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SOME RESULTS ON QUASI ROUGH IDEAL STATISTICAL CONVERGENCE VIA NEUTROSOPHIC NORMED SPACES

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Abstract. This paper explores quasi rough ideal statistical convergence for sequences in neutrosophic normed spaces. Illustrative examples are provided to clarify the related topological and geometrical properties. By analyzing the criteria associated with quasi rough ideal statistical convergence in these spaces, the study aims to establish a comprehensive set of equivalent conditions for sequences that exhibit quasi rough ideal statistical convergence.

1. Introduction

The study of summability theory and sequence convergence has long been regarded as a significant and dynamic area in mathematics for many decades. The concept of convergence serves as a fundamental tool for extracting information from existing data. There are various procedures for achieving convergence, encompassing both analytical and non-analytical approaches. One such approach is statistical convergence [20] which is employed when the traditional notions of convergence fails to yield satisfactory results.

A sequence $s = \{s_k\}$ converges statistically to s^*, if $A(\varepsilon) = \{k \in \mathbb{N} : |s_k - s^*| > \varepsilon, \varepsilon > 0\}$ has zero natural density [21]. The natural density of set A which is a subset of $\mathbb{N}$, is expressed by $d(A) = \lim_{n\to\infty} \frac{1}{n} \left|\{a \leq n : a \in A\}\right|$.

Several authors([8, 29, 35]) have proposed generalizations of statistical convergence; among these, Özgüc and Yurdakadim [37] used quasi-density in place of natural density to generalize statistical convergence of sequences to quasi statistical convergence.

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The quasi-density for set M, which is a subset of $\mathbb{N}$, is denoted by

$$d_q(M) = \lim_{n\to\infty} \frac{1}{q_n} \mid \{m \le n : m \in M\} \mid,$$

where sequence $q = \{q_n\} \in \mathbb{R}$ satisfying $q_n > 0\ \forall n \in \mathbb{N}$, $\lim_{n\to\infty} q_n = \infty$ and $\limsup_{n\to\infty} \frac{q_n}{n} < \infty$.

If q_n is chosen to be for all n, quasi-density becomes exactly the same as natural density, and quasi statistical convergence reduces to standard statistical convergence. This generalization has attracted considerable attention due to its capacity to bridge classical and statistical convergence, effectively capturing behaviors that neither fully conform to statistical convergence nor exhibit complete divergence. Quasi-statistical convergence is characterized by convergence with respect to a non-dense subsequence or through a control sequence, thereby offering a more flexible framework for applications in summability theory, functional analysis, and sequence spaces [15, 18, 22, 38, 42, 44].

Definition 1.1 ([37]). *A sequence $s = \{s_k\} \in \mathbb{R}$ is said to be quasi statistical convergent to number s^* if for each $\varepsilon > 0$, we have*

$$\lim_{n\to\infty} \frac{1}{q_n} |\{k \le n : |s_k - s^*| \ge \varepsilon\}| = 0,$$

$$i.e\ d_q(\{k \in \mathbb{N} : |s_k - s^*| \ge \varepsilon\}) = 0.$$

It is denoted by $St_q - \lim_{k\to\infty} s_k = s^*$ or $s_k \xrightarrow{St_q} s^*$.

The concept of convergence holds central importance in analysis, particularly for understanding the behavior of sequences, functions, and operators. Classical normed spaces, together with extensions such as probabilistic normed spaces and intuitionistic fuzzy normed spaces, have provided rigorous frameworks for such investigations. However, as modern applications increasingly involve uncertainty, vagueness, and incomplete information, the need for more flexible mathematical structures capable of accommodating these realities, has become more pronounced. In this context, Smarandache [43] introduced neutrosophic sets as a significant advancement over intuitionistic fuzzy sets, particularly for non-standard analysis under real-world conditions that involve vague data, uncertain environments, or incomplete knowledge situations where classical approaches often fail to provide satisfactory solutions. Several researchers have investigated neutrosophic structures in conjunction with functional spaces. These efforts resulted in the formulation of neutrosophic continuity, convergence, and compactness, as well as to their applications in decision-making, pattern recognition, and computational intelligence. Subsequently, Kirişci and Şimşek introduced the concept of neutrosophic normed spaces as a significant contribution to the study of generalized metric spaces [30].

Definition 1.2 ([30]). *An algebraic structure $(\mathbb{V}, \mathcal{G}, \oplus, \circledast)$ is regarded as neutrosophic normed space (NNS) with vector space $\mathbb{V}$, normed space $\mathcal{G} = \{< \varphi(x), \psi(x), \varrho(x) > : x \in \mathbb{V}\}$ where $\varphi(x)$, $\psi(x)$ and $\varrho(x)$ represents truth, indeterminacy and falsity membership functions respectively, such that $\mathcal{G} : \mathbb{V} \times \mathbb{R}^+ \to [0, 1]$, continuous t-conorm $\circledast$ and continuous t-norm $\oplus$, if for $u, v \in \mathbb{V}$ and $s, t > 0$, the following axioms hold:*

(1) $\varphi(u, t), \psi(u, t), \varrho(u, t) \in [0, 1]$;

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(2) $\varphi(u,t)+\psi(u,t)+\varrho(u,t)\leq 3$;
(3) $\varphi(u,t)=1$, $\psi(u,t)=0$ *and* $\varrho(u,t)=0$ *for* $t>0$ *iff* $u=0$;
(4) $\varphi(u,t)=0$, $\psi(u,t)=1$ *and* $\varrho(u,t)=1$ *for* $t\leq 0$;
(5) $\varphi(\alpha u,t)=\varphi\left(u,\frac{t}{|\alpha|}\right)$; $\psi(\alpha u,t)=\psi\left(u,\frac{t}{|\alpha|}\right)$ *and* $\varrho(\alpha u,t)=\varrho\left(u,\frac{t}{|\alpha|}\right)$ *for* $\alpha\neq 0$;
(6) $\varphi(u,\cdot)$ *is a non-decreasing continuous function;*
(7) $\varphi(u,s)\oplus\varphi(v,t)\leq\varphi(u+v,s+t)$;
(8) $\psi(u,\cdot)$ *is a non-increasing continuous function;*
(9) $\psi(u,s)\circledast\psi(v,t)\geq\psi(u+v,s+t)$;
(10) $\varrho(u,\cdot)$ *is a non-increasing continuous function;*
(11) $\varrho(u,s)\circledast\varrho(v,t)\geq\varrho(u+v,s+t)$;
(12) $\lim\limits_{t\to\infty}\varphi(u,t)=1$, $\lim\limits_{t\to\infty}\psi(u,t)=0$ *and* $\lim\limits_{t\to\infty}\varrho(u,t)=0$.

Then (φ,ψ,ϱ) *is a neutrosophic norm.*

Example 1.3 ([30]). *Consider* $(\mathbb{V},\|\cdot\|)$ *as a normed space. To every* $t>0$ *and* $u\in\mathbb{V}$, *if*

(i) $\varphi(u,t)=\frac{t}{t+\|u\|}$, $\psi(u,t)=\frac{\|x\|}{t+\|u\|}$ *and* $\varrho(u,t)=\frac{\|u\|}{t}$; $t>\|u\|$,
(ii) $\varphi(u,t)=0$, $\psi(u,t)=1$ *and* $\varrho(u,t)=1$; $t\leq\|u\|$.

Also $\alpha_1\oplus\alpha_2=\alpha_1\alpha_2$ *and* $\alpha_1\circledast\alpha_2=\alpha_1+\alpha_2-\alpha_1\alpha_2$ $\forall$ $\alpha_1,\alpha_2\in[0,1]$.
Then, 4-tuple $(\mathbb{V},\mathcal{G},\oplus,\circledast)$ *is a NNS.*

Furthermore, Kirişci and Şimşek [30] established statistical convergence concept for sequences by employing natural density in neutrosophic normed spaces. Currently, neutrosophic normed spaces and the theory of statistical convergence of sequences constitute a fundamental component of various fields in mathematics, economics, science, and technology ([1, 5, 10, 11, 13, 16, 24, 26–28, 31, 33, 40]).

Definition 1.4 ([30]). *Let* $(\mathbb{V},\mathcal{G},\oplus,\circledast)$ *be defined as NNS associated with neutrosophic norm* (φ,ψ,ϱ). *Then, a sequence* $s=\{s_k\}\in\mathbb{V}$ *is regarded as statistically convergent to* $s^*\in\mathbb{V}$ *with respect to the norm* (φ,ψ,ϱ), *if for every* $\sigma>0$ *and* $\varepsilon>0$, *the following condition holds*

$$d(\{k\in\mathbb{N}:\varphi(s_k-s^*,\varepsilon)\leq 1-\sigma \text{ or } \psi(s_k-s^*,\varepsilon)\geq\sigma,\ \varrho(s_k-s^*,\varepsilon)\geq\sigma\})=0.$$

It is denoted by $St_{(\varphi,\psi,\varrho)}-\lim\limits_{k\to\infty}s_k=s^*$ *or* $s_k\xrightarrow{St_{(\varphi,\psi,\varrho)}}s^*$.

The concept of ideal convergence [34], also referred to as $\mathcal{I}$-convergence, has been rigorously formalized for the sequences by extending the well-established notion of statistical convergence [20]. This advancement has much contributed to a deeper understanding of convergence phenomena within mathematical analysis. Moreover, each ideal $\mathcal{I}$ is inherently associated with a filter, which is defined by $\mathcal{F}(\mathcal{I})=\{M\subseteq\mathbb{V}:M^c\in\mathcal{I}\}$. For the purposes of this article, the symbol $\mathcal{I}$ is adopted to denote a non-trivial admissible ideal. Subsequently, statistical convergence is examined within the framework of ideals.

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Definition 1.5 ([34]). *A sequence $s = \{ s_k \} \in \mathbb{V}$ is regarded as statistically ideal convergent ($\mathcal{I}$-St-convergent) to $s^* \in \mathbb{V}$, if for every $\varepsilon > 0$ and $\mu > 0$, the following condition holds*

$$\left\{ n \in \mathbb{N} : \frac{1}{n} |\, \{ n \in \mathbb{N} : |s_k - s^*| \geq \varepsilon \} \,| \geq \mu \right\} \in \mathcal{I}.$$

Here, s^* is identified as $\mathcal{I}$-St-limit of a sequence $s = \{s_k\}$.

In a recent study, Kirişci [31] established a significant relationship between ideal convergence and neutrosophic normed spaces, thereby contributing to a deeper understanding of these mathematical structures.

Definition 1.6 ([31]). *Let $(\mathbb{V}, \mathcal{G}, \oplus, \circledast)$ be defined as a NNS associated with neutrosophic norm (φ, ψ, ϱ). Then, a sequence $s = \{s_k\} \in \mathbb{V}$ is regarded as $\mathcal{I}$-convergent to $s^* \in \mathbb{V}$ with respect to the norm (φ, ψ, ϱ), if for every $\sigma \in (0, 1)$ and $\varepsilon > 0$, the following condition holds*

$$\{k \in \mathbb{N} : \varphi(s_k - s^*, \varepsilon) \leq 1 - \sigma \text{ or } \psi(s_k - s^*, \varepsilon) \geq \sigma,\ \varrho(s_k - s^*, \varepsilon) \geq \sigma\} \in \mathcal{I}.$$

It is denoted by $\mathcal{I}_(\varphi, \psi, \varrho)\text{-}\lim_{k \to \infty} s_k = s^*$ or $s_k \xrightarrow{(\varphi,\psi,\varrho)} s^*$.

There has been growing interest ([2–4, 17, 25, 32, 36]) in the study of rough convergence theory, which serves to validate the accuracy of approximate solutions for problems whose data are derived from computational methods and scientific experiments. Rough convergence in some instances may be preferable compared to other convergences since it reflects the actual behavior of the studied system.

Definition 1.7 ([39]). *Assume that $(\mathbb{V}, \| \, . \, \|)$ to be a normed linear space and $r \geq 0$ be a real number. Then, a sequence $s = \{s_k\} \in \mathbb{V}$ is regarded as rough convergent to $s^* \in \mathbb{V}$, if for every $\varepsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that $\|s_k - s^*\| < r + \varepsilon$ for $k \geq k_0$.*

Aytar in [6] applied the existing concept of rough convergence to introduce a statistical analogue, referred to as rough statistical convergence of sequences. In a subsequent study, Aytar [7] investigated certain convexity-related and closeness-related criteria in relation to a set of rough statistical limit points of a sequence within the framework of normed linear spaces.

Definition 1.8 ([6]). *Assume, $(\mathbb{V}, \| \cdot \|)$ to be a normed linear space and $r \geq 0$ be a real number. Then, a sequence $s = \{s_k\} \in \mathbb{V}$ is regarded as rough statistically convergent to $s^* \in \mathbb{V}$, if for every $\varepsilon > 0$, the following condition holds*

$$d(\{k \in \mathbb{N} : \|s_k - s^*\| \geq r + \varepsilon\}) = 0.$$

This paper aims to establish a generalized statistical ideal convergence using quasi-density within the framework of neutrosophic normed linear spaces. It is expected that the results provided in the present paper can serve as a useful tool for analyzing sequences in which uncertainty is multi-dimensional and convergence behavior is non-classical.

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2. Main results

The notion of quasi-ideal statistical convergence of sequences is introduced and examined in the context of neutrosophic normed spaces (NNS), several analytical results are established.

Definition 2.1. *Let* $(\mathbb{V}, \mathcal{G}, \oplus, \circledast)$ *be defined as NNS associated with neutrosophic norm* (φ, ψ, ϱ)*. Then, a sequence* $s = \{s_k\} \in \mathbb{V}$ *is regarded as quasi-*$\mathcal{I}$*-statistically convergent to* $s^* \in \mathbb{V}$ *with respect to the norm* (φ, ψ, ϱ)*, if for every* $\sigma \in (0,1)$*,* $\mu > 0$ *and* $\varepsilon > 0$*, the set*

$$\left\{ n \in \mathbb{N} : \frac{1}{q_n} |\{k \leq n : \varphi(s_k - s^*, \varepsilon) \leq 1 - \sigma \text{ or } \psi(s_k - s^*, \varepsilon) \geq \sigma,\ \varrho(s_k - s^*, \varepsilon) \geq \sigma\}| \geq \mu \right\}$$

belongs to $\mathcal{I}$*. It is denoted by* $St_q - \mathcal{I}_{(\varphi,\psi,\varrho)} - \lim\limits_{k\to\infty} s_k = s^*$ *or* $s_k \xrightarrow{St_q - \mathcal{I}_{(\varphi,\psi,\varrho)}} s^*$.

Definition 2.2. *Let* $(\mathbb{V}, \mathcal{G}, \oplus, \circledast)$ *be defined as NNS associated with neutrosophic norm* (φ, ψ, ϱ) *and* $r \geq 0$ *be a real number. Then, a sequence* $s = \{s_k\} \in \mathbb{V}$ *is regarded as quasi rough* $\mathcal{I}$*-statistically convergent to* $s^* \in \mathbb{V}$ *with respect to the norm* (φ, ψ, ϱ) *if for every* $\sigma \in (0,1)$*,* $\mu > 0$ *and* $\varepsilon > 0$*, the set*

$$\left\{ n \in \mathbb{N} : \frac{1}{q_n} |\{k \in \mathbb{N} : \varphi(s_k - s^*, r+\varepsilon) \leq 1 - \sigma \text{ or} \psi(s_k - s^*, r+\varepsilon) \geq \sigma,\ \varrho(s_k - s^*, r+\varepsilon) \geq \sigma\}| \geq \mu \right\}$$

belongs to $\mathcal{I}$*. It is represented by* $r - St_q - \mathcal{I}_{(\varphi,\psi,\varrho)} - \lim\limits_{k\to\infty} s_k = s^*$ *or* $s_k \xrightarrow{r - St_q - \mathcal{I}_{(\varphi,\psi,\varrho)}} s^*$.

The $r - St_q - \mathcal{I}_{(\varphi,\psi,\varrho)}$-limit of a sequence is not necessarily to be unique. For that reason, the set $r - St_q - \mathcal{I}_{(\varphi,\psi,\varrho)}$-limit set of $s = \{s_k\}$ is considered as St_q-$\mathcal{I}_{(\varphi,\psi,\varrho)}$-$LIM_s^r = \{s^* : s_k \xrightarrow{r - St_q - \mathcal{I}_{(\varphi,\psi,\varrho)}} s^*\}$. This will be clear from the next example.

Example 2.3. *Take NNS* $(\mathbb{V}, \mathcal{G}, \oplus, \circledast)$ *as illustrated in Example 1.3. Let us examine a nontrivial ideal* $\mathcal{I}$ *that which is an admissible ideal. Take an infinite set* $M \in \mathcal{I}$ *for which* $d_{\mathcal{I}}(M) = 0$*, even though* $d(M)$ *does not exist. Consider the sequence*

$$s_k = \begin{cases} (-1)^k & k \notin M \\ k & \text{otherwise.} \end{cases}$$

Then

$$St_q\text{-}\mathcal{I}_{(\varphi,\psi,\varrho)}\text{-}LIM_s^r = \begin{cases} \emptyset & r < 1 \\ [1-r, r-1] & \text{otherwise.} \end{cases}$$

For any sub-sequence $s' = \{s_{j_k}\}$ of $s = \{s_k\}$ in NNS $(\mathbb{V}, \mathcal{G}, \oplus, \circledast)$ we have $LIM_{s_k}^{r(\varphi,\psi,\varrho)} \subset LIM_{s_{j_k}}^{r(\varphi,\psi,\varrho)}$, where $LIM_{s_k}^{r(\varphi,\psi,\varrho)}$ and $LIM_{s_{j_k}}^{r(\varphi,\psi,\varrho)}$ are rough limit sets of sequence $\{s_k\}$ and $\{s_{j_k}\}$, respectively. However, the aforementioned result does not hold in rough ideal statistical convergence of sequences in NNS, as demonstrated in the next example.

Example 2.4. *Take NNS* $(\mathbb{V}, \mathcal{G}, \oplus, \circledast)$ *as illustrated in the Example 1.3. Consider an infinite set* $J = \{j_1 < j_2 < j_3 < \dots\} \in \mathcal{I}$ *satisfying* $d_{\mathcal{I}}(J) = 0$*. Also,* $d(J)$ *fails to exist. Take a sequence*

$$s_k = \begin{cases} k & k \in J \\ 0 & \text{otherwise,} \end{cases}$$

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with St_q-$\mathcal{I}_{(\varphi,\psi,\varrho)}$-$LIM_s^r = [-r, r]$ *whenever* $r > 0$. *Further, it's sub-sequence* $s' = \{s_{j_k}\}$ *where* $j_k \in J$, *having* St_q-$\mathcal{I}_{(\varphi,\psi,\varrho)}$-$LIM_{s'}^r = \emptyset$.

Theorem 2.5. St_q-$\mathcal{I}_{(\varphi,\psi,\varrho)}$-$LIM_s^r$ *of a sequence* $s = \{s_k\}$ *in NNS* $(\mathbb{V}, \mathcal{G}, \oplus, \circledast)$ *is a closed set.*

Proof. If we take St_q-$\mathcal{I}_{(\varphi,\psi,\varrho)}$-$LIM_s^r = \emptyset$ then there is nothing to prove.
Consider St_q-$\mathcal{I}_{(\varphi,\psi,\varrho)}$-$LIM_s^r \neq \emptyset$. Take convergent sequence $t = \{t_k\}$ from St_q-$\mathcal{I}_{(\varphi,\psi,\varrho)}$-$LIM_s^r$ which converges to $t^* \in \mathbb{V}$.
For $\nu \in (0, 1)$ choose $\sigma \in (0, 1)$ such that $(1 - \sigma) \oplus (1 - \sigma) > 1 - \nu$ and $\sigma \circledast \sigma < \nu$. Also, for $\varepsilon > 0$ and $\sigma \in (0, 1)$ there is $k_1 \in \mathbb{N}$ satisfying

$$\varphi\left(t_k - t^*; \frac{\varepsilon}{2}\right) > 1 - \sigma, \psi\left(t_k - t^*; \frac{\varepsilon}{2}\right) < \sigma \text{ and } \varrho\left(t_k - t^*; \frac{\varepsilon}{2}\right) < \sigma \text{ for all } \ k \geq k_1.$$

Now select $\mu > 0$ and $t_m \in St_q$-$\mathcal{I}_{(\varphi,\psi,\varrho)}$-$LIM_s^r$ with $m > k_1$ that gives $A \in \mathcal{I}$ where $A = \left\{n \in \mathbb{N} : \frac{1}{q_n}|S| \geq \mu\right\}$ and $S = \{k \in \mathbb{N} : \varphi\left(s_k - t_m; r + \frac{\varepsilon}{2}\right) \leq 1 - \sigma,\ \psi\left(s_k - t_m; r + \frac{\varepsilon}{2}\right) \geq \sigma, \text{or } \varrho\left(s_k - t_m; r + \frac{\varepsilon}{2}\right) \geq \sigma\}$. Being $\mathcal{I}$ an admissible ideal, set $M = A^c \neq \emptyset$. Take $n \in M$ which provides

$$\frac{1}{q_n}|\{k \in \mathbb{N} : \varphi\left(s_k - t_m; r + \frac{\varepsilon}{2}\right) \leq 1 - \sigma, \psi\left(s_k - t_m; r + \frac{\varepsilon}{2}\right) \geq \sigma \text{ and } \varrho\left(s_k - t_m; r + \frac{\varepsilon}{2}\right) \geq \sigma\}| < \mu$$

i.e

$$\frac{1}{q_n}|\{k \in \mathbb{N} : \varphi\left(s_k - t_m; r + \frac{\varepsilon}{2}\right) > 1 - \sigma, \psi\left(s_k - t_m; r + \frac{\varepsilon}{2}\right) < \sigma \text{ and } \varrho\left(s_k - t_m; r + \frac{\varepsilon}{2}\right) < \sigma\}| \geq 1 - \mu. \tag{2.1}$$

Let $B = \{k \in \mathbb{N} : \varphi\left(s_k - t_m; r + \frac{\varepsilon}{2}\right) > 1 - \sigma, \psi\left(s_k - t_m; r + \frac{\varepsilon}{2}\right) < \sigma \text{ and } \varrho\left(s_k - t_m; r + \frac{\varepsilon}{2}\right) < \sigma\}$.
The following inequalities holds for $j \in B$,

$$\begin{aligned}
\varphi(s_j - t^*; r + \varepsilon) &\geq \varphi\left(s_j - t_m; r + \frac{\varepsilon}{2}\right) \oplus \varphi\left(t_m - t^*; \frac{\varepsilon}{2}\right) \\
&> (1 - \sigma) \oplus (1 - \sigma) \\
&> 1 - \nu, \\
\psi(s_j - t^*; r + \varepsilon) &\leq \psi\left(s_j - t_m; r + \frac{\varepsilon}{2}\right) \circledast \psi\left(t_m - t^*; \frac{\varepsilon}{2}\right) \\
&< \sigma \circledast \sigma \\
&< \nu,
\end{aligned}$$

and

$$\begin{aligned}
\varrho(s_j - t^*; r + \varepsilon) &\leq \varrho\left(s_j - t_m; r + \frac{\varepsilon}{2}\right) \circledast \varrho\left(t_m - t^*; \frac{\varepsilon}{2}\right) \\
&< \sigma \circledast \sigma \\
&< \nu.
\end{aligned}$$

Therefore, $j \in D = \{k \in \mathbb{N} : \varphi(s_k - t^*; r + \varepsilon) > 1 - \nu,\ \psi\left(s_k - t^*; r + \varepsilon\right) < \nu \text{ and } \varrho\left(s_k - t^*; r + \varepsilon\right) < \nu\}$.
Further, $B \subseteq D$ and equation (2.1) provides $1 - \mu \leq \frac{|B|}{q_n} \leq \frac{|D|}{q_n}$. Thus,

$$\frac{1}{q_n}\left|\{k \in \mathbb{N} : \varphi(s_k - t^*; r + \varepsilon) \leq 1 - \nu,\ \psi\left(s_k - t^*; r + \varepsilon\right) \geq \nu \text{ and } \varrho\left(s_k - t^*; r + \varepsilon\right) \geq \nu\}\right| \geq \mu,$$

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$$\left\{n\in \mathbb{N}: \frac{1}{q_n}|\{k\in \mathbb{N}: \varphi(s_k - t^*; r+\varepsilon) \leq 1-\nu,\ \psi\left(s_k - t^*; r+\varepsilon\right) \geq \nu \,\text{and}\, \varrho\left(s_k - t^*; r+\varepsilon\right) \geq t| \geq \mu\right\} \in \mathcal{I}.$$

Therefore, $t^* \in St_q\text{-}\mathcal{I}_{(\varphi,\psi,\varrho)}\text{-}LIM_s^r$. □

In the next result, the convexity of $St_q\text{-}\mathcal{I}_{(\varphi,\psi,\varrho)}\text{-}LIM_s^r$ will be established.

Theorem 2.6. *Let $s=\{s_k\}$ be a sequence in NNS $(\mathbb{V},\mathcal{G},\oplus,\circledast)$ and $r\geq 0$. Then, set $St_q\text{-}\mathcal{I}_{(\varphi,\psi,\varrho)}\text{-}LIM_s^r$ is convex set.*

Proof. Consider $\wp_1,\wp_2\in St_q\text{-}\mathcal{I}_{(\varphi,\psi,\varrho)}\text{-}LIM_s^r$. For convexity's existence for $St_q\text{-}\mathcal{I}_{(\varphi,\psi,\varrho)}\text{-}LIM_s^r$, it needs to be proved that $[(1-\lambda)\wp_1+\lambda\wp_2]\in St_q\text{-}\mathcal{I}_{(\varphi,\psi,\varrho)}\text{-}LIM_s^r$; $\lambda\in(0,1)$.
As $\nu\in(0,1)$, choose $\sigma\in(0,1)$ such that $(1-\sigma)\oplus(1-\sigma)>1-\nu$ and $\sigma\circledast\sigma<\nu$. For $\varepsilon>0$ and $\sigma\in(0,1)$, take

$$M_1=\left\{k\in\mathbb{N}:\varphi(s_k-\wp_1;\frac{r+\varepsilon}{2(1-\lambda)})\leq 1-\sigma \,\text{or}\, \psi(s_k-\wp_1;\frac{r+\varepsilon}{2(1-\lambda)})\geq\sigma,\ \varrho(s_k-\wp_1;\frac{r+\varepsilon}{2(1-\lambda)})\geq\sigma\right\},$$

$$M_2=\left\{k\in\mathbb{N}:\varphi\left(s_k-\wp_2;\frac{r+\varepsilon}{2\lambda}\right)\leq 1-\sigma \text{ or } \psi\left(s_k-\wp_2;\frac{r+\varepsilon}{2\lambda}\right)\geq\sigma,\ \varrho\left(s_k-\wp_2;\frac{r+\varepsilon}{2\lambda}\right)\geq\sigma\right\}.$$

Then,

$$\frac{1}{q_n}\left|\{k\leq n: k\in M_1\cup M_2\}\right|\leq\frac{1}{q_n}\left|\{k\leq n:k\in M_1\}\right|+\frac{1}{q_n}\left|\{k\leq n:k\in M_2\}\right|.$$

Applying $\mathcal{I}-$ convergence characteristics, we obtain $\left\{n\in\mathbb{N}:\frac{1}{q_n}|\{k\leq n:k\in M_1\cup M_2\}|\geq\mu\right\}\in\mathcal{I}$ for $\mu>0$. Select $0<\mu_1<1$ such that $0<1-\mu_1<\mu$.
Consider $M=\left\{n\in\mathbb{N}:\frac{1}{n}\left|\{k\leq n:k\in M_1\cup M_2\}\right|\geq\mu_1\right\}$ and $M\in\mathcal{I}$. If $n\notin M$, then

$$\frac{1}{q_n}\left|\{k\leq n:k\in M_1\cup M_2\}\right|<1-\mu_1 \implies \frac{1}{q_n}\left|\{k\leq n:k\notin M_1\cup M_2\}\right|\geq\mu_1.$$

Consequently, $\{k\in\mathbb{N}:k\notin M_1\cup M_2\}\neq\emptyset$. If $k\in M_1^c\cap M_2^c$, the following holds

$$\begin{aligned}
\varphi(s_k-[(1-\lambda)\wp_1+\lambda\wp_2];r+\varepsilon)&\geq\varphi((1-\lambda)(s_k-\wp_1)+\lambda(s_k-\wp_2);r+\varepsilon)\\
&\geq\varphi\left((1-\lambda)(s_k-\wp_1);\frac{r+\varepsilon}{2}\right)\oplus\varphi\left(\lambda(s_k-\wp_2);\frac{r+\varepsilon}{2}\right)\\
&\geq\varphi\left(s_k-\wp_1;\frac{r+\varepsilon}{2(1-\lambda)}\right)\oplus\varphi\left(s_k-\wp_2;\frac{r+\varepsilon}{2\lambda}\right)\\
&>(1-\sigma)\oplus(1-\sigma)\\
&>1-\nu,
\end{aligned}$$

$$\begin{aligned}
\psi(s_k-[(1-\lambda)\wp_1+\lambda\wp_2];r+\varepsilon)&\leq\psi((1-\lambda)(s_k-\wp_1)+\lambda(s_k-\wp_2);r+\varepsilon)\\
&\leq\psi\left((1-\lambda)(s_k-\wp_1);\frac{r+\varepsilon}{2}\right)\circledast\psi\left(\lambda(s_k-\wp_2);\frac{r+\varepsilon}{2}\right)\\
&\leq\psi\left(s_k-\wp_1;\frac{r+\varepsilon}{2(1-\lambda)}\right)\circledast\psi\left(s_k-\wp_2;\frac{r+\varepsilon}{2\lambda}\right)\\
&<\sigma\circledast\sigma\\
&<\nu.
\end{aligned}$$

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and

$$\begin{aligned}\varrho(s_k-[(1-\lambda)\wp_1+\lambda\wp_2];r+\varepsilon) &\leq \varrho((1-\lambda)(s_k-\wp_1)+\lambda(s_k-\wp_2);r+\varepsilon)\\ &\leq \varrho\left((1-\lambda)(s_k-\wp_1);\frac{r+\varepsilon}{2}\right)\circledast\varrho\left(\lambda(s_k-\wp_2);\frac{r+\varepsilon}{2}\right)\\ &\leq \varrho\left(s_k-\wp_1;\frac{r+\varepsilon}{2(1-\lambda)}\right)\circledast\varrho\left(s_k-\wp_2;\frac{r+\varepsilon}{2\lambda}\right)\\ &< \sigma\circledast\sigma\\ &< \nu.\end{aligned}$$

Clearly, $M_1^c\cap M_2^c\subseteq Q^c$ where

$$Q=\{k\in\mathbb{N}:\varphi(s_k-\wp;r+\varepsilon)\leq 1-\nu \text{ or } \psi(s_k-\wp;r+\varepsilon)\geq 1-\nu,\ \varrho(s_k-\wp;r+\varepsilon)\geq 1-\nu\}$$

and $\wp=(1-\lambda)\wp_1+\lambda\wp_2$.

Thus, if $n\notin M$ then we have $\mu_1\leq\frac{|M_1^c\cap M_2^c|}{q_n}\leq\frac{|Q^c|}{q_n}$ *i.e.* $\frac{1}{q_n}\,|\{k\leq n:k\in Q\}|<1-\mu_1<\mu$.

Therefore, $M^c\subseteq\left\{k\in\mathbb{N}:\frac{1}{q_n}\,|\{k\leq n:k\in Q\}|<\mu\right\}$.

As $M\in\mathcal{I}$, consequently $\{k\in\mathbb{N}:\frac{1}{n}\,|\{k\leq n:k\in Q\}|\geq\mu\}\in\mathcal{I}$.

So, $[(1-\lambda)\wp_1+\lambda\wp_2]\in St_q\text{-}\mathcal{I}_{(\varphi,\psi,\varrho)}\text{-}LIM_s^r$ *i.e* $St_q\text{-}\mathcal{I}_{(\varphi,\psi,\varrho)}\text{-}LIM_s^r$ is a convex set. □

Theorem 2.7. *Any sequence $s=\{s_k\}$ of NNS $(\mathbb{V},\mathcal{G},\oplus,\circledast)$ regarded as quasi rough $\mathcal{I}$-statistically convergent to $s^*\in\mathbb{V}$ with respect to the norm (φ,ψ,ϱ) for some $r\geq 0$, if some sequence $t=\{t_k\}\in\mathbb{V}$ exists, which is $\mathcal{I}$-statistically convergent to $s^*\in\mathbb{V}$ along with the same norm (φ,ψ,ϱ). Also for $\sigma\in(0,1)$ it holds that $\varphi(s_k-t_k;r)>1-\sigma$, $\psi(s_k-t_k;r)<\sigma$ and $\varrho(s_k-t_k;r)<\sigma$, where $k\in\mathbb{N}$.*

Proof. Let $t_k\xrightarrow{r-St_q-\mathcal{I}_{(\varphi,\psi,\varrho)}}s^*$. Also, $\varphi(s_k-t_k;r)<\sigma$, $\psi(s_k-t_k;r)>1-\sigma$ and $\varrho(s_k-t_k;r)>1-\sigma$ for $k\in\mathbb{N}$. Take $\varepsilon>0$. Also, for fixed $\sigma\in(0,1)$ choose $\nu\in(0,1)$ such that $(1-\nu)\oplus(1-\nu)>1-\sigma$ and $\sigma\circledast\sigma<\nu$. Furthermore, $\varphi(s_k-t_k;r)\leq 1-\nu$ or $\psi(s_k-t_k;r)\geq\nu,\ \varrho(s_k-t_k;r)\geq\nu$ for $k\in\mathbb{N}$. Consider

$$A=\{k\in\mathbb{N}:\varphi(t_k-s^*;\varepsilon)\leq 1-\nu \text{ or } \psi(t_k-s^*;\varepsilon)\geq\nu,\ \varrho(t_k-s^*;\varepsilon)\geq\nu\}.$$

Clearly, the set $\left\{n\in\mathbb{N}:\frac{1}{n}|\{k\leq n:k\in A\}|\geq\mu\right\}\in\mathcal{I}$; for $\mu>0$. Moreover,

$$B=\{k\in\mathbb{N}:\varphi(t_k-s^*;\varepsilon)\leq 1-\nu \text{ or } \psi(t_k-s^*;\varepsilon)\geq\nu,\ \varrho(t_k-s^*;\varepsilon)\geq\nu\}$$

For $k\in B^c$,

$$\begin{aligned}\varphi(s_k-s^*;r+\varepsilon)&\geq\varphi(s_k-t_k;r)\oplus\varphi(t_k-s^*;\varepsilon)\\ &>(1-\nu)\oplus(1-\nu)\\ &>1-\sigma,\end{aligned}$$

$$\begin{aligned}\psi(s_k-s^*;r+\varepsilon)&\leq\psi(s_k-t_k;r)\circledast\psi(t_k-s^*;\varepsilon)\\ &<\nu\circledast\nu\\ &<\sigma,\end{aligned}$$

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and

$$\begin{aligned}\varrho(s_k - s^*; r+\varepsilon) &\le \varrho(s_k - t_k; r) \circledast \varrho(t_k - s^*; \varepsilon)\\ &< \nu \circledast \nu\\ &< \sigma.\end{aligned}$$

Then, $\varphi(s_k - s^*; r+\varepsilon) > 1-\sigma$, $\psi(s_k - s^*; r+\varepsilon) < \sigma$ and $\varrho(s_k - s^*; r+\varepsilon) < \sigma$ whenever $k \in B^c$. Thus,

$$\{k \in \mathbb{N} : \varphi(s_k - s^*; r+\varepsilon) \le 1-\sigma \text{ or } \psi(s_k - s^*; r+\varepsilon) \ge \sigma,\ \varrho(s_k - s^*; r+\varepsilon) \ge \sigma\} \subseteq B.$$

So,

$$\frac{1}{q_n}|\{k \in \mathbb{N} : \varphi(s_k - s^*; r+\varepsilon) \le 1-\sigma \text{ or } \psi(s_k - s^*; r+\varepsilon) \ge \sigma,\ \varrho(s_k - s^*; r+\varepsilon) \ge \sigma\}| \le \frac{1}{q_n}|B|.$$

Further,

$$\frac{1}{q_n}|\{k \in \mathbb{N} : \varphi(s_k - s^*; r+\varepsilon) \le 1-\sigma \text{ or } \psi(s_k - s^*; r+\varepsilon) \ge \sigma,\ \varrho(s_k - s^*; r+\varepsilon) \ge \sigma\}| \le \mu.$$

Hence,

$$\left\{k \in \mathbb{N} : \frac{1}{q_n}|\{k \in \mathbb{N} : \varphi(s_k - s^*; r+\varepsilon) \le 1-\sigma \text{ or } \psi(s_k - s^*; r+\varepsilon) \ge \sigma, \varrho(s_k - s^*; r+\varepsilon) \ge \sigma\}| \le \mu\right\} \in \mathcal{I}.$$

Therefore, $s_k \xrightarrow{r-St_q-\mathcal{I}_{(\varphi,\psi,\varrho)}} s^*$. □

Theorem 2.8. *Let $s = \{s_k\}$ be sequence in NNS $(\mathbb{V}, \mathcal{G}, \oplus, \circledast)$. Elements $u, v \in St_q$-$\mathcal{I}_{(\varphi,\psi,\varrho)}$-$LIM_s^r$ fails to exist for some $r > 0$ and every $\sigma \in (0,1)$ with $\varphi(u-v; mr) \le 1-\sigma$ or $\psi(u-v; mr) \ge \sigma$, $\varrho(u-v; mr) \ge \sigma$ for $m > 2$.*

Proof. It is proved by assuming the contrary. Suppose elements $u, v \in St_q$-$\mathcal{I}_{(\varphi,\psi,\varrho)}$-$LIM_s^r$ exists with

$$\varphi(u-v; mr) \le 1-\sigma \text{ or } \psi(u-v; mr) \ge \sigma,\ \varrho(u-v; mr) \ge \sigma \text{ for } m > 2 \tag{2.2}$$

For $\sigma \in (0,1)$ choose $\nu \in (0,1)$ such that $(1-\nu) \oplus (1-\nu) > 1-\sigma$ and $\sigma \circledast \sigma < \nu$. Then, for every $\varepsilon > 0$ and $\nu \in (0,1)$.
Take $M_1 = \{k \in \mathbb{N} : \varphi\left(s_k - u; r+\frac{\varepsilon}{2}\right) \le 1-\nu$ or $\psi\left(s_k - u; r+\frac{\varepsilon}{2}\right) \ge \nu,\ \varrho\left(s_k - u; r+\frac{\varepsilon}{2}\right) \ge \nu\}$ and $M_2 = \{k \in \mathbb{N} : \varphi\left(s_k - v; r+\frac{\varepsilon}{2}\right) \le 1-\nu$ or $\psi\left(s_k - v; r+\frac{\varepsilon}{2}\right) \ge \nu, \varrho\left(s_k - v; r+\frac{\varepsilon}{2}\right) \ge \nu\}$. Then,

$$\frac{1}{q_n}\left|\{k \le n : k \in M_1 \cup M_2\}\right| \le \frac{1}{q_n}\left|\{k \le n : k \in M_1\}\right| + \frac{1}{q_n}\left|\{k \le n : k \in M_2\}\right|$$

Using $\mathcal{I}-$convergence characteristics, for $\mu > 0$ we have $\{n \in \mathbb{N} : \frac{1}{q_n}\left|\{k \le n : k \in M_1 \cup M_2\}\right| \ge \mu\} \in \mathcal{I}$. For $k \in M_1^c \cap M_2^c$ we have

$$\begin{aligned}\varphi(u-v; 2r+\varepsilon) &\ge \varphi\left(s_k - v; r+\frac{\varepsilon}{2}\right) \oplus \varphi\left(s_k - u; r+\frac{\varepsilon}{2}\right)\\ &> (1-\nu) \oplus (1-\nu)\\ &> 1-\sigma,\end{aligned}$$

$$\begin{aligned}\psi(u-v;2r+\varepsilon) &\leq \psi\left(s_k-v;r+\frac{\varepsilon}{2}\right)\circledast\psi\left(s_k-u;r+\frac{\varepsilon}{2}\right)\\ &<\nu\circledast\nu\\ &<\sigma,\end{aligned}$$

and

$$\begin{aligned}\varrho(u-v;2r+\varepsilon) &\leq \varrho\left(s_k-v;r+\frac{\varepsilon}{2}\right)\circledast\varrho\left(s_k-u;r+\frac{\varepsilon}{2}\right)\\ &<\nu\circledast\nu\\ &<\sigma.\end{aligned}$$

Hence,

$$\varphi(u-v;2r+\varepsilon)>1-\sigma,\ \psi(u-v;2r+\varepsilon)<\sigma \text{ and } \varrho(u-v;2r+\varepsilon)<\sigma. \tag{2.3}$$

Using (2.3),

$$\varphi(u-v;mr)>1-\sigma,\ \psi(u-v;mr)<\sigma \text{ and } \varrho(u-v;mr)<\sigma \text{ for } m>2.$$

which give counterpoint to (2.2).
Hence, $u,v\notin St_q\text{-}\mathcal{I}_{(\varphi,\psi,\varrho)}\text{-}LIM_s^r$ that gives $\varphi(u-v;mr)\leq 1-\sigma$ or $\psi(u-v;mr)\geq\sigma$, $\varrho(u-v;mr)\geq\sigma$ for $m>2$. □

Additionally, we define quasi $\mathcal{I}$-statistically bounded sequence in the framework of NNS as below.

Definition 2.9. *Let $(\mathbb{V},\mathcal{G},\oplus,\circledast)$ be defined as NNS along with neutrosophic norm (φ,ψ,ϱ). Then, a sequence $s=\{s_k\}\in\mathbb{V}$ is regarded as quasi $\mathcal{I}$-statistically bounded with respect to the neutrosophic norm (φ,ψ,ϱ), if for every $\sigma\in(0,1)$, $\mu>0$ and $\varepsilon>0$ there exists $H>0$ real number such that*

$$\left\{n\in\mathbb{N}:\frac{1}{n}|\{k\in\mathbb{N}:\varphi(s_k,H)\leq 1-\sigma \textit{ or } \psi(s_k,H)\geq\sigma,\ \varrho(s_k,H)\geq\sigma\}|\geq\mu\right\}\in\mathcal{I}.$$

Theorem 2.10. *Consider sequence $s=\{s_k\}$ in a NNS $(\mathbb{V},\mathcal{G},\oplus,\circledast)$. Then, a sequence $s=\{s_k\}$ is quasi $\mathcal{I}$-statistically bounded with respect to the norm (φ,ψ,ϱ), iff $St_q\text{-}\mathcal{I}_{(\varphi,\psi,\varrho)}\text{-}LIM_s^r\neq\emptyset$ for some $r>0$.*

Proof. Necessary part:
Let sequence $s=\{s_k\}$ be quasi $\mathcal{I}$-statistically bounded with neutrosophic norm (φ,ψ,ϱ) in NNS $(\mathbb{V},\mathcal{G},\oplus,\circledast)$. For every $\varepsilon>0$ and $\sigma\in(0,1)$, we get $H>0$ which satisfies the condition

$$\left\{n\in\mathbb{N}:\frac{1}{q_n}|\{k\in\mathbb{N}:\varphi(s_k,H)\leq 1-\sigma \text{ or } \psi(s_k,H)\geq\sigma,\ \varrho(s_k,H)\geq\sigma\}|\geq\mu\right\}\in\mathcal{I}.$$

Assume that $K=\{k\in\mathbb{N}:\varphi(s_k;H)\leq 1-\sigma \text{ or } \psi(s_k,H)\geq\sigma,\ \varrho(s_k,H)\geq\sigma\}$.
For $k\in K^c$ we have $\varphi(s_k;H)>1-\sigma$, $\psi(s_k,H)<\sigma$ and $\varrho(s_k,H)<\sigma$.
Also,

$$\begin{aligned}\varphi(s_k;r+H) &\geq \varphi(0;r)\oplus\varphi(s_k;H)\\ &>1\oplus(1-\sigma)\\ &=1-\sigma,\end{aligned}$$

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$$\begin{aligned}\psi(s_k; r+H) &< \psi(0;r) \circledast \psi(s_k;H)\\ &< 0 \circledast \sigma\\ &= \sigma,\end{aligned}$$

and

$$\begin{aligned}\varrho(s_k; r+H) &< \varrho(0;r) \circledast \varrho(s_k;H)\\ &< 0 \circledast \sigma\\ &= \sigma.\end{aligned}$$

Hence, $0 \in St_q\text{-}\mathcal{I}_{(\varphi,\psi,\varrho)}\text{-}LIM_s^r$. Therefore, $St_q\text{-}\mathcal{I}_{(\varphi,\psi,\varrho)}\text{-}LIM_s^r \neq \emptyset$.

Sufficient Part:

Let $St_q\text{-}\mathcal{I}_{(\varphi,\psi,\varrho)}\text{-}LIM_s^r \neq \emptyset$ for some $r > 0$. So some $s^* \in \mathbb{V}$ exists such that $s^* \in St_q\text{-}\mathcal{I}_{(\varphi,\psi,\varrho)}\text{-}LIM_s^r$. Consequently, whenever $\varepsilon > 0$, $\mu > 0$ and $\sigma \in (0,1)$ the following holds

$$\left\{n \in \mathbb{N} : \frac{1}{q_n}|\{\in \mathbb{N} : \varphi(s_k - s^*; r+\varepsilon) \leq 1-\sigma \,\text{or}\, \psi(s_k - s^*, r+\varepsilon) \geq \sigma, \varrho(s_k - s^*, r+\varepsilon) \geq \sigma\}| \geq \mu\right\} \in \mathcal{I}.$$

Here, mostly s_k's belongs to ball having center at s^* *i.e* $s = \{s_k\}$ is quasi $\mathcal{I}$-statistically bounded in the given NNS $(\mathbb{V}, \mathcal{G}, \oplus, \circledast)$. □

Additionally, we describe quasi rough $\mathcal{I}$-statistical cluster point for a sequence in the framework of NNS and establish several related results.

Definition 2.11. *Let $(\mathbb{V}, \mathcal{G}, \oplus, \circledast)$ be defined as NNS along with neutrosophic norm (φ, ψ, ϱ). An element $\zeta \in \mathbb{V}$ is said to be quasi rough $\mathcal{I}$-statistical cluster point of a sequence $s = \{s_k\}$ in $\mathbb{V}$ with respect to the norm (φ, ψ, ϱ) for some $r \geq 0$, if for every $\sigma \in (0,1)$, $\mu > 0$ and $\varepsilon > 0$, satisfying $d^q{}_{\mathcal{I}}(\{k \in \mathbb{N} : \varphi(s_k - \zeta; r+\varepsilon) > 1-\sigma$ and $\psi(s_k - \zeta; r+\varepsilon) < \sigma,\ \varrho(s_k - \zeta; r+\varepsilon) < \sigma\} \neq 0$, where $d^q{}_{\mathcal{I}}(\{.\}) = \mathcal{I} - \lim_{n\to\infty} \frac{1}{q_n} \mid \{.\} \mid$ represents quasi $\mathcal{I}$-asymptotic density of any set $\{.\}$.*
Here, ζ denotes r-$\mathcal{I}$-$St_{(\varphi,\psi,\varrho)}$-cluster point of sequence $s = \{s_k\}$.

Let $\Gamma^{r\text{-}\mathcal{I}}_{(\varphi,\psi,\varrho)}(s)$ denotes the collection of all possible r-$\mathcal{I}$-$St_{(\varphi,\psi,\varrho)}$-cluster points with respect to the norm (φ, ψ, ϱ) for sequence $s = \{s_k\}$ in NNS $(\mathbb{V}, \mathcal{G}, \oplus, \circledast)$. When $r = 0$ the rough $\mathcal{I}$-statistical cluster point becomes $\mathcal{I}$-statistical cluster point with respect to the norm (φ, ψ, ϱ) in NNS $(\mathbb{V}, \mathcal{G}, \oplus, \circledast)$ *i.e.* $\Gamma^{r\text{-}\mathcal{I}}_{(\varphi,\psi,\varrho)}(s) = \Gamma^{\mathcal{I}}_{(\varphi,\psi,\varrho)}(s)$.

Theorem 2.12. *The set $\Gamma^{r\text{-}\mathcal{I}}_{(\varphi,\psi,\varrho)}(s)$ of all r-$St_{(\varphi,\psi,\varrho)}$-cluster points with respect to the norm (φ, ψ, ϱ) for some $r \geq 0$ of a sequence $s = \{s_k\}$ in $\mathbb{V}$, is closed.*

Proof. (i) For $\Gamma^{r\text{-}\mathcal{I}}_{(\varphi,\psi,\varrho)}(s) = \emptyset$, the proof is omitted as it is straightforward.

(ii) For $\Gamma^{r\text{-}\mathcal{I}}_{(\varphi,\psi,\varrho)}(s) \neq \emptyset$ take a sequence $t = \{t_k\} \in \Gamma^{r\text{-}\mathcal{I}}_{(\varphi,\psi,\varrho)}(s)$ such that $t_k \xrightarrow{(\varphi,\psi,\varrho)} t^*$. Here, enough to prove that $t^* \in \Gamma^{r\text{-}\mathcal{I}}_{(\varphi,\psi,\varrho)}(s)$. Now, for $\nu \in (0,1)$ choose $\sigma \in (0,1)$ such that $(1-\sigma) \oplus (1-\sigma) > (1-\nu)$ and $\sigma \circledast \sigma < \nu$.

As $t_k \xrightarrow{(\varphi,\psi,\varrho)} t^*$, then for every $\varepsilon > 0$ and $\sigma \in (0,1)$ we get $k_\varepsilon \in \mathbb{N}$ satisfying $\varphi\left(t_k - t^*; \frac{\varepsilon}{2}\right) > 1-\sigma$,

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$\psi\left(t_k - t^*; \frac{\varepsilon}{2}\right) < \sigma$ and $\varrho\left(t_k - t^*; \frac{\varepsilon}{2}\right) < \sigma$ for $k \geq k_\varepsilon$.
Now, choose $k_0 \in \mathbb{N}$ such that $k_0 \geq k_\varepsilon$. Then, we have

$$\varphi\left(t_{k_0} - t^*; \frac{\varepsilon}{2}\right) > 1 - \sigma, \psi\left(t_{k_0} - t^*; \frac{\varepsilon}{2}\right) < \sigma \text{ and } \varrho\left(t_{k_0} - t^*; \frac{\varepsilon}{2}\right) < \sigma.$$

Again as $t = \{t_k\} \subseteq \mathit{\Gamma}^{r\text{-}\mathcal{I}}_{(\varphi,\psi,\varrho)}(s)$, we have $y_{k_0} \in \mathit{\Gamma}^{r\text{-}\mathcal{I}}_{(\varphi,\psi,\varrho)}(s)$. Then,

$$M = \left\{k \in \mathbb{N} : \varphi\left(s_k - t_{k_0}; r + \frac{\varepsilon}{2}\right) > 1 - \sigma,\ \psi\left(s_k - t_{k_0}; r + \frac{\varepsilon}{2}\right) < \sigma \text{ and } \varrho\left(s_k - t_{k_0}; r + \frac{\varepsilon}{2}\right) < \sigma\right\}$$

with

$$d^q{}_{\mathcal{I}}(M) > 0. \tag{2.4}$$

Let $j \in M = \{k \in \mathbb{N} : \varphi\left(s_k - t_{k_0}; r + \frac{\varepsilon}{2}\right) > 1 - \sigma,\ \psi\left(s_k - t_{k_0}; r + \frac{\varepsilon}{2}\right) < \sigma$ and $\varrho(s_k - t_{k_0}; r + \frac{\varepsilon}{2}) < \sigma\}$, then we have $\varphi\left(s_j - t_{k_0}; r + \frac{\varepsilon}{2}\right) > 1 - \sigma$, $\psi\left(s_j - t_{k_0}; r + \frac{\varepsilon}{2}\right) < \sigma$ and $\varrho\left(s_j - t_{k_0}; r + \frac{\varepsilon}{2}\right) < \sigma$.

$$\begin{aligned}
\varphi(s_j - t^*; r + \varepsilon) &\geq \varphi\left(s_j - t_{k_0}; r + \frac{\varepsilon}{2}\right) \oplus \varphi\left(t_{k_0} - t^*; \frac{\varepsilon}{2}\right) \\
&> (1 - \sigma) \oplus (1 - \sigma) \\
&> 1 - \nu, \\
\psi(s_j - t^*; r + \varepsilon) &\geq \psi\left(s_j - t_{k_0}; r + \frac{\varepsilon}{2}\right) \circledast \psi\left(t_{k_0} - t^*; \frac{\varepsilon}{2}\right) \\
&< \sigma \circledast \sigma \\
&< \nu,
\end{aligned}$$

and

$$\begin{aligned}
\varrho(s_j - t^*; r + \varepsilon) &\geq \varrho\left(s_j - t_{k_0}; r + \frac{\varepsilon}{2}\right) \circledast \varrho\left(t_{k_0} - t^*; \frac{\varepsilon}{2}\right) \\
&< \sigma \circledast \sigma \\
&< \nu.
\end{aligned}$$

Therefore, $j \in \{k \in \mathbb{N} : \varphi(s_k - t^*; r + \varepsilon) > 1 - \nu,\ \psi(s_k - t^*; r + \varepsilon) < \nu$ and $\varrho(s_k - t^*; r + \varepsilon) < \nu\}$.
Hence,

$$\begin{aligned}
&\{k \in \mathbb{N} : \varphi\left(s_k - t_{k_0}; r + \frac{\varepsilon}{2}\right) > 1 - \sigma,\ \psi\left(s_k - t_{k_0}; r + \frac{\varepsilon}{2}\right) < \sigma \text{ and } \varrho\left(s_k - t_{k_0}; r + \frac{\varepsilon}{2}\right) < \sigma\} \\
&\qquad \subseteq \{k \in \mathbb{N} : \varphi(s_k - t^*; r + \varepsilon) > 1 - \nu,\ \psi(s_k - t^*; r + \varepsilon) < \nu \text{ and } \varrho(s_k - t^*; r + \varepsilon) < \nu\}.
\end{aligned}$$

Now,

$$\begin{aligned}
&d^q{}_{\mathcal{I}}(\{k \in \mathbb{N} : \varphi\left(s_k - t_{k_0}; r + \frac{\varepsilon}{2}\right) > 1 - \sigma,\ \psi\left(s_k - t_{k_0}; r + \frac{\varepsilon}{2}\right) < \sigma \text{ and } \varrho\left(s_k - t_{k_0}; r + \frac{\varepsilon}{2}\right) < \sigma\}) \\
&\qquad \leq d^q{}_{\mathcal{I}}(\{k \in \mathbb{N} : \varphi(s_k - t^*; r + \varepsilon) > 1 - \nu,\ \psi(s_k - t^*; r + \varepsilon) < \nu \text{ and } \varrho(s_k - t^*; r + \varepsilon) < \nu\}).
\end{aligned} \tag{2.5}$$

Then, using equations (2.4) and (2.5), the condition following holds

$$d^q{}_{\mathcal{I}}(\{k \in \mathbb{N} : \varphi(s_k - t^*; r + \varepsilon) > 1 - \nu,\ \psi(s_k - t^*; r + \varepsilon) < \nu \text{ and } \varrho(s_k - t^*; r + \varepsilon) < \nu\}) > 0.$$

Evidently, $t^* \in \mathit{\Gamma}^{r\text{-}\mathcal{I}}_{(\varphi,\psi,\varrho)}(s)$. $\square$

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Theorem 2.13. *Let $\Gamma^{\mathcal{I}}_{(\varphi,\psi,\varrho)}(s)$ be defined as set of $\mathcal{I}$-statistical cluster points with respect to the norm (φ,ψ,ϱ) for sequence $s=\{s_k\}$ in $\mathbb{V}$ and $r\geq 0$. Then, for arbitrary $\zeta\in\Gamma^{\mathcal{I}}_{(\varphi,\psi,\varrho)}(s)$ and $\sigma\in(0,1)$ we have $\varphi(\wp-\zeta;r)-\zeta;r)>1-\sigma$, $\psi(\wp-\zeta;\varepsilon)<\sigma$ and $\varrho(\wp-\zeta;r)<\sigma$ for all $\wp\in\Gamma^{r\text{-}\mathcal{I}}_{(\varphi,\psi,\varrho)}(s)$.*

Proof. For $\sigma\in(0,1)$ choose $\nu\in(0,1)$ such that $(1-\nu)\oplus(1-\nu)>1-\sigma$ and $\nu\circledast\nu<\sigma$. If $\zeta\in\Gamma^{\mathcal{I}}_{(\varphi,\psi,\varrho)}(s)$ then for $\varepsilon>0$ and $\nu\in(0,1)$ satisfies the following condition

$$d^q{}_{\mathcal{I}}(\{k\in\mathbb{N}:\varphi(s_k-\zeta;\varepsilon)>1-\nu,\ \psi(s_k-\zeta;\varepsilon)<\nu \text{ and } \varrho(s_k-\zeta;\varepsilon)<\nu\})>0. \tag{2.6}$$

Further, we prove if $\wp\in\mathbb{V}$ with $\varphi(\wp-\zeta;r)>1-\nu$, $\psi(\wp-\zeta;r)<\nu$ and $\varrho(\wp-\zeta;r)<\nu$ the element $\wp\in\Gamma^{r\text{-}\mathcal{I}}_{(\varphi,\psi,\varrho)}(s)$.
Take $j\in\{k\in\mathbb{N}:\varphi(s_k-\zeta;\varepsilon)>1-\nu,\ \psi(s_k-\zeta;\varepsilon)<\nu$ and $\varrho(s_k-\zeta;\varepsilon)<\nu\}$, then $\varphi(s_j-\zeta;\varepsilon)>1-\nu$, $\psi(s_j-\zeta;\varepsilon)<\nu$ and $\varrho(s_j-\zeta;\varepsilon)<\nu$. Now,

$$\begin{aligned}\varphi(s_j-\wp;r+\varepsilon)&\geq\varphi(s_j-\zeta;\varepsilon)\oplus\varphi(\wp-\zeta;r)\\&>(1-\nu)\oplus(1-\nu)\\&>1-\sigma,\end{aligned}$$

$$\begin{aligned}\psi(s_j-\wp;r+\varepsilon)&\leq\psi(s_j-\zeta;\varepsilon)\circledast\psi(\wp-\zeta;r)\\&<\nu\circledast\nu\\&<\sigma,\end{aligned}$$

and

$$\begin{aligned}\varrho(s_j-\wp;r+\varepsilon)&\leq\varrho(s_j-\zeta;\varepsilon)\circledast\varrho(\wp-\zeta;r)\\&<\nu\circledast\nu\\&<\sigma.\end{aligned}$$

Therefore, $j\in\{k\in\mathbb{N}:\varphi(s_k-\wp;r+\varepsilon)>1-\sigma,\ \psi(s_k-\wp;\varepsilon)<\sigma \text{ and } \varrho(s_k-\wp;\varepsilon)<\sigma\}$.
Consequently,

$$\begin{aligned}&\{k\in\mathbb{N}:\varphi(s_k-\zeta;\varepsilon)>1-\nu,\ \psi(s_k-\zeta;\varepsilon)<\nu \text{ and } \varrho(s_k-\zeta;\varepsilon)<\nu\}\\&\subseteq\{k\in\mathbb{N}:\varphi(s_k-\wp;r+\varepsilon)>1-\sigma,\ \psi(s_k-\wp;r+\varepsilon)<\sigma \text{ and } \varrho(s_k-\wp;r+\varepsilon)<\sigma\}.\end{aligned}$$

Then,

$$\begin{aligned}&d^q{}_{\mathcal{I}}(\{k\in\mathbb{N}:\varphi(s_k-\zeta;\varepsilon)>1-\nu,\ \psi(s_k-\zeta;\varepsilon)<\nu \text{ and } \varrho(s_k-\zeta;\varepsilon)<\nu\})\\&\leq d^q{}_{\mathcal{I}}(\{k\in\mathbb{N}:\varphi(s_k-\wp;r+\varepsilon)>1-\sigma,\ \psi(s_k-\wp;\varepsilon)<t \text{ and } \varrho(s_k-s^*;r+\varepsilon)<\sigma\}).\end{aligned}$$

By equation (2.6), we have

$$d^q{}_{\mathcal{I}}(\{k\in\mathbb{N}:\varphi(s_k-\wp;r+\varepsilon)>1-\sigma,\ \psi(s_k-\wp;r+\varepsilon)<\sigma \text{ and } \varrho(s_k-\wp;r+\varepsilon)<\sigma\})>0.$$

Therefore, $\wp\in\Gamma^{r\text{-}\mathcal{I}}_{(\varphi,\psi,\varrho)}(s)$. □

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3. Conclusion

In this paper, we introduce a significant convergence structure known as quasi-ideal statistical convergence on sequences in neutrosophic normed spaces. We investigate some of its basic properties and justfied with suitable examples. The importance of introducing ideal convergence within this framework lies in providing a new mathematical tool for addressing convergence problems, particularly those arising from practical issues such as factual incompleteness, indeterminacy, and data inconsistency. Future research may focus on exploring further extensions of quasi-ideal statistical convergence to higher-dimensional settings, as well as examining its applicability to practical problems in decision theory, optimization, and related fields (see [9, 12, 14, 19, 23, 41]).

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