion for a g.o. (α_1, α_2)-space [21].

In the next year, An, Yan, and Zhang gave a complete description of invariant (α_1, α_2)-metrics on spheres with zero S-curvature and investigated the geodesic orbit property and S-curvature of these invariant metrics on spheres and found that there are many Finsler spheres with zero S-curvature that are not geodesic orbit spaces [2]. Furthermore, in 2023, Tan and Xu explored the concept of Naturally Reductive (α_1, α_2)-metrics from both geometric and algebraic perspectives [18]. From a geometric viewpoint, they noted that every non-Riemannian naturally reductive homogeneous (α_1, α_2)-metric is locally isometric to an f-product formed by two naturally reductive Riemannian metrics. On the algebraic side, they demonstrated that if a non-Riemannian homogeneous (α_1, α_2)-metric is derived from a homogeneous Riemannian metric α, then the natural reductivity of the (α_1, α_2)-metric is inherited from α. At the end, the authors presented an explicit formula for the flag curvature of a naturally reductive (α_1, α_2)-metric of the form $F = \alpha\phi\left(\frac{\alpha_2}{\alpha}\right)$.

As previously mentioned, the (α_1, α_2)-metrics can be regarded as a generalization of (α, β)-metrics, suggesting that results established for (α, β)-metrics could also be extended to (α_1, α_2)-metrics. For instance, Huang and Mo [13] constructed several homogeneous Einstein metrics that fall within the (α_1, α_2) category.

We begin by computing the formula for the Cartan tensor associated with an(α_1, α_2)-metric. Then, we show that if $(M = G/H, g)$ is a naturally reductive homogeneous Riemannian manifold and $\mathfrak{g}'$ is a subset of $\mathfrak{h} + \mathfrak{m}_1$, then the (α_1, α_2)-metric F defined by g and the decomposition $\mathfrak{m} = \mathfrak{m}_1 + \mathfrak{m}_2$ is also naturally reductive. Additionally, we investigate the relationship between geodesic vector fields on homogeneous Riemannian spaces and those on homogeneous (α_1, α_2)-spaces. Finally, using left-invariant Randers metrics on the base Lie group, we construct left-invariant (α_1, α_2)-metrics on the tangent bundle of Lie groups and study some geometric relations between these two classes of metrics.

2. Preliminaries

This section provides basic definitions, notations, and results related to Finsler geometry, particularly concerning (α_1, α_2)-metrics and also the lifting of a Riemannian metric on the tangent bundle of a manifold.

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Let (M, F) be a Finsler manifold. We recall that the fundamental tensor and the Cartan tensor are symmetric sections of the vector bundles $(\pi^*T^*M)^{\otimes 2}$ and $(\pi^*T^*M)^{\otimes 3}$, respectively, defined as follows:

$$g_y(u,v)=\frac{1}{2}\frac{\partial^2}{\partial s\partial t}\left[F^2\left(y+su+tv\right)\right]_{s=t=0}=g_{ij}(y)u^iv^j,$$

$$C_y(u,v,w)=\frac{1}{2}\frac{d}{dt}[g_{y+tw}]_{t=0}=\frac{1}{4}\frac{\partial^3}{\partial r\partial s\partial t}\left[F^2(y+ru+sv+tw\right]_{r=s=t=0}=c_{ijk}(y)u^iv^jw^k,$$

where $y\neq 0$, and $g_{ij}=\frac{1}{2}\frac{\partial^2 F}{\partial y^i\partial y^j}$ and $c_{ijk}=\frac{1}{2}\frac{\partial g_{ij}}{\partial y^k}$ denote their components in local coordinates.

The spray coefficients G_i of the spray vector field $G=y^i\frac{\partial}{\partial x^i}-2G^i\frac{\partial}{\partial y^i}$ on $TM\setminus\{0\}$, in a standard local coordinate system (x^i, y^i) of TM, are defined by $G^i(x,y)=\frac{1}{4}g^{il}[\left(F^2\right)_{x^ky^l}y^k-\left(F^2\right)_{x^l}]$. If there exists a function $P(x,y)$ which is positively y-homogeneous of degree one, such that $G^i=\frac{1}{2}\Gamma^i_{jk}(x)y^jy^k+P(x,y)y^i$, then the Finsler metric F is called of Douglas type. In particular, if $P(x,y)=0$, then the Finsler metric F is called a Berwald metric.

Randers metrics are a special class of Finsler metrics defined by the following expression:

$$F_x(y)=F(x,y)=\alpha(x,y)+\beta(x,y)=\alpha(x,y)+g(X_x,y),$$

where $\alpha(x,y)=\sqrt{g_x(y,y)}$ represents a Riemannian norm, β is a one-form, X is the vector field corresponding to the one-form β, and $\|\beta\|_\alpha=\|X\|_\alpha<1$ (see [5, 16]).

It can be shown that for a Randers metric F, the condition that F is of Berwald type is equivalent to the condition that X is a parallel vector field with respect to the Riemannian connection g (see [6]).

We will now introduce the preliminaries for (α_1,α_2)-metrics, the main metrics explored in this article.

Definition 2.1 ([10])**.** *A Finsler metric F on a manifold M is called an (α_1,α_2)-metric, if there exists a Riemannian metric α on M and an α-orthogonal bundle decomposition $TM=\mathcal{V}_1\bigoplus\mathcal{V}_2$ with respect to (possibly non-integrable) distributions $\mathcal{V}_i$ $(i=1,2)$ such that for all $x\in M$ and $y=y_1+y_2\in T_xM$, where $y_i\in\mathcal{V}_i$, F can be represented as:*

$$F_x(y)=f(\alpha_1(x,y_1),\alpha_2(x,y_2)),$$

in which each α_i is defined on a distribution $\mathcal{V}_i$ and $\alpha_i(y)=\alpha(y_i)=\alpha|_{\mathcal{V}_i}$ and also $f:\mathbb{R}^2\to\mathbb{R}$ is a positive smooth function. The pair (n_1,n_2) such that $dimTM=n$, $dimV_i=n_i$ $(i=1,2)$, and $n=n_1+n_2$ is called the dimension decomposition for the (α_1,α_2)-metric.

Remark 2.2. *An (α_1,α_2)-metric F can be written as $F=\alpha\phi(\frac{\alpha_2}{\alpha})$ (or $F=\alpha\psi(\frac{\alpha_1}{\alpha})$) , where $\phi(s)$ $(\psi(s))$ is a positive smooth function on $(0,1)$ and $\phi(s)=\psi\left(\sqrt{1-s^2}\right)$. Sometimes it is necessary to write an (α_1,α_2)-metric as $F=\sqrt{L(\alpha_1^2,\alpha_2^2)}$, where $L\colon\mathbb{R}^+\cup\{0\}\times\mathbb{R}^+\cup\{0\}\to\mathbb{R}^+\cup\{0\}$ is a positive smooth real function homogeneous of degree 1. With a little calculation, we have $L(a,b)=(a+b)\,\phi^2\left(\sqrt{\frac{b}{a+b}}\right)=(a+b)\,\psi^2\left(\sqrt{\frac{a}{a+b}}\right)$ where $a,b\in\mathbb{R}^+\times\mathbb{R}^+$.*

Remark 2.3. (i) *For the cases $n_2=0$ and $n_2=1$, the (α_1,α_2) metric F is either an Euclidean metric or an (α,β)-metric, respectively.*

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(ii) *An* (α_1, α_2)*-metric* F *is Riemannian, if and only if for positive constants* $k, l > 0$*,* F *has the form* $f(u_1, u_2) = \sqrt{ku_1^2 + lu_2^2}$*.*

Theorem 2.4 ([10])**.** *Assume* ϕ *and* ψ *are two positive functions on the interval* $[0,1]$ *such that* $\phi(s) = \psi\left(\sqrt{1-s^2}\right)$*. Also, let* $F = \alpha\phi(\frac{\alpha_2}{\alpha}) = \alpha\psi(\frac{\alpha_1}{\alpha})$ *be a* (α_1, α_2)*-norm on* $\mathbb{R}^n$ *with dimension decomposition* (n_1, n_2) *such that* $n_1 \geq n_2 \geq 1$*. Then* ϕ *and* ψ *are both positive smooth functions on* $[0,1]$*, and for any* s *and* b *with the condition* $0 \leq s \leq b \leq 1$*, we have*

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0,$$
$$\psi(s) - s\psi'(s) + (b^2 - s^2)\psi''(s) > 0.$$

Conversely, if $\phi(s) - s\phi'(s) + (1 - s^2)\phi''(s) > 0$*, then* $F(\alpha) = \alpha\phi(\frac{\alpha_1}{\alpha})$ *defines a* (α_1, α_2)*-norm with dimension decomposition* (n_1, n_2) *where* $n_1 \geq n_2 \geq 1$*.*

There are two definitions for a naturally reductive Finsler space as follows.

Definition 2.5 ([4, 15])**.** *A homogeneous manifold* $M = G/H$ *with an invariant Finsler metric* F *is called naturally reductive, if there exists an* $\mathrm{Ad}(H)$*-invariant decomposition* $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ *such that*

$$g_y\left(u, [z, x]_{\mathfrak{m}}\right) + g_y\left([z, u]_{\mathfrak{m}}, x\right) + 2C_y\left([z, y], u, x\right) = 0,$$

where $y \neq 0$*,* u*,* $u, x, z \in \mathfrak{m}$*, and* C_y *is the Cartan tensor at* y*.*

Definition 2.6 ([8, 9])**.** *A homogeneous manifold* M *equipped with an invariant Finsler metric* F *is called naturally reductive, if there exists an invariant Riemannian metric* g *on* M *such that the space* $(M = G/H, g)$ *is naturally reductive and the Riemannian connection and the Chern connection coincide.*

Zhang, Yan, and Deng proved in [22] that these two definitions are equivalent. Also, according to [9] and [12], another equivalent definition of a naturally reductive homogeneous Finsler space is presented in terms of the corresponding spray vector field [18].

Xu and Tan derived in their paper [18] the fundamental tensor and flag curvature for a naturally reductive metric of the form $F = \alpha\phi\left(\frac{\alpha_2}{\alpha}\right) = |y|\phi\left(\frac{|y_2|}{|y|}\right)$ $\left(F = \alpha\psi\left(\frac{\alpha_1}{\alpha}\right) = |y|\psi\left(\frac{|y_1|}{|y|}\right)\right)$, where $|\cdot| = \langle\cdot,\cdot\rangle^{\frac{1}{2}}$ is the $\mathrm{Ad}(H)$-invariant Euclidean norm on $\mathfrak{m}$ and $y = y_1 + y_2 \in \mathfrak{m}$ with $y_i \in \mathfrak{m}_i$, for an $\mathrm{Ad}(H)$-invariant $\langle\cdot,\cdot\rangle$-orthogonal decomposition $\mathfrak{m} = \mathfrak{m}_1 + \mathfrak{m}_2$, on the manifold $M = G/H$ with respect to the reductive decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$.

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$$
\begin{aligned}
g_y(u,v) &= \frac{1}{2}\frac{\partial^2}{\partial s\partial t}F^2\left(y+su+tv\right)\Big|_{s=t=0}\\
&= \langle u,v\rangle\phi^2(b)\\
&\quad +\phi(b)\phi'(b)\left(\frac{\langle y,v\rangle\langle y_2,u_2\rangle}{|y||y_2|}+\frac{\langle y_2,v_2\rangle\langle y,u\rangle}{|y||y_2|}-\frac{\langle y,u\rangle\langle y,v\rangle|y_2|}{|y|^3}+\frac{\langle u_2,v_2\rangle|y|}{|y_2|}\right.\\
&\quad \left.-\frac{\langle u,v\rangle|y_2|}{|y|}-\frac{\langle u_2,y_2\rangle\langle v_2,y_2\rangle|y|}{|y_2|^3}\right)\\
&\quad +\left(\phi'^2(b)+\phi(b)\phi''(b)\right)\left(\frac{\langle u_2,y_2\rangle}{|y||y_2|}-\frac{\langle u,y\rangle|y_2|}{|y|^3}\right)\left(\frac{\langle v_2,y_2\rangle|y|}{|y_2|}-\frac{\langle v,y\rangle|y_2|}{|y|}\right)\\
&= \langle u,v\rangle\psi^2(b_1)\\
&\quad +\psi(b_1)\psi'(b_1)\left(\frac{\langle y,v\rangle\langle y_1,u_1\rangle}{|y||y_1|}+\frac{\langle y_1,v_1\rangle\langle y,u\rangle}{|y||y_1|}-\frac{\langle y,u\rangle\langle y,v\rangle|y_1|}{|y|^3}+\frac{\langle u_1,v_1\rangle|y|}{|y_1|}\right.\\
&\quad \left.-\frac{\langle u,v\rangle|y_1|}{|y|}-\frac{\langle u_1,y_1\rangle\langle v_1,y_1\rangle|y|}{|y_1|^3}\right)\\
&\quad +\left(\psi'^2(b_1)+\psi(b_1)\psi''(b_1)\right)\left(\frac{\langle u_1,y_1\rangle}{|y||y_1|}-\frac{\langle u,y\rangle|y_1|}{|y|^3}\right)\left(\frac{\langle v_1,y_1\rangle|y|}{|y_1|}-\frac{\langle v,y\rangle|y_1|}{|y|}\right),
\end{aligned}
$$

where $b=\frac{|y_2|}{|y|}$ and $b_1=\frac{|y_1|}{|y|}$.

Also, with similar calculations, the formula for the fundamental tensor of a (α_1,α_2)-metric of the form $F=\sqrt{L\left(\alpha_1^2,\alpha_2^2\right)}$ is obtained as follows:

$$
\begin{aligned}
g_y(u,v) &= \frac{1}{2}\frac{\partial^2}{\partial s\partial t}F^2\left(y+su+tv\right)\big|_{s=t=0}\\
&= \frac{1}{2}\frac{\partial^2}{\partial s\partial t}\,L\left(\alpha_1^2\left(y_1+su_1+tv_1\right),\alpha_2^2\left(y_2+su_2+tv_2\right)\right)\big|_{s=t=0}\\
&= \Big[L_{11}\left(|y_1|^2,|y_2|^2\right)\langle u_1,y_1\rangle\langle v_1,y_1\rangle+L_{21}\left(|y_1|^2,|y_2|^2\right)\langle u_2,y_2\rangle\langle v_1,y_1\rangle\\
&\quad +L_{12}\left(|y_1|^2,|y_2|^2\right)\langle u_1,y_1\rangle\langle v_2,y_2\rangle+L_{22}\left(|y_1|^2,|y_2|^2\right)\langle u_2,y_2\rangle\langle v_2,y_2\rangle\Big]\\
&\quad +\Big[L_1\left(|y_1|^2,|y_2|^2\right)\langle u_1,v_1\rangle+L_2\left(|y_1|^2,|y_2|^2\right)\langle u_2,v_2\rangle\Big],
\end{aligned}
$$

where $L_1(\cdot,\cdot)$, $L_2(\cdot,\cdot)$, $L_{11}(\cdot,\cdot)$, etc., are partial derivatives with respect to the variables indicated by the lower indices and also u_i and v_i are the components of u and v in $\mathfrak{m}_i$, respectively.

Since we want to work on the tangent Lie group TG, here we mention some preliminaries about such spaces.

Theorem 2.7 ([3]). *Let X and Y be two left-invariant vector fields on a Lie group G, and let X^c and Y^c, and X^v and Y^v denote their corresponding complete and vertical lifts on the tangent bundle*

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TG, respectively. Then for their Lie bracket we have:

$$\begin{aligned}
[X^c, Y^c] &= [X, Y]^c, \\
[X^c, Y^v] &= [X, Y]^v, \\
[X^v, Y^v] &= 0.
\end{aligned}$$

Theorem 2.8 ([3]). *If $\{X_1, X_2, \ldots, X_n\}$ is a basis consisting of left-invariant vector fields for the Lie algebra of a Lie group G, then $\{X_1^c, X_1^v, X_2^c, X_2^v, \ldots, X_n^c, X_n^v\}$ is a basis consisting of left-invariant vector fields for the Lie algebra of the tangent Lie group TG.*

Definition 2.9 ([3]). *Let g be a left-invariant Riemannian metric on a Lie group G. A natural Riemannian metric $\tilde{g}$ on the tangent bundle TG is defined as follows, and is clearly left-invariant:*

$$\begin{aligned}
\widetilde{g}\left(X^c, Y^c\right) &= g\left(X, Y\right), \\
\widetilde{g}\left(X^v, Y^v\right) &= g\left(X, Y\right), \\
\widetilde{g}\left(X^c, Y^v\right) &= 0.
\end{aligned}$$

3. On the natural reductivity of (α_1, α_2)-Metrics

As we have seen in Definition 2.5, besides the fundamental tensor, the Cartan tensor is an important quantity to study the natural reductivity of a homogeneous Finsler space. So, at first, we derive the formula for the Cartan tensor of an (α_1, α_2)-metric of the form $F = \alpha\phi\left(\frac{\alpha_2}{\alpha}\right) = |y|\phi\left(\frac{|y_2|}{|y|}\right)$ on a manifold $M = G/H$ with respect to the reductive decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$, where $y \in \mathfrak{m} \setminus \{0\}$ and $u, v, w \in \mathfrak{m}$ such that $y = y_1 + y_2$ and $u = u_1 + u_2$ and $v = v_1 + v_2$ and $w = w_1 + w_2$ with $y_i, u_i, v_i, w_i \in \mathfrak{m}_i$ $(i = 1, 2)$ and $\mathfrak{m} = \mathfrak{m}_1 + \mathfrak{m}_2$ is an $\mathrm{Ad}(H)$-invariant $\langle\cdot,\cdot\rangle$-orthogonal, and $|\cdot| = \langle\cdot,\cdot\rangle^{\frac{1}{2}}$ is an $\mathrm{Ad}(H)$-invariant Euclidean norm on $\mathfrak{m}$.

$$\begin{aligned}
C_y(u, v, w) &= \frac{1}{2}\frac{d}{dt}\bigg|_{t=0} g_{y+tw}(u, v) = \frac{1}{4}\,\frac{\partial^3}{\partial r\partial s\partial t}\bigg|_{r=s=t=0} F^2\left(y + ru + sv + tw\right) \\
&= \frac{1}{4}\,\frac{\partial^3}{\partial r\partial s\partial t}\bigg|_{r=s=t=0} \langle y + ru + sv + tw, y + ru + sv + tw\rangle \\
&\quad \times\phi^2\left(\frac{\langle y_2 + ru_2 + sv_2 + tw_2, y_2 + ru_2 + sv_2 + tw_2\rangle^{\frac{1}{2}}}{\langle y + ru + sv + tw, y + ru + sv + tw\rangle^{\frac{1}{2}}}\right).
\end{aligned}$$

To simplify the calculations, we set $K = F^2\left(y + ru + sv + tw\right)$, $A = y + ru + sv + tw$, $B = y_2 + ru_2 + sv_2 + tw_2$, $M = \langle y + ru + sv + tw, y + ru + sv + tw\rangle$, and $N = \frac{\langle y_2+ru_2+sv_2+tw_2, y_2+ru_2+sv_2+tw_2\rangle^{\frac{1}{2}}}{\langle y+ru+sv+tw, y+ru+sv+tw\rangle^{\frac{1}{2}}}$.

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$$
\begin{aligned}
\frac{1}{2}\frac{\partial}{\partial s}\frac{\partial K}{\partial t} &= \frac{1}{2}\frac{\partial}{\partial s}\left(\frac{\partial M}{\partial t}\phi^2(N) + 2M\phi(N)\phi'(N)\frac{\partial N}{\partial t}\right)\\
&= \frac{1}{2}\frac{\partial}{\partial s}\left(2\left\langle\frac{\partial A}{\partial t}, A\right\rangle\phi^2(N) + 2\langle A, A\rangle\phi(N)\phi'(N)\frac{\partial N}{\partial t}\right)\\
&= \frac{\partial}{\partial s}\left(\left\langle\frac{\partial A}{\partial t}, A\right\rangle\right)\phi^2(N) + 2\left\langle\frac{\partial A}{\partial t}, A\right\rangle\phi(N)\phi'(N)\frac{\partial N}{\partial s}\\
&\quad + \frac{\partial}{\partial s}\left[\phi(N)\phi'(N)\left(\frac{\langle w_2, B\rangle\langle A, A\rangle^{\frac{1}{2}}}{\langle B, B\rangle^{\frac{1}{2}}} - \frac{\langle w, A\rangle\langle B, B\rangle^{\frac{1}{2}}}{\langle A, A\rangle^{\frac{1}{2}}}\right)\right]\\
&= \langle v, w\rangle\phi^2(N) + 2\langle w, A\rangle\phi(N)\phi'(N)(E-F) + \frac{\partial}{\partial s}(\phi(N)\phi'(N)(C-D))\\
&= \Bigg(\langle v, w\rangle\phi^2(N) + 2\langle w, A\rangle\phi(N)\phi'(N)(E-F) + {\phi'}^2(N)\frac{\partial N}{\partial s}(C-D)\\
&\qquad + \phi(N)\phi''(N)\frac{\partial N}{\partial s}(C-D) + \phi(N)\phi'(N)(\frac{\partial C}{\partial s} - \frac{\partial D}{\partial D}\Big)\Bigg),
\end{aligned}
$$

where $C = \frac{\langle w_2,B\rangle\langle A,A\rangle^{\frac{1}{2}}}{\langle B,B\rangle^{\frac{1}{2}}}$, $D = \frac{\langle w,A\rangle\langle B,B\rangle^{\frac{1}{2}}}{\langle A,A\rangle^{\frac{1}{2}}}$, $E = \frac{\langle v_2,B\rangle}{\langle A,A\rangle^{\frac{1}{2}}\langle B,B\rangle^{\frac{1}{2}}}$, and $F = \frac{\langle v,A\rangle\langle B,B\rangle^{\frac{1}{2}}}{\langle A,A\rangle^{\frac{3}{2}}}$.
So we have:

$$
\begin{aligned}
\frac{1}{2}\frac{\partial}{\partial r}\left(\frac{\partial^2 K}{\partial s\partial t}\right) &= \frac{1}{2}\frac{\partial}{\partial r}\Bigg[\langle v, w\rangle\phi^2(N) + 2\left\langle w, A\right\rangle\phi(N)\phi'(N)(E-F) + {\phi^{'}}^{2}(N)(C-D)(E-F)\\
&\qquad + \phi(N)\phi''(N)(C-D)(E-F) + \phi(N)\phi'(N)\left(\frac{\partial C}{\partial s} - \frac{\partial D}{\partial s}\right)\Bigg]\\
&= \langle v, w\rangle\phi(N)\phi'(N)\frac{\partial N}{\partial r} + \left\langle u, w\right\rangle\phi(N)\phi'(N)(E-F) + \left\langle w, A\right\rangle{\phi'}^2(N)(E-F)\\
&\quad + \left\langle w, A\right\rangle\phi(N)\phi''(N)\frac{\partial N}{\partial r}(E-F) + \left\langle w, A\right\rangle\phi(N)\phi'(N)\left(\frac{\partial E}{\partial r} - \frac{\partial F}{\partial r}\right)\\
&\quad + \phi'(N)\phi''(N)\frac{\partial N}{\partial r}(C-D)(E-F) + \frac{1}{2}{\phi'}^2(N)\left(\frac{\partial C}{\partial r} - \frac{\partial D}{\partial r}\right)(E-F)\\
&\quad + \frac{1}{2}{\phi'}^2(N)(C-D)\left(\frac{\partial E}{\partial r} - \frac{\partial F}{\partial r}\right) + \frac{1}{2}\phi'(N)\phi''(N)\frac{\partial N}{\partial r}(C-D)(E-F)\\
&\quad + \frac{1}{2}\phi(N)\phi^{'''}(N)\frac{\partial N}{\partial r}(C-D)(E-F) + \frac{1}{2}\phi(N)\phi''(N)\left(\frac{\partial C}{\partial r} - \frac{\partial D}{\partial r}\right)(E-F)\\
&\quad + \frac{1}{2}\phi(N)\phi''(N)(C-D)\left(\frac{\partial E}{\partial r} - \frac{\partial F}{\partial r}\right) + \frac{1}{2}{\phi'}^2(N)\frac{\partial N}{\partial r}\left(\frac{\partial C}{\partial s} - \frac{\partial D}{\partial s}\right)\\
&\quad + \frac{1}{2}\phi(N)\phi''(N)\frac{\partial N}{\partial r}\left(\frac{\partial C}{\partial s} - \frac{\partial D}{\partial s}\right) + \frac{1}{2}\phi(N)\phi'(N)(\frac{\partial^2 C}{\partial r\partial s} - \frac{\partial^2 D}{\partial r\partial s})
\end{aligned}
$$

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$$\begin{aligned}
&= \phi(N)\phi'(N)\Bigg[\langle v,w\rangle\frac{\partial N}{\partial r}+\langle u,w\rangle(E-F)+\langle w,A\rangle\left(\frac{\partial E}{\partial r}-\frac{\partial F}{\partial r}\right)\\
&\quad+\frac{1}{2}\left(\frac{\partial^2 C}{\partial r\partial s}-\frac{\partial^2 D}{\partial r\partial s}\right)\Bigg]\\
&\quad+(\phi^{'2}(N)+\phi(N)\phi''(N))\Bigg[\langle w,A\rangle\frac{\partial N}{\partial r}(E-F)+\frac{1}{2}\left(\frac{\partial C}{\partial r}-\frac{\partial D}{\partial r}\right)(E-F)\\
&\quad+\frac{1}{2}(C-D)\left(\frac{\partial E}{\partial r}-\frac{\partial F}{\partial r}\right)+\frac{1}{2}\frac{\partial N}{\partial r}\left(\frac{\partial C}{\partial s}-\frac{\partial D}{\partial s}\right)\Bigg]\\
&\quad+\frac{1}{2}\left(3\phi'(N)\phi''(N)+\phi(N)\phi^{'''}(N)\right)\left[\frac{\partial N}{\partial r}(C-D)(E-F)\right].
\end{aligned}$$

Hence for the Cartan tensor we have:

$$\begin{aligned}
C_y(u,v,w) &= \frac{1}{4}\frac{\partial^3}{\partial r\partial s\partial t}\bigg|_{r=s=t=0}F^2(y+ru+sv+tw)\\
&= \phi(b)\phi'(b)\Bigg[\langle v,w\rangle\left(\frac{\langle u_2,y_2\rangle}{|y_2||y|}-\frac{\langle u,y\rangle|y_2|}{|y|^3}\right)+\langle u,w\rangle\left(\frac{\langle v_2,y_2\rangle}{|y_2||y|}-\frac{\langle v,y\rangle|y_2|}{|y|^3}\right)\\
&\quad+\langle w,y\rangle\Bigg(\frac{\langle u_2,v_2\rangle}{|y_2||y|}-\frac{\langle v_2,y_2\rangle\langle u,y\rangle}{|y_2||y|^3}-\frac{\langle v_2,y_2\rangle\langle u_2,y_2\rangle}{|y||y_2|^3}-\frac{\langle u,v\rangle|y_2|}{|y|^3}-\frac{\langle v,y\rangle\langle u_2,y_2\rangle}{|y_2||y|^3}\\
&\quad+\frac{3\langle u,y\rangle\langle v,y\rangle|y_2|}{|y|^5}\Bigg)\\
&\quad+\frac{1}{2}\Bigg(\frac{2\langle w_2,v_2\rangle\langle u,y\rangle+\langle u,v\rangle\langle w_2,y_2\rangle+\langle v,y\rangle\langle u_2,w_2\rangle}{|y_2||y|}\\
&\quad-\frac{\left(\langle u,y\rangle|y_2|^2+\langle u_2,y_2\rangle|y|^2\right)\left(\langle w_2,v_2\rangle|y|^2+\langle v,y\rangle\langle w_2,y_2\rangle\right)}{|y_2|^3|y|^3}\\
&\quad-\frac{2\langle w,v\rangle\langle u_2,y_2\rangle+\langle u_2,v_2\rangle\langle w,y\rangle+\langle v_2,y_2\rangle\langle u,w\rangle}{|y_2||y|}\\
&\quad+\frac{\left(\langle u,y\rangle|y_2|^2+\langle u_2,y_2\rangle|y|^2\right)\left(\langle w,v\rangle|y_2|^2+\langle v_2,y_2\rangle\langle w,y\rangle\right)}{|y_2|^3|y|^3}\\
&\quad-\frac{(\langle u_2,v_2\rangle\langle w_2,y_2\rangle+\langle v_2,y_2\rangle\langle u_2,w_2\rangle)|y|}{|y_2|^3}-\frac{\langle v_2,y_2\rangle\langle w_2,y_2\rangle\langle u,y\rangle}{|y||y_2|^3}\\
&\quad+\frac{3|y|\langle u_2,y_2\rangle\langle v_2,y_2\rangle\langle w_2,y_2\rangle}{|y_2|^5}\\
&\quad+\frac{(\langle u,v\rangle\langle w,y\rangle+\langle v,y\rangle\langle u,w\rangle)|y_2|}{|y|^3}+\frac{\langle v,y\rangle\langle w,y\rangle\langle u_2,y_2\rangle}{|y_2||y|^3}-\frac{3|y_2|\langle u,y\rangle\langle v,y\rangle\langle w,y\rangle}{|y|^5}\Bigg)\Bigg]\\
&\quad+(\phi'^2(b)+\phi(b)\phi''(b))\Bigg[\langle w,y\rangle\left(\frac{\langle u_2,y_2\rangle}{|y_2||y|}-\frac{\langle u,y\rangle|y_2|}{|y|^3}\right)\left(\frac{\langle v_2,y_2\rangle}{|y_2||y|}-\frac{\langle v,y\rangle|y_2|}{|y|^3}\right)
\end{aligned}$$

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$$
\begin{aligned}
&+\frac{1}{2}\Bigg(\frac{\langle u_2,w_2\rangle\,|y|}{|y_2|}+\frac{\langle u,y\rangle\,\langle w_2,y_2\rangle}{|y_2||y|}-\frac{\langle u_2,y_2\rangle\,\langle w_2,y_2\rangle\,|y|}{|y_2|^3}-\frac{\langle u,w\rangle\,|y_2|}{|y|}-\frac{\langle u_2,y_2\rangle\,\langle w,y\rangle}{|y_2||y|}\\
&+\frac{\langle u,y\rangle\,\langle w,y\rangle\,|y_2|}{|y|^3}\Bigg)\left(\frac{\langle v_2,y_2\rangle}{|y_2||y|}-\frac{\langle v,y\rangle|y_2|}{|y|^3}\right)\\
&+\frac{1}{2}\left(\frac{\langle w_2,y_2\rangle\,|y|}{|y_2|}-\frac{\langle w,y\rangle\,|y_2|}{|y|}\right)\Bigg(\frac{\langle u_2,v_2\rangle}{|y_2||y|}-\frac{\langle v_2,y_2\rangle\,\langle u,y\rangle}{|y_2||y|^3}-\frac{\langle v_2,y_2\rangle\,\langle u_2,y_2\rangle}{|y||y_2|^3}\\
&-\frac{\langle u,v\rangle\,|y_2|}{|y|^3}-\frac{\langle v,y\rangle\,\langle u_2,y_2\rangle}{|y_2||y|^3}+\frac{3\,\langle u,y\rangle\,\langle v,y\rangle|y_2|}{|y|^5}\Bigg)\\
&+\frac{1}{2}\left(\frac{\langle u_2,y_2\rangle}{|y_2||y|}-\frac{\langle u,y\rangle\,|y_2|}{|y|^3}\right)\Bigg(\frac{\langle w_2,v_2\rangle\,|y|}{|y_2|}+\frac{\langle v,y\rangle\,\langle w_2,y_2\rangle}{|y_2||y|}-\frac{\langle v_2,y_2\rangle\,\langle w_2,y_2\rangle\,|y|}{|y_2|^3}\\
&-\frac{\langle w,v\rangle\,|y_2|}{|y|}-\frac{\langle v_2,y_2\rangle\,\langle w,y\rangle}{|y_2||y|}+\frac{\langle v,y\rangle\,\langle w,y\rangle\,|y_2|}{|y|^3}\Bigg)\Bigg]\\
&+\frac{1}{2}\left(3\phi'(b)\phi''(b)+\phi(b)\phi'''(b)\right)\Bigg[\left(\frac{\langle u_2,y_2\rangle}{|y||y_2|}-\frac{\langle u,y\rangle\,|y_2|}{|y|^3}\right)\left(\frac{\langle w_2,y_2\rangle\,|y|}{|y_2|}-\frac{\langle w,y\rangle\,|y_2|}{|y|}\right)\\
&\left(\frac{\langle v_2,y_2\rangle|y|}{|y_2|}-\frac{\langle v,y\rangle|y_2|}{|y|}\right)\Bigg],
\end{aligned}
$$

in which $b=\frac{|y_2|}{|y|}$.

We note that in [1], it has been shown under certain conditions that the tangent Lie groups are naturally reductive. Also, in [18], it has been proven that if a (non-Riemannian) homogeneous (α_1,α_2)-metric F on $M=G/H$ is naturally reductive with respect to the reductive decomposition $\mathfrak{g}=\mathfrak{m}+\mathfrak{h}$, then the Riemannian metric $\alpha=\sqrt{\alpha_1^2+\alpha_2^2}$ is also naturally reductive with respect to the same decomposition. The following theorem shows that, under certain conditions, if $(M=G/H,g)$ is a naturally reductive homogeneous Riemannian manifold, then the (α_1,α_2)-metric F is also naturally reductive.

Theorem 3.1. *Suppose $(M=G/H,F)$ is a homogeneous Finsler space where F is an invariant (α_1,α_2)-metric defined by the invariant Riemannian metric $\alpha=\sqrt{g(\cdot,\cdot)}$. If (M,g) is naturally reductive and $\mathfrak{g}'=[\mathfrak{g},\mathfrak{g}]\subseteq\mathfrak{h}+\mathfrak{m}_1$, then (M,F) is also naturally reductive.*

Proof. Assume (M,g) is naturally reductive. We show that:

$$g_y\left([z,u]_{\mathfrak{m}},v\right)+g_y\left(u,[z,v]_{\mathfrak{m}}\right)+2C_y\left([z,y]_{\mathfrak{m}},u,v\right)=0,\qquad y\neq 0,z,u,v\in\mathfrak{m}$$

First, by calculating each term of the left side expression above, we observe that

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$$
\begin{aligned}
g_y\left([z,u]_{\mathfrak{m}},v\right) &= \left\langle [z,u]_{\mathfrak{m}},v\right\rangle \phi^2(b)+\phi(b)\phi'(b)\Bigg(\frac{\langle y,v\rangle \left\langle y_2,[z,u]_{\mathfrak{m}_2}\right\rangle}{|y||y_2|}+\frac{\langle y_2,v_2\rangle \left\langle y,[z,u]_{\mathfrak{m}}\right\rangle}{|y||y_2|}\\
&\quad -\frac{\left\langle y,[z,u]_{\mathfrak{m}}\right\rangle \langle y,v\rangle |y_2|}{|y|^3}+\frac{\left\langle [z,u]_{\mathfrak{m}_2},v_2\right\rangle |y|}{|y_2|}-\frac{\left\langle [z,u]_{\mathfrak{m}},v\right\rangle |y_2|}{|y|}\\
&\quad -\frac{\left\langle [z,u]_{\mathfrak{m}_2},y_2\right\rangle \langle v_2,y_2\rangle |y|}{|y_2|^3}\Bigg)\\
&\quad +\left(\phi'^2(b)+\phi(b)\phi''(b)\right)\left(\frac{\left\langle [z,u]_{\mathfrak{m}_2},y_2\right\rangle}{|y||y_2|}-\frac{\left\langle [z,u]_{\mathfrak{m}},y\right\rangle |y_2|}{|y|^3}\right)\\
&\quad \left(\frac{\langle v_2,y_2\rangle |y|}{|y_2|}-\frac{\langle v,y\rangle |y_2|}{|y|}\right),
\end{aligned}
$$

$$
\begin{aligned}
g_y\left([z,v]_{\mathfrak{m}},u\right) &= -\left\langle [z,u]_{\mathfrak{m}},v\right\rangle \phi^2(b)+\phi(b)\phi'(b)\Bigg(\frac{\langle y,u\rangle \left\langle y,[z,v]_{\mathfrak{m}2}\right\rangle}{|y||y_2|}+\frac{\langle y_2,u_2\rangle \left\langle y,[z,v]_{\mathfrak{m}}\right\rangle}{|y||y_2|}\\
&\quad -\frac{\left\langle y,[z,v]_{\mathfrak{m}}\right\rangle \langle y,u\rangle |y_2|}{|y|^3}+\frac{\left\langle [z,v]_{\mathfrak{m}_2},u_2\right\rangle |y|}{|y_2|}-\frac{\left\langle [z,v]_{\mathfrak{m}},u\right\rangle |y_2|}{|y|}\\
&\quad -\frac{\left\langle [z,v]_{\mathfrak{m}_2},y_2\right\rangle \langle u_2,y_2\rangle |y|}{|y_2|^3}\Bigg)\\
&\quad +\left(\phi'^2(b)+\phi(b)\phi''(b)\right)\left(\frac{\left\langle [z,v]_{\mathfrak{m}_2},y_2\right\rangle}{|y||y_2|}-\frac{\left\langle [z,v]_{\mathfrak{m}},y\right\rangle |y_2|}{|y|^3}\right)\left(\frac{\langle u_2,y_2\rangle |y|}{|y_2|}-\frac{\langle u,y\rangle |y_2|}{|y|}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
2C_y\left([z,y]_{\mathfrak{m}},u,v\right) &= 2\phi(b)\phi'(b)\Bigg[\left\langle [z,y]_{\mathfrak{m}},v\right\rangle \left(\frac{\langle u_2,y_2\rangle}{|y||y_2|}-\frac{\langle u,y\rangle |y_2|}{|y|^3}\right)\\
&\quad +\langle v,y\rangle \left(\frac{\left\langle [z,y]_{\mathfrak{m}_2},u_2\right\rangle}{|y||y_2|}-\frac{\left\langle [z,y]_{\mathfrak{m}},u\right\rangle |y_2|}{|y|^3}\right)\\
&\quad +\frac{1}{2}\Bigg(\frac{\left\langle [z,y]_{\mathfrak{m}},u\right\rangle \langle v_2,y_2\rangle}{|y||y_2|}+\frac{\left\langle [z,y]_{\mathfrak{m}_2},v_2\right\rangle \langle u,y\rangle}{|y||y_2|}-\frac{\left\langle [z,y]_{\mathfrak{m}_2},u_2\right\rangle \langle v,y\rangle}{|y||y_2|}\\
&\quad -\frac{\left\langle [z,y]_{\mathfrak{m}},v\right\rangle}{|y||y_2|}-\frac{\left\langle [z,y]_{\mathfrak{m}_2},u_2\right\rangle \langle v_2,y_2\rangle |y|}{|y_2|^3}-\frac{\left\langle [z,y]_{\mathfrak{m}_2},v_2\right\rangle \langle u_2,y_2\rangle |y|}{|y_2|^3}\\
&\quad +\frac{\left\langle [z,y]_{\mathfrak{m}_2},u\right\rangle \langle v,y\rangle |y_2|}{|y|^3}+\frac{\left\langle [z,y]_{\mathfrak{m}},v\right\rangle \langle u,y\rangle |y_2|}{|y|^3}\Bigg)\Bigg]+2\left(\phi'^2(b)+\phi(b)\phi''(b)\right)\\
&\quad \Bigg[\frac{1}{2}\left(\frac{\left\langle [z,y]_{\mathfrak{m}_2},v_2\right\rangle |y|}{|y_2|}-\frac{\left\langle [z,y]_{\mathfrak{m}},v\right\rangle |y_2|}{|y|}\right)\left(\frac{\langle u_2,y_2\rangle}{|y||y_2|}-\frac{\langle u,y\rangle |y_2|}{|y|^3}\right)\\
&\quad +\frac{1}{2}\left(\frac{\left\langle [z,y]_{\mathfrak{m}_2},u_2\right\rangle}{|y||y_2|}-\frac{\left\langle [z,y]_{\mathfrak{m}},u\right\rangle |y_2|}{|y|^3}\right)\left(\frac{\langle v_2,y_2\rangle |y|}{|y_2|}-\frac{\langle v,y\rangle |y_2|}{|y|}\right)\Bigg].
\end{aligned}
$$

We note by assumption of the natural reducibility that $\langle [z,v]_{\mathfrak{m}},u\rangle = -\langle [z,u]_{\mathfrak{m}},v\rangle$ and $\langle [z,y]_{\mathfrak{m}},y\rangle = 0$. On the other hand, from the condition $\mathfrak{g}' \subseteq \mathfrak{h}+\mathfrak{m}_1$, one can deduce that $\langle [z,y]_{\mathfrak{m}_2},y_2\rangle = 0$. Therefore,

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with some lengthy calculations and simplification, we have:

$$\begin{aligned}
g_y & \left([z,u]_{\mathfrak{m}},v\right)+g_y\left(u,[z,v]_{\mathfrak{m}}\right)+2C_y\left([z,y]_{\mathfrak{m}},u,v\right)=\\
&\phi(b)\phi'(b)\left[\frac{-\left\langle [z,y]_{\mathfrak{m}_2},u_2\right\rangle\langle v,y\rangle}{|y||y_2|}-\frac{\langle [z,y]_{\mathfrak{m}},u\rangle\langle v_2,y_2\rangle}{|y||y_2|}+\frac{\langle [z,y]_{\mathfrak{m}},u\rangle\langle y,v\rangle|y_2|}{|y|^3}\right.\\
&\left.+\frac{\left\langle [z,u]_{\mathfrak{m}_2},v_2\right\rangle|y|}{|y_2|}-\frac{\langle [z,u]_{\mathfrak{m}},v\rangle|y_2|}{|y|}+\frac{\left\langle [z,y]_{\mathfrak{m}_2},u_2\right\rangle\langle v_2,y_2\rangle|y|}{|y_2|^3}\right]\\
&+\phi(b)\phi'(b)\left[\frac{-\left\langle [z,y]_{\mathfrak{m}_2},v_2\right\rangle\langle u,y\rangle}{|y||y_2|}-\frac{\langle [z,y]_{\mathfrak{m}},v\rangle\langle u_2,y_2\rangle}{|y||y_2|}+\frac{\langle [z,y]_{\mathfrak{m}},v\rangle\langle u,v\rangle|y_2|}{|y|^3}\right.\\
&\left.-\frac{\left\langle [z,v]_{\mathfrak{m}_2},u_2\right\rangle|y|}{|y_2|}+\frac{\left\langle [z,y]_{\mathfrak{m}_2},v_2\right\rangle\langle u_2,y_2\rangle|y|}{|y_2|^3}\right]\\
&+\phi(b)\phi'(b)\left[\frac{2\langle [z,y]_{\mathfrak{m}},v\rangle\langle u_2,y_2\rangle}{|y||y_2|}-\frac{2\langle [z,y]_{\mathfrak{m}},v\rangle\langle u,y\rangle}{|y|^3}+\frac{2\left\langle [z,y]_{\mathfrak{m}_2},u_2\right\rangle\langle v,y\rangle}{|y||y_2|}\right.\\
&-\frac{2\langle [z,y]_{\mathfrak{m}},u\rangle\langle v,y\rangle|y_2|}{|y|^3}+\frac{\langle [z,y]_{\mathfrak{m}},u\rangle\langle v_2,y_2\rangle}{|y||y_2|}+\frac{\left\langle [z,y]_{\mathfrak{m}_2},v_2\right\rangle\langle u,y\rangle}{|y||y_2|}\\
&-\frac{\left\langle [z,y]_{\mathfrak{m}_2},u_2\right\rangle\langle v,y\rangle}{|y||y_2|}-\frac{\langle [z,y]_{\mathfrak{m}},v\rangle\langle u_2,y_2\rangle}{|y||y_2|}-\frac{\left\langle [z,y]_{\mathfrak{m}_2},u_2\right\rangle\langle v_2,y_2\rangle|y|}{|y_2|^3}\\
&\left.-\frac{\left\langle [z,y]_{\mathfrak{m}_2},v_2\right\rangle\langle u_2,y_2\rangle|y|}{|y_2|^3}+\frac{\langle [z,y]_{\mathfrak{m}},u\rangle\langle v,y\rangle|y_2|}{|y|^3}+\frac{\langle [z,y]_{\mathfrak{m}},v\rangle\langle u,y\rangle|y_2|}{|y|^3}\right]\\
&+\left(\phi'^2(b)+\phi(b)\phi''(b)\right)\left(-\frac{\left\langle [z,y]_{\mathfrak{m}_2},u_2\right\rangle}{|y||y_2|}+\frac{\langle [z,y]_{\mathfrak{m}},u\rangle|y_2|}{|y|^3}\right)\left(\frac{\langle v_2,y_2\rangle}{|y_2|}-\frac{\langle v,y\rangle|y_2|}{|y|}\right)\\
&+\left(\phi'^2(b)+\phi(b)\phi''(b)\right)\left(-\frac{\left\langle [z,y]_{\mathfrak{m}_2},v_2\right\rangle}{|y||y_2|}+\frac{\langle [z,y]_{\mathfrak{m}},v\rangle|y_2|}{|y|^3}\right)\left(\frac{\langle u_2,y_2\rangle|y|}{|y_2|}-\frac{\langle u,y\rangle|y_2|}{|y|}\right)\\
&+\left(\phi'^2(b)+\phi(b)\phi''(b)\right)\left(\frac{\left\langle [z,y]_{\mathfrak{m}_2},v_2\right\rangle|y|}{|y_2|}+\frac{\langle [z,y]_{\mathfrak{m}},v\rangle|y_2|}{|y|}\right)\left(\frac{\langle u_2,y_2\rangle}{|y||y_2|}-\frac{\langle u,y\rangle|y_2|}{|y|^3}\right)\\
&+\left(\phi'^2(b)+\phi(b)\phi''(b)\right)\left(\frac{\left\langle [z,y]_{\mathfrak{m}_2},u_2\right\rangle}{|y||y_2|}-\frac{\langle [z,y]_{\mathfrak{m}},u\rangle|y_2|}{|y|^3}\right)\left(\frac{\langle v_2,y_2\rangle}{|y_2|}-\frac{\langle v,y\rangle|y_2|}{|y|}\right)\\
= &\ \phi(b)\phi'(b)\left(\frac{\left\langle [z,u]_{\mathfrak{m}_2},v_2\right\rangle|y|}{|y_2|}-\frac{\langle [z,u]_{\mathfrak{m}},v\rangle|y_2|}{|y|}+\frac{\left\langle [z,v]_{\mathfrak{m}_2},u_2\right\rangle|y|}{|y_2|}-\frac{\langle [z,v]_{\mathfrak{m}},u\rangle|y_2|}{|y|}\right)\\
&+\left(\phi'^2(b)+\phi(b)\phi''(b)\right)\left(-\frac{\left\langle [z,y]_{\mathfrak{m}_2},v_2\right\rangle\langle u_2,y_2\rangle}{|y_2|^2}+\frac{\left\langle [z,y]_{\mathfrak{m}_2},v_2\right\rangle\langle u,y\rangle}{|y|^2}+\frac{\langle [z,y]_{\mathfrak{m}},v\rangle\langle u_2,y_2\rangle}{|y|^2}\right.\\
&\left.-\frac{\langle [z,y]_{\mathfrak{m}},v\rangle\langle u,y\rangle|y_2|^2}{|y|^4}\right)\left(\frac{\left\langle [z,y]_{\mathfrak{m}_2},v_2\right\rangle\langle u_2,y_2\rangle}{|y_2|^2}-\frac{\left\langle [z,y]_{\mathfrak{m}_2},v_2\right\rangle\langle u,y\rangle}{|y|^2}-\frac{\langle [z,y]_{\mathfrak{m}},v\rangle\langle u_2,y_2\rangle}{|y|^2}\right.\\
&\left.+\frac{\langle [z,y]_{\mathfrak{m}},v\rangle\langle u,y\rangle|y_2|^2}{|y|^4}\right)=0. \quad \square
\end{aligned}$$

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Corollary 3.2. *Let (G, g) be naturally reductive, $H = \{e\}$ (In other words, $\mathfrak{h} = \{0\}$), and let $\mathfrak{g}' \subseteq \mathfrak{m}_1$, then $\mathfrak{m}_2 \subseteq Z(\mathfrak{g})$.*

Proof. If the above assumptions hold, then (M, F) is naturally reductive. Now, using Lemma 3.2 in [18], we have $[\mathfrak{m}_2, \mathfrak{m}_2] \subseteq \mathfrak{m}_1$and $[\mathfrak{m}_2, \mathfrak{m}_2] \subseteq \mathfrak{m}_2$. Therefore, $[\mathfrak{m}_2, \mathfrak{m}_2] = 0$. On the other hand, the second part of the same Lemma shows that $[\mathfrak{m}_1, \mathfrak{m}_2] \subseteq \mathfrak{h} = \{0\}$. Thus $\mathfrak{m}_2 \subseteq Z(\mathfrak{g})$. □

Remark 3.3. *As a result, if g is a bi-invariant Riemannian metric on a connected Lie group G such that in the reductive decomposition $\mathfrak{g} = \mathfrak{m} = \mathfrak{m}_1 + \mathfrak{m}_2$, we have $\mathfrak{g}' \subseteq \mathfrak{m}_1$, then (G, F) is naturally reductive.*

We recall that a vector field $X \in \mathfrak{g} \setminus \{0\}$ for a homogeneous Riemannian manifold $(M = G/H, g)$ (resp. a homogeneous Finsler manifold $(M = G/H, F)$) is called a geodesic vector, if the curve $\sigma(t) = \exp(tX)_o$ is a geodesic on $(G/H, g)$ (resp. on $(G/H, F)$). For a homogeneous Riemannian manifold $(G/H, g)$ with a reductive decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$, a vector $X \in \mathfrak{g}$ is a geodesic vector, if and only if $g\left(X_{\mathfrak{m}}, [X, Y]_{\mathfrak{m}}\right) = 0$ for all $Y \in \mathfrak{m}$ (see [14]). It can be shown that, similarly for a homogeneous Finsler manifold $(M = G/H, F)$ (see [15]), a vector $X \in \mathfrak{g}$ is a geodesic vector if and only if

$$g_{X_{\mathfrak{m}}}\left(X_{\mathfrak{m}}, [X, Z]_{\mathfrak{m}}\right) = 0, \qquad \forall Z \in \mathfrak{g}.$$

In [17], the relationship between geodesic vector fields on Riemannian and Finslerian homogeneous spaces in the case of an (α, β)-metric is demonstrated. The following proposition shows the relationship between geodesic vector fields on these spaces in the case of an (α_1, α_2)-metric.

Proposition 3.4. *Let $(M = G/H, F)$ be a homogeneous Finsler manifold where $F = \alpha\phi(\frac{\alpha_2}{\alpha})$ is an invariant (α_1, α_2)-metric defined by an invariant Riemannian metric $\alpha = \sqrt{g(\cdot, \cdot)}$ such that $\mathfrak{g}' \subseteq \mathfrak{h} + \mathfrak{m}_1$. If $X \in \mathfrak{g} \setminus \{0\}$ is a geodesic vector field for $(G/H, g)$, then X is a geodesic vector field for $(G/H, F)$. Conversely, if X is a geodesic vector field for $(G/H, F)$ such that $\left(\phi^2(b) - \phi(b)\phi'(b)b\right) \neq 0$ and $\phi''(b) \leq 0$, where $b = \frac{|X_{\mathfrak{m}_2}|_g}{|X_{\mathfrak{m}}|_g}$, then X is a geodesic vector field for $(G/H, g)$.*

Proof. Let X be a geodesic vector field for $(G/H, g)$. Given the assumption $\mathfrak{g}' \subseteq \mathfrak{h} + \mathfrak{m}_1$, it can be said $\left\langle X_{\mathfrak{m}2}, [X, Z]_{\mathfrak{m}_2} \right\rangle = 0$. Therefore, we have

$$\begin{aligned} g_{X_{\mathfrak{m}}} \quad & \left(X_{\mathfrak{m}}, [X, Z]_{\mathfrak{m}}\right) = \left\langle X_{\mathfrak{m}}, [X, Z]_{\mathfrak{m}} \right\rangle \phi^2(b) \\ & + \phi(b)\phi'(b)\Bigg(\frac{\left\langle X_{\mathfrak{m}}, [X, Z]_{\mathfrak{m}} \right\rangle \left\langle X_{\mathfrak{m}_2}, X_{\mathfrak{m}_2} \right\rangle}{|X_{\mathfrak{m}}|\,|X_{\mathfrak{m}_2}|} + \frac{\left\langle X_{\mathfrak{m}2}, [X, Z]_{\mathfrak{m}_2} \right\rangle \left\langle X_{\mathfrak{m}}, X_{\mathfrak{m}} \right\rangle}{|X_{\mathfrak{m}}|\,|X_{\mathfrak{m}_2}|} \\ & - \frac{\left\langle X_{\mathfrak{m}}, X_{\mathfrak{m}} \right\rangle \left\langle X_{\mathfrak{m}}, [X, Z]_{\mathfrak{m}} \right\rangle |X_{\mathfrak{m}2}|}{|X_{\mathfrak{m}}|^3} + \frac{\left\langle X_{\mathfrak{m}2}, [X, Z]_{\mathfrak{m}_2} \right\rangle |X_{\mathfrak{m}}|}{|X_{\mathfrak{m}_2}|} - \frac{\left\langle X_{\mathfrak{m}2}, [X, Z]_{\mathfrak{m}_2} \right\rangle |X_{\mathfrak{m}_2}|}{|X_{\mathfrak{m}}|} \end{aligned}$$

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$$
\begin{aligned}
& -\frac{\left\langle X_{\mathfrak{m}_2}, X_{\mathfrak{m}_2}\right\rangle \left\langle X_{\mathfrak{m}2}, [X,Z]_{\mathfrak{m}_2}\right\rangle |X_{\mathfrak{m}}|}{|X_{\mathfrak{m}_2}|^3}\Bigg) \\
& +\left(\phi'^2(b)+\phi(b)\,\phi''(b)\right)\left(\frac{\left\langle X_{\mathfrak{m}_2}, X_{\mathfrak{m}_2}\right\rangle}{|X_{\mathfrak{m}}|\,|X_{\mathfrak{m}_2}|}-\frac{\left\langle X_{\mathfrak{m}}, X_{\mathfrak{m}}\right\rangle |X_{\mathfrak{m}_2}|}{|X_{\mathfrak{m}}|^3}\right) \\
& \left(\frac{\left\langle [X,Z]_{\mathfrak{m}_2}, X_{\mathfrak{m}2}\right\rangle}{|X_{\mathfrak{m}_2}|}-\frac{\left\langle [X,Z]_{\mathfrak{m}}, X_{\mathfrak{m}2}\right\rangle |X_{\mathfrak{m}_2}|}{|X_{\mathfrak{m}}|}\right) \\
= \quad & \left\langle X_{\mathfrak{m}}, [X,Z]_{\mathfrak{m}}\right\rangle \phi^2(b)+\phi(b)\phi'(b)\left(\frac{\left\langle X_{\mathfrak{m}2}, [X,Z]_{\mathfrak{m}_2}\right\rangle |X_{\mathfrak{m}}|}{|X_{\mathfrak{m}_2}|}-\frac{\left\langle X_{\mathfrak{m}}, [X,Z]_{\mathfrak{m}}\right\rangle |X_{\mathfrak{m}_2}|}{|X_{\mathfrak{m}}|}\right),
\end{aligned}
$$

which results in $g_{X_{\mathfrak{m}}}\left(X_{\mathfrak{m}}, [X,Z]_{\mathfrak{m}}\right)=0$. To prove the converse, we note that the last sentence in above can be written as follows:

$$
\left\langle X_{\mathfrak{m}}, [X,Z]_{\mathfrak{m}}\right\rangle\left(\phi^2(b)-\phi(b)\phi'(b)\frac{|X_{\mathfrak{m}_2}|}{|X_{\mathfrak{m}}|}\right)+\left\langle X_{\mathfrak{m}2}, [X,Z]_{\mathfrak{m}_2}\right\rangle \phi(b)\phi'(b)\frac{|X_{\mathfrak{m}}|}{|X_{\mathfrak{m}_2}|}.
$$

Therefore, given the condition in the assumption and that $\left\langle X_{\mathfrak{m}2}, [X,Z]_{\mathfrak{m}_2}\right\rangle=0$, the statement holds. □

Remark 3.5. *In the above proposition, suppose that $\phi''(b)\leq 0$, $\phi'(b)\neq 0$, and $\phi(b)>0$. Then, by Theorem 2.4, it follows that*

$$
\phi^2(b)-\phi(b)\phi'(b)\frac{|X_{\mathfrak{m}2}|}{|X\mathfrak{m}|}\neq 0.
$$

Consequently, the final assumption in the above proposition may be replaced by the simpler conditions $\phi'(b)\neq 0$ and $\phi''(b)\leq 0$.

In the remainder of this section, we analyze the (α_1,α_2)-metrics on the tangent bundle of a Lie group under certain conditions. We first present the following remark.

Remark 3.6. *Let G be a Lie group equipped with a bi-invariant Riemannian metric g. Then the Riemannian metric $\tilde{g}$, as defined in Definition 2.9, on the tangent bundle TG is bi-invariant if and only if G is commutative.*

Proof. Let $g\left([X,Y],Z\right)+g\left(X,[Z,Y]\right)=0$ for every $X,Y,Z\in\mathfrak{g}$. To show that the statement holds, it suffices to note that

$$
\begin{aligned}
\tilde{g}\left([X^c,Y^v],Z^v\right)+\tilde{g}\left(X^c,[Z^v,Y^v]\right) &= \tilde{g}\left([X,Y]^v,Z^v\right)=g\left([X,Y],Z\right), \\
\tilde{g}\left([X^v,Y^v],Z^c\right)+\tilde{g}\left(X^v,[Z^c,Y^v]\right) &= \tilde{g}\left(X^v,[Z,Y]^v\right)=g\left(X,[Z,Y]\right).
\end{aligned}
$$

Furthermore, for the remaining cases, the sum vanishes. Consequently, the equality holds if and only if $\mathfrak{g}$ is an Abelian Lie algebra. □

Here we give a procedure to construct an (α_1,α_2)-metric on the tangent bundle TG using a Randers metric on the Lie group G. Let $F(x,y)=\alpha(x,y)+\beta(x,y)=\sqrt{g_x(y,y)}+g(X\left(x\right),y)$ be a left-invariant

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Randers metric on an n-dimensional connected Lie group G, where $x \in M$, $0 \neq y \in T_xM$, g is a left-invariant Riemannian metric and X is a left-invariant vector field ($\|X\|_\alpha < 1$). Then, there exists an α-orthogonal decomposition $T_eG = \mathfrak{g} = \mathcal{V}_1 \oplus \mathcal{V}_2$ ($\mathcal{V}_1 \perp_g \mathcal{V}_2$) in which $\dim \mathcal{V}_1 = n-1$ and $\dim \mathcal{V}_2 = 1$.

Let $\mathcal{V}_1 = \operatorname{span}\{Y_1, Y_2, \ldots, Y_{n-1}\}$ and $\mathcal{V}_2 = \operatorname{span}\{X\}$ so that $\{Y_i\}_{i=1}^{n-1}$ are orthonormal with respect to the Riemannian metric α and also X be α-orthogonal to $\{Y_i\}_{i=1}^{n-1}$. With the lift from G to the tangent bundle TG by the vertical and complete vector fields, there exists an $\widetilde{\alpha}$-orthogonal decomposition $T_{(e,y)}TG = \widetilde{\mathfrak{g}} = \widetilde{\mathcal{V}_1} \oplus \widetilde{\mathcal{V}_2}$ ($\widetilde{\mathcal{V}_1} \perp_{\tilde{g}} \widetilde{\mathcal{V}_2}$) such that $\widetilde{\mathcal{V}_1} = \operatorname{span}\{Y_1^c, Y_2^c, \ldots, Y_{n-1}^c, Y_1^v, Y_2^v, \ldots, Y_{n-1}^v\}$ and $\widetilde{\mathcal{V}_2} = \operatorname{span}\{X^c, X^v\}$, where $\widetilde{\alpha} = \sqrt{\tilde{g}(\cdot,\cdot)}$ and $\tilde{g}$ is the left-invariant lifted Riemannian metric on TG according to Definition 2.9.

It can be seen that if $\tilde{Z} \in \widetilde{\mathfrak{g}}$, then

$$\underset{1 \le i \le n-1}{\exists \mu_i^c, \mu_i^v, \lambda^c, \lambda^v}; \quad \tilde{Z} = \left(\sum_{i=1}^{n-1} (\mu_i^c Y_i^c + \mu_i^v Y_i^v)\right) + \lambda^c X^c + \lambda^v X^v$$

Therefore, the (α_1, α_2)-metric on the tangent bundle TG is as follows:

$$\begin{aligned}
\tilde{F}\left((x,y), \tilde{Z}\right) &= \widetilde{\alpha}\phi\left(\frac{\widetilde{\alpha_2}}{\widetilde{\alpha}}\right) = \left\|\tilde{Z}\right\|_{\widetilde{\alpha}} \phi\left(\frac{\|\lambda^c X^c + \lambda^v X^v\|_{\widetilde{\alpha_2}}}{\left\|\tilde{Z}\right\|_{\widetilde{\alpha}}}\right) \\
&= \widetilde{\alpha}\psi\left(\frac{\widetilde{\alpha_1}}{\widetilde{\alpha}}\right) = \left\|\tilde{Z}\right\|_{\widetilde{\alpha}} \psi\left(\frac{\left\|\sum_{i=1}^{n-1} (\mu_i^c Y_i^c + \mu_i^v Y_i^v)\right\|_{\widetilde{\alpha_1}}}{\left\|\tilde{Z}\right\|_{\widetilde{\alpha}}}\right).
\end{aligned}$$

Here we recall the concept of S-curvature. Let (M, F) be a n-dimensional Finsler manifold, and let $\{b_i\}$ be a basis for T_xM. For a vector $y \in T_xM \setminus \{0\}$, let $\gamma(t)$ be a geodesic with conditions $\gamma(0) = x$, and $\gamma'(0) = y$. The S-curvature, first introduced by Z. Shen to measure the rate of change of the volume form of a Finsler space along geodesics, is a function on $TM \setminus \{0\}$ defined as follows:

$$S(x,y) = \frac{d}{dt}\left[\tau\left(\gamma(t), \gamma'(t)\right)\right]_{t=0},$$

where $\tau = \tau(y)$ is called the distortion function associated with F, which is independent of the choice of the basis $\{b_i\}$ and also is homogeneous of degree zero with respect to y, and is defined as $\tau(y) = Ln\frac{\sqrt{\det(g_{ij}(y)}}{\sigma_F}$. It should be noted that in the latter formula, σ_F is defined as $\sigma_F(x) = \frac{Vol(B^n(1))}{Vol(B_x^n)}$, where Vol denotes the volume of a subset in Euclidean space, $\mathbb{R}^n$, $B^n(1)$ is an open ball of radius 1, and $B_x^n = \left\{(y^i \in \mathbb{R}^n | F(y^i b_i) < 1\right\}$ is a strongly convex open set of $\mathbb{R}^n$.

Proposition 3.7. *Let G be a nilpotent connected Lie group, and F be a Randers metric generated by a left-invariant Riemannian metric g and a left-invariant vector field X. If $S_F = 0$, then, $S_{\tilde{F}} = 0$, where S_F and $S_{\tilde{F}}$ are the S-curvature of the Randers metric F on G and the S-curvature of the (α_1, α_2)-metric $\tilde{F}$ on TG constructed by the Randers metric F, respectively.*

Proof. According to Proposition 7.6 in Chapter 7 of [7]

$$S_F = 0 \Leftrightarrow X \in Z(\mathfrak{g}) \Leftrightarrow X^c, X^v \in Z(\widetilde{\mathfrak{g}}).$$

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On the other hand, $\widetilde{\mathcal{V}_2} = \mathrm{span}\,\{X^c, X^v\}$. Therefore, according to Corollary 4.4 in [10], the statement holds. $\square$

Proposition 3.8. *Let G be an arbitrary connected Lie group, and $S_{\tilde{F}} = 0$. Then, $S_F = 0$ and the base Randers metric F is of Berwald type.*

Proof. According to Corollary 4.4 in [9],

$$\begin{cases} \tilde{g}\left(\left[Y_i^c, X^c\right], Y_i^c\right) = 0 \\ \tilde{g}\left(\left[Y_i^v, X^c\right], Y_i^v\right) = 0 \end{cases} \Longrightarrow g\left(\left[Y_i, X\right], Y_i\right) = 0. \tag{3.1}$$

Also,

$$\begin{cases} \tilde{g}\left(\left[Y_i^c, X^c\right], X^c\right) = 0 \\ \tilde{g}\left(\left[Y_i^v, X^v\right], X^v\right) = 0 \end{cases} \Longrightarrow g\left(\left[Y_i, X\right], X\right) = 0. \tag{3.2}$$

Therefore, from (3.1) and (3.2), it follows that $S_F = 0$. We also note that $(\star\star)$ implies that F is of Douglas type, and according to Proposition 7.4 in [7], F is of Berwald type. $\square$

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