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SIMPLICITY OF FULL CENTRALIZER NEARRINGS AND EXPONENT-PRESERVING GROUPS

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ABSTRACT. The simplicity of $M_S(G)$, the centralizer nearring determined by a finite group G and a semigroup S of endomorphisms of G, is determined for the cases of $S = \operatorname{Inn} G$ and $S = \operatorname{Aut} G$. The concept of exponent-preserving (EP) groups is defined and used to study the simplicity of $M_E(G)$, where $E = \operatorname{End} G$ and G has a cyclic Sylow p-subgroup.

1. **Introduction**

Let G be a group written additively with identity 0, but not necessarily abelian. For a semigroup S of endomorphisms of G, the set of functions $M_S(G) = \{f : G \to G \mid f(0) = 0 \text{ and } f \circ s = s \circ f \text{ for all } s \in S\}$ forms a right nearring under function addition and composition, called the centralizer nearring determined by G and S. Every nearring with identity is isomorphic to an $M_S(G)$ for some G and S ([3, Theorem 14.3]). Thus, centralizer nearrings are integral in nearring theory. For more information on nearrings, see [3, 8], and [9].

For $E = \operatorname{End} G$, the nearrings $M_E(G)$ are the smallest centralizer nearrings for a given group G and were investigated in [1] and [2]. The structure of $M_E(G)$ was explored for various classes of finite groups, and the localness and simplicity of $M_E(G)$ were also considered. Fundamental results about centralizer nearrings determined by general automorphism groups and endomorphism semigroups can be found in [6] and [7]. We continue the study of the simplicity of centralizer nearrings, but restrict

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our attention to the full centralizer nearrings, i.e., the centralizer nearrings determined by $I = \operatorname{Inn} G$, $A = \operatorname{Aut} G$, and $E = \operatorname{End} G$.

We assume that G is a finite group. Throughout, we use the common notations $Z(G) = \{a \in G \mid a + g = g + a \text{ for all } g \in G\}$ for the center of G and $C_G(g) = \{a \in G \mid a + g = g + a\}$ for the centralizer of g in G. When discussing centralizer nearrings, we use 0 for the additive identity of G and by extension, the additive identity of the nearring $M_S(G)$. We let G^* be the nonzero elements of G, id denote the identity function from G to G, and $\exp G$ be the exponent of G.

2. **Full centralizer nearrings determined by automorphisms**

In this section we consider the nearrings $M_I(G)$ and $M_A(G)$, where $I = \operatorname{Inn} G$ and $A = \operatorname{Aut} G$. This first result appears in [7].

Theorem 2.1. *Let G be a finite group, and let S be a group of automorphisms of G. If $M_S(G)$ is simple and S does not act fixed point free on G, then G is an elementary abelian p-group for some prime p.*

Lemma 2.2. *Let G be a finite group, and let S be a group of automorphisms of G such that $\operatorname{Inn} G \subseteq S$. If $M_S(G)$ is simple, then G is abelian.*

Proof. Assume that $M_S(G)$ is simple. If $\operatorname{Inn} G$ does not act fixed point free on G, then S does not act fixed point free on G. Thus, G is elementary abelian by Theorem 2.1. Now assume that $\operatorname{Inn} G$ does act fixed point free on G. Consider the inner automorphism on G determined by the element g, i.e., $\varphi_g(x) = -g + x + g$. Then $\varphi_x(x) = -x + x + x = x$, and x is a fixed point of φ_x for each $x \in G$. Since $\operatorname{Inn} G$ acts fixed point free on G, then $\varphi_x = id$ for each $x \in G$. So for any $y \in G$, $y = \varphi_x(y) = -x + y + x$ and $y + x = x + y$. Thus G is an abelian group in this case as well. □

With these results, we can characterize when $M_I(G)$ is simple.

Theorem 2.3. *Let G be a finite group. The following are equivalent:*

(i) *$M_I(G)$ is simple;*
(ii) *G is an abelian group;*
(iii) *$M_I(G) = M_0(G)$, where $M_0(G) = \{f : G \to G \mid f(0) = 0\}$.*

Proof. If $M_I(G)$ is simple, then G is abelian by Lemma 2.2. So (i) implies (ii). If G is abelian, then $\operatorname{Inn} G = \{id\}$ and $M_I(G) = M_0(G)$. So (ii) implies (iii). If $M_I(G) = M_0(G)$, then $M_I(G)$ is simple by Theorem 1.40 of [8]. We conclude that (iii) implies (i), and the proof is complete. □

Now we consider $M_A(G)$.

Lemma 2.4. *Let $x \in G^*$ and let S be a group of automorphisms of G. If G^* is a single orbit under the action of S on G, then a function $f \in M_S(G)$ is completely determined by the single value $f(x)$.*

Proof. Fix $x, y \in G^*$ and let $f \in M_S(G)$. Since G^* is a single orbit, there is an automorphism $\theta \in S$ such that $\theta(x) = y$. So $f(y) = f\theta(x) = \theta f(x)$ and f is completely determined by $f(x)$. □

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Lemma 2.5. *Let $G = \mathbb{Z}_p$ where p is a prime number. Then $M_A(G) \cong \mathbb{Z}_p$.*

Proof. For $k \in G^*$, we define $\phi_k : G \to G$ by $\phi_k(x) = kx$, the product of elements in the ring $\mathbb{Z}_p$. Then $\phi_k \in \operatorname{Aut} G$ and $\phi_k(1) = k$. Thus every element in G^* is in the same orbit as 1, making G^* a single orbit. By the previous lemma, $f \in M_A(G)$ is completely determined by $f(1)$. So there are at most p functions in $M_A(G)$. But since $id \in M_A(G)$, we have $\langle id \rangle \subseteq M_A(G)$, and there are at least p functions in $M_A(G)$. We conclude that $M_A(G) = \langle id \rangle \cong \mathbb{Z}_p$. □

Lemma 2.6. *Let G be a finite abelian group such that $\operatorname{Aut} G$ acts fixed point free on G. Then $G = \mathbb{Z}_p$ for some prime p.*

Proof. Assume $G \neq \mathbb{Z}_p$. Suppose that $G = \mathbb{Z}_{p^n}$ for some prime p and integer $n \geq 2$. Then the mapping $\alpha : G \to G$ given by $\alpha(x) = (p^{n-1} + 1)x$ is a nonidentity automorphism with fixed point p, a contradiction.

Suppose $G = \mathbb{Z}_{p_1^{n_1}} \times \mathbb{Z}_{p_2^{n_2}} \times \cdots \times \mathbb{Z}_{p_k^{n_k}}$ where the p_i are not necessarily distinct primes, $n_i \geq 1$, $k \geq 2$, and at least one direct factor is not $\mathbb{Z}_2$. Without loss of generality, assume $\mathbb{Z}_{p_1^{n_1}} \neq \mathbb{Z}_2$. Consider any invertible element $1 \neq s \in \mathbb{Z}_{p_1^{n_1}}$. Then the mapping $\beta : G \to G$ given by $\beta(x_1, x_2, \ldots, x_k) = (sx_1, x_2, \ldots, x_k)$ is a nonidentity automorphism with fixed point $(0, 1, 1, \ldots, 1)$, a contradiction.

The only case left to consider is $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ with $k \geq 2$ direct factors. Define a mapping $\gamma : G \to G$ by $\gamma(x_1, x_2, x_3, \ldots, x_k) = (x_k, x_1, x_2, \ldots, x_{k-1})$. Then γ is a nonidentity automorphism of G with fixed point $(1, 1, \ldots, 1)$, a contradiction.

In all cases we reach a contradiction. Therefore, $G = \mathbb{Z}_p$ for some prime p. □

Fixed points of endomorphisms will be useful in determining the simplicity of $M_A(G)$ and describing necessary conditions for the situation $M_E(G)$ is simple. First, we need a definition.

Definition 2.7. *For $\varphi \in \operatorname{End} G$, we define $\operatorname{Fix} \varphi = \{x \in G \mid \varphi(x) = x\}$.*

Lemma 2.8. *Let $\varphi \in S \subseteq \operatorname{End} G$ and $f \in M_S(G)$. Then $f(\operatorname{Fix} \varphi) \subseteq \operatorname{Fix} \varphi$.*

Proof. Let $f \in M_S(G)$, $\varphi \in S$, and $x \in \operatorname{Fix} \varphi$. Then $x = \varphi(x)$ and $f(x) = f\varphi(x) = \varphi f(x)$. Thus $f(x) \in \operatorname{Fix} \varphi$. □

We can now characterize when $M_A(G)$ is simple.

Theorem 2.9. *Let G be a finite group. The following are equivalent:*

(i) *$M_A(G)$ is simple;*

(ii) *G is an elementary abelian p-group for some prime p;*

(iii) *$M_A(G) \cong \mathbb{Z}_p$ for some prime p.*

Proof. Assume that $M_A(G)$ is simple. By Lemma 2.2, G is abelian. If $\operatorname{Aut} G$ acts fixed point free on G, then $G = \mathbb{Z}_p$ for some prime p by Lemma 2.6. If $\operatorname{Aut} G$ does not act fixed point free on G, then G is an elementary abelian p-group for some prime p by Theorem 2.1. In either case, G is an elementary abelian p-group. Thus (i) implies (ii).

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Now assume that G is elementary abelian. If $G = \mathbb{Z}_p$, then $M_A(G) \cong \mathbb{Z}_p$ by Lemma 2.5. So assume there are $n \geq 2$ copies of $\mathbb{Z}_p$ in the direct product for G. Consider G as a vector space of dimension n over the field $\mathbb{Z}_p$. Let $b = (1, 1, \dots, 1) \in G$ and $c \in G^*$. Then we can create an automorphism α of G by linearly extending a mapping from one basis of G containing b to another basis containing c with $\alpha(b) = c$. Since $c \in G^*$ is chosen arbitrarily and $\alpha(b) = c$, G^* consists of one orbit under the action of $\operatorname{Aut} G$ on G. Thus for each $f \in M_A(G)$, f is completely determined by $f(b)$ by Lemma 2.4.

Define a mapping $\phi : G \to G$ by $\phi(x_1, x_2, \dots, x_{n-1}, x_n) = (x_n, x_1, x_2, \dots, x_{n-1})$. Then ϕ is an automorphism of G and $\operatorname{Fix} \phi = \{(x, x, \dots, x) \mid x \in \mathbb{Z}_p\}$. Note that $b = (1, 1, \dots, 1) \in \operatorname{Fix} \phi$. For any $f \in M_A(G)$, $f(b) \in \operatorname{Fix} \phi$ by Lemma 2.8. As f is completely determined by $f(b)$ and $|\operatorname{Fix} \phi| = p$, we conclude that $|M_A(G)| \leq p$. Since $id \in M_A(G)$, it follows that $\langle id \rangle \subseteq M_A(G)$. Hence there are at least p functions in $M_A(G)$. Therefore $M_A(G) = \langle id \rangle \cong \mathbb{Z}_p$, and (ii) implies (iii).

Condition (iii) clearly implies (i). This completes the proof. □

3. Exponent-preserving groups

In studying the simplicity of $M_E(G)$, the exponents of certain subsets of G will play a major role. We formalize this in the next definition.

Definition 3.1. *A finite group G with identity 0 is an exponent-preserving (EP) group if G satisfies*

(i) $\exp[\operatorname{Ker} \phi] = \exp[\operatorname{Im} \phi] = \exp G$ *for all* $0 \neq \phi \in \operatorname{End} G \setminus \operatorname{Aut} G$*;*

(ii) $\exp[\operatorname{Fix} \varphi] = \exp G$ *for all* $\varphi \in \operatorname{End} G$ *with* $\operatorname{Fix} \varphi \neq \{0\}$*; and*

(iii) $\exp[H_1 \cap H_2 \cap \cdots \cap H_k] = \exp G$ *where* $H_1 \cap H_2 \cap \cdots \cap H_k \neq \{0\}$ *and each* $H_i \in \{\operatorname{Ker} \alpha, \operatorname{Im} \beta, \operatorname{Fix} \gamma\}$ *with* $0 \neq \alpha, \beta \in \operatorname{End} G \setminus \operatorname{Aut} G$ *and* $\operatorname{Fix} \gamma \neq \{0\}$.

In particular, the subgroups $Z(G)$ and $C_G(g)$ will be of special interest.

Lemma 3.2. *Let G be a finite group. If G is an EP group, then*

(i) $\exp[C_G(g)] = \exp G$ *for all* $g \in G$*; and*

(ii) *if* $Z(G) \neq \{0\}$*, then* $\exp[Z(G)] = \exp G$.

Proof. (i) Let φ_g be the inner automorphism of G determined by g. Then $\operatorname{Fix} \varphi_g = C_G(g) \neq \{0\}$. Thus $\exp[C_G(g)] = \exp G$ by condition (ii) of the definition of an EP group.

(ii) Note that $Z(G) = \bigcap_{g \in G} C_G(g) = \bigcap_{g \in G} \operatorname{Fix} \varphi_g$. Since $Z(G) \neq \{0\}$, by condition (iii) of the definition of an EP group, we get $\exp[Z(G)] = \exp G$. □

Note that if there exists $z \in Z(G)$ with $|z| = \exp G$, then both conditions of Lemma 3.2 are satisfied. This does not imply, however, that G is an EP group, for if $G = \mathbb{Z}_2 \times \mathbb{Z}_3$, then the projection map π_1 from G onto the first component $\mathbb{Z}_2$ is a nonzero endomorphism with $\exp[\operatorname{Im} \pi_1] \neq \exp G$.

Definition 3.3. *Let $X \subseteq G$. Then the annihilator of X in $M_E(G)$ is* $\operatorname{Ann} X = \{f \in M_E(G) \mid f(x) = 0$ *for all* $x \in X\}$.

Lemma 3.4. *Let* $\{0\} \neq X \subseteq G$ *such that* $f(X) \subseteq X$ *for all* $f \in M_E(G)$*. If* $M_E(G)$ *is simple, then* $\exp X = \exp G$.

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Proof. It is straightforward to verify that $\operatorname{Ann} X$ is a left ideal of $M_E(G)$. Since $f(X) \subseteq X$ for all $f \in M_E(G)$, it follows that $\operatorname{Ann} X$ is a right ideal of $M_E(G)$, hence an ideal. As $X \neq \{0\}$, we conclude that $id \notin \operatorname{Ann} X$, and $\operatorname{Ann} X$ is a proper ideal of $M_E(G)$.

Let $n = \exp X$ and assume that $n < \exp G$. Then $0 \neq n \cdot id \in \operatorname{Ann} X$, and $\operatorname{Ann} X$ is a nonzero, proper ideal of $M_E(G)$, contradicting the simplicity of $M_E(G)$. Thus $\exp X = \exp G$. □

We end this section by showing G being an EP group is a necessary condition for $M_E(G)$ to be simple.

Theorem 3.5. *Let G be a finite group. If $M_E(G)$ is simple, then G is an EP group.*

Proof. Let $0 \neq \phi \in \operatorname{End} G \setminus \operatorname{Aut} G$. Then $\operatorname{Ker} \phi \neq \{0\}$ and $\operatorname{Im} \phi \neq \{0\}$. Since $f(\operatorname{Ker} \phi) \subseteq \operatorname{Ker} \phi$ and $f(\operatorname{Im} \phi) \subseteq \operatorname{Im} \phi$ for all $f \in M_E(G)$, condition (i) of the definition of an EP group follows from Lemma 3.4. Let $\varphi \in \operatorname{End} G$ with $\operatorname{Fix} \varphi \neq \{0\}$. Then $f(\operatorname{Fix} \varphi) \subseteq \operatorname{Fix} \varphi$ for all $f \in M_E(G)$ by Lemma 2.8, and condition (ii) follows from Lemma 3.4. For condition (iii) of the definition of an EP group, for each of the given H_i, $f(H_i) \subseteq H_i$ for all $f \in M_E(G)$. Thus $f[H_1 \cap H_2 \cap \cdots \cap H_k] \subseteq H_1 \cap H_2 \cap \cdots \cap H_k$. Condition (iii) now follows from the assumption that $H_1 \cap H_2 \cap \cdots \cap H_k \neq \{0\}$ and Lemma 3.4. □

4. Full centralizer nearrings determined by endomorphisms

For the remainder of the paper, we consider finite groups G with a nontrivial cyclic Sylow p-subgroup for some prime p. For example, if x is a 5-cycle in S_5, then $\langle x \rangle$ is a cyclic Sylow 5-subgroup of S_5. Also, if $G = \mathbb{Z}_{p^n} \times H$ where $p \nmid |H|$, then $\mathbb{Z}_{p^n}$ is a cyclic Sylow p-subgroup of G.

If G is a finite simple group, then $M_E(G)$ is simple if and only $G = \mathbb{Z}_p$ for some prime p (see [1]). Therefore, we also assume that G has a nontrivial proper normal subgroup. We begin the study of the simplicity of $M_E(G)$ when the cyclic Sylow p-subgroup is normal in G.

Theorem 4.1. *Let G be a finite group with a nontrivial normal cyclic Sylow p-subgroup for some prime p. If $M_E(G)$ is simple, then $G \cong \mathbb{Z}_p$.*

Proof. Let G be a finite group with a nontrivial normal cyclic Sylow p-subgroup, $S = \langle s \rangle$, and assume that $M_E(G)$ is simple. Then G is an EP group by Theorem 3.5. If $\operatorname{End} G = \operatorname{Aut} G \cup \{0\}$, then G is an elementary abelian p-group by Theorem 2.9. Since S is cyclic, it follows that $G \cong \mathbb{Z}_p$. So assume that there exists $0 \neq \varphi \in \operatorname{End} G \setminus \operatorname{Aut} G$. If G is a p-group, then $\exp G = p$ by Lemma 2.4 of [2]. Since S is cyclic, we conclude that $G \cong \mathbb{Z}_p$. So assume that G is not a p-group. We note that $|G| = |\operatorname{Ker} \varphi||\operatorname{Im} \varphi|$. If $s \in \operatorname{Ker} \varphi$, then $S \subseteq \operatorname{Ker} \varphi$ and p does not divide $|\operatorname{Im} \varphi|$. Thus $\exp[\operatorname{Im} \phi] \neq \exp G$, contradicting that G is an EP group. If $s \notin \operatorname{Ker} \varphi$, then $s \in \operatorname{Im} \varphi$. Thus $S \subseteq \operatorname{Im} \varphi$ and p does not divide $|\operatorname{Ker} \varphi|$. Thus $\exp[\operatorname{Ker} \phi] \neq \exp G$, again contradicting that G is an EP group. This completes the proof. □

The next result lists elementary facts from group theory. We leave their proofs to the reader.

Lemma 4.2. *Let N be a normal subgroup of G. Then $C_G(N)$, $Z(N)$, and $Z(C_G(N))$ are normal subgroups of G.*

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Lemma 4.3. *Let G be an EP group with $\exp G = m$ and $Z(G) = \{0\}$.*

(i) *For all $S \subseteq G$ such that $C_G(S) \neq \{0\}$, there is an element $x \in C_G(S)$ such that $|x| = m$.*

(ii) *Furthermore, $y \in C_G(S)$ with $|y| = m$ can be chosen so that $C_G(y)$ is maximal in $L = \{C_G(g) \mid g \in G^*\}$.*

Proof. (i) Suppose that $m = p_1^{k_1} p_2^{k_2} \cdots p_n^{k_n}$ where $p_1, \ldots, p_n$ are distinct primes. Consider $S \subseteq G$ such that $C_G(S) \neq \{0\}$. Thus $\exp[C_G(S)] = m$, and there exists $x_1 \in C_G(S)$ such that $|x_1| = p_1^{k_1}$. Since $x_1 \in C_G(S \cup \{x_1\})$, it follows that $C_G(S \cup \{x_1\}) \neq \{0\}$. Since G is an EP group, we conclude that $\exp[C_G(S \cup \{x_1\})] = m$. Hence, we can find $x_2 \in C_G(S \cup \{x_1\})$ so that $|x_2| = p_2^{k_2}$. Since $x_2 \in C_G(S \cup \{x_1, x_2\})$, it follows that $C_G(S \cup \{x_1, x_2\}) \neq \{0\}$. Since G is an EP group, we conclude that $\exp[C_G(S \cup \{x_1, x_2\})] = m$. Continuing in this manner, we eventually choose $x_n \in C_G(S \cup \{x_1, \cdots, x_{n-1}\})$ so that $|x_n| = p_n^{k_n}$.

Let $x = x_1 + x_2 + \cdots + x_n$. Since each $x_i \in C_G(S)$, it follows that $x \in C_G(S)$. Furthermore, for $i \geq 2$, x_i commutes with all elements $x_1, \cdots, x_{i-1}$. Using this fact, it is straightforward to verify that $|x| = m = \exp G$.

(ii) Using S and $x \in C_G(S)$ as in part (i), we get $S \subseteq C_G(x)$. As L is finite, there exists $g \in G^*$ such that $S \subseteq C_G(x) \subseteq C_G(g)$ and $C_G(g)$ is maximal in L. Note that for all $x \in G^*$, $C_G(x) \neq G$ since $x \notin Z(G) = \{0\}$.

Consider $S_1 = C_G(g)$. Then $C_G(S_1) \neq \{0\}$ since $g \in C_G(S_1)$. By part (i), there exists $y \in C_G(S_1)$ such that $|y| = m$. Hence $S_1 \subseteq C_G(y)$ and $C_G(g) = S_1 \subseteq C_G(y)$. By the maximality of $C_G(g)$ in L, we must have $C_G(g) = C_G(y)$. So $S \subseteq C_G(x) \subseteq C_G(g) = C_G(y)$. Therefore, $y \in C_G(S)$, $|y| = m$, and $C_G(y)$ is maximal in L. □

Definition 4.4. *As described in the previous lemma, for each $S \subseteq G$ such that $C_G(S) \neq \{0\}$, we can choose $y \in C_G(S)$ with $|y| = m = \exp G$ and $C_G(y)$ is maximal in L. Let $C_G(y_1), C_G(y_2), \ldots, C_G(y_k)$ be all distinct sets with these properties. We define $Y_L = \{y_1, y_2, \ldots, y_k\}$.*

We next consider the case where G has a normal subgroup that is not necessarily a Sylow p-subgroup.

Theorem 4.5. *Let G be a finite group having a nontrivial cyclic Sylow p-subgroup for some prime p, and suppose that there exists a normal subgroup $\{0\} \neq N$ of G such that $C_G(N) \neq \{0\}$. If $M_E(G)$ is simple, then $G = \mathbb{Z}_p$.*

Proof. Assume that $M_E(G)$ is simple. By Theorem 3.5, G is an EP group. Assume $Z(G) \neq \{0\}$. By Lemma 3.2, $\exp[Z(G)] = \exp G$. Let p^n be the maximum power of p dividing $\exp G$. Then there exists an element $g \in Z(G)$ with $|g| = p^n$. Then $\langle g \rangle \subseteq Z(G)$ is a nontrivial normal cyclic Sylow p-subgroup. By Theorem 4.1, $G = \mathbb{Z}_p$.

Now assume $Z(G) = \{0\}$. Suppose that $\{0\} \neq N$ is a normal subgroup of G with $C_G(N) \neq \{0\}$. By Lemma 4.3, there exists $y_1 \in Y_L$ such that $y_1 \in C_G(N)$. Let $S = C_G(N)$. Since $N \leq C_G(C_G(N)) = C_G(S)$, we conclude that $C_G(S) \neq \{0\}$. By Lemma 4.3, there exists $y_2 \in Y_L$ with $y_2 \in C_G(C_G(N))$. It follows that y_1 and y_2 commute.

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Assume $\langle y_1\rangle\cap\langle y_2\rangle = \{0\}$. Since $|y_1| = |y_2| = \exp G$, there exist integers n_1 and n_2 such that $|n_1y_1| = |n_2y_2| = p$. As y_1 and y_2 commute, it follows that n_1y_1 and n_2y_2 commute. Hence $\langle n_1y_1, n_2y_2\rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p$, and the Sylow p-subgroup is not cyclic, a contradiction. Thus $\langle y_1\rangle \cap \langle y_2\rangle \neq \{0\}$, say $0 \neq w \in \langle y_1\rangle \cap \langle y_2\rangle$. Thus $C_G(y_1) \subseteq C_G(w)$ and $C_G(y_2) \subseteq C_G(w)$. Since $y_1, y_2 \in Y_L$, it follows that $C_G(y_1) = C_G(w) = C_G(y_2)$. Furthermore, $y_1 = y_2$ since representatives in Y_L are distinct.

Since $y_1 \in C_G(N)$ and $y_2 \in C_G(C_G(N))$, we conclude that $y_1 = y_2 \in Z(C_G(N))$. Let $\exp G = p^n t$ where $p \nmid t$. As $|y_1| = \exp G$, we have $|ty_1| = p^n$ and $Z(C_G(N))$ has a Sylow p-subgroup, S, containing ty_1. Since S is contained in a Sylow p-subgroup of G, we conclude that S is cyclic. Furthermore, $\exp S = p^n$ so that $S = \langle ty_1\rangle$ and S is a Sylow p-subgroup of G. It follows that $\langle ty_1\rangle$ is a characteristic subgroup of the abelian group $Z(C_G(N))$. By Lemma 4.2, $Z(C_G(N))$ is a normal subgroup of G. Since $\langle ty_1\rangle$ is characteristic in $Z(C_G(N))$, we conclude $\langle ty_1\rangle$ is normal in G. By Theorem 4.1, $G = \mathbb{Z}_p$. □

Definition 4.6 ([10], p. 26)**.** *For a group G, the outer automorphism group of G is* $\operatorname{Out} G = \operatorname{Aut} G/\operatorname{Inn} G$.

We continue with two lemmas concerning groups.

Lemma 4.7. *Let A be a finite simple nonabelian group.*

- (i) *There exist primes p and q which divide $|A|$, but do not divide $|\operatorname{Out} A|$, and such that A contains no element of order pq.*
- (ii) *If $p \nmid |A|$ and $p \nmid |\operatorname{Out} A|$ for some prime p, then $p \nmid |\operatorname{Inn} A|$ and $p \nmid |\operatorname{Aut} A|$.*

Proof. Item (i) is a restatement of Theorem 2 in [11] in our context. For (ii), since A is a simple group, we have $Z(A) = \{0\}$. Hence $A \cong A/Z(A) \cong \operatorname{Inn} A$. As $p \nmid |A|$, we conclude that $p \nmid |\operatorname{Inn} A|$. Since $p \nmid |\operatorname{Out} A|$, it follows that $p \nmid |\operatorname{Aut} A|$. □

Lemma 4.8. *Let G be a finite group with a nonabelian normal subgroup N such that $C_G(N) = \{0\}$. Also, for any $g \in G$, let φ_g be the inner automorphism of G determined by g. Then there exists an embedding $G \hookrightarrow \operatorname{Aut} N$ via the assignment $g \mapsto \varphi_g|_N$.*

Proof. Since N is normal in G, $\varphi_g(N) = N$ and $\varphi_g|_N \in \operatorname{Aut} N$. Let $f : G \to \operatorname{Aut} N$ be defined by $f(g) = \varphi_g|_N$. Then f is a homomorphism of groups and $\operatorname{Ker} f = C_G(N) = \{0\}$. So f is a monomorphism and $G \hookrightarrow \operatorname{Aut} N$. □

In light of Theorem 4.5, we only need to consider normal subgroups N of G with $C_G(N) = \{0\}$. The next result adds the further restriction that N is a simple subgroup of G.

Lemma 4.9. *Let G be a finite group with a nonabelian normal simple subgroup A such that $C_G(A) = \{0\}$. Then $M_E(G)$ is not simple.*

Proof. Arguing by contradiction, we assume $M_E(G)$ is simple. Thus G is an EP group by Theorem 3.5. By Lemma 4.7, there exist primes p and q which divide $|A|$, but do not divide $|\operatorname{Out} A|$, and such that A contains no element of order pq. Since $p \,|\, |A|$, there exists $a \in A$ with $|a| = p$. We know

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$C_G(a) \neq \{0\}$ since $a \in C_G(a)$. Since G is an EP group, $\exp[C_G(a)] = \exp G = pqt$ for some $t \in \mathbb{Z}^+$. Thus, there exists $g \in C_G(a)$ such that $|g| = q$.

Using the embedding from Lemma 4.8, consider $\varphi_g|_A \in \operatorname{Aut} A$. If $\varphi_g|_A$ is not an inner automorphism of A, then $\operatorname{Inn} A \neq \varphi_g|_A(\operatorname{Inn} A) \in \operatorname{Out} A$. It follows that $|\varphi_g|_A(\operatorname{Inn} A)| = q$, a contradiction, since $q \nmid |\operatorname{Out} A|$. Hence $\varphi_g|_A \in \operatorname{Inn} A$. Therefore $\varphi_g|_A = \varphi_x$ for some $x \in A$, where φ_x is the inner automorphism of A determined by x. Thus $\varphi_g|_A(a) = \varphi_x(a)$ implies $-g + a + g = -x + a + x$. Since $g \in C_G(a)$, we get $a = -x + a + x$ and $x + a = a + x$. So $x \in C_G(a)$.

Since $|g| = q$, we conclude that $(\varphi_g|_A)^q = id_A$. Thus $|\varphi_g|_A| \in \{1, q\}$. Hence $|\varphi_x| \in \{1, q\}$ and $|x| \in \{1, q\}$. We have $x + a \in A$. If $|x| = q$, then $|x + a| = pq$, a contradiction. So $|x| = 1$ and $\varphi_g|_A = \varphi_0 = id_A$. Therefore for all $b \in A$, $-g + b + g = \varphi_g|_A(b) = id_A(b) = b$. We conclude that $b + g = g + b$ and $g \in C_G(A) = \{0\}$. This contradicts $|g| = q$. Therefore $M_E(G)$ is not simple. □

Lemma 4.10. *Let G be a finite group with a nonabelian minimal normal subgroup N. Then*

(i) $N = \underbrace{A \times A \times \cdots \times A}_{k \text{ times}}$, *where A is a nonabelian simple group; and*

(ii) $\operatorname{Aut} N = [\underbrace{\operatorname{Aut} A \times \operatorname{Aut} A \times \cdots \times \operatorname{Aut} A}_{k \text{ times}}]S_k$, *where the symmetric group S_k acts on the direct product by permuting the factors* $\operatorname{Aut} A$.

Proof. Condition (i) appears in [10] (p. 87), while condition (ii) is given in [5]. □

Lemma 4.11. *Let G be a finite EP group with a minimal normal subgroup $N = \underbrace{A \times A \times \cdots \times A}_{k \text{ times}}$, where A is a nonabelian simple group, $C_G(N) = \{0\}$, and p is a prime number with $p \nmid |A|$ and $p \nmid |\operatorname{Aut} A|$. Then*

(i) $k - (p-1)(|A| - 1) \geq p$; *and*

(ii) $k > 2(p-1)(|A| - 1)$.

Proof. (i) Assume $k \leq (p-1)|A|$. Then there exist non-negative integers r_1, r_2 such that $k = r_1|A| + r_2$ with $0 \leq r_2 < |A|$. Let $A = \{a_1, \ldots, a_m\}$ and consider the element $x \in N$ given by $x = (\underbrace{a_1, \ldots, a_m, \ldots, a_1, \ldots, a_m}_{r_1 \text{ repetitions}}, a_1, \ldots, a_{r_2})$.

Let $g \in G$ with $|g| = p$. Using the embedding from Lemma 4.8, we get $|\varphi_g|_N| \in \{1, p\}$. If $|\varphi_g|_N| = 1$, then $\varphi_g|_N = id_N$. In this case, $g \in C_G(N) = \{0\}$, a contradiction since $|g| = p$. Thus, $|\varphi_g|_N| = p$.

Since $p \nmid |\operatorname{Aut} A|$, we can write $\varphi_g|_N = (id_A, \ldots, id_A)\sigma_p$ where σ_p is a product of disjoint p-cycles in S_k. Also, σ_p cannot fix $x = (a_1, \ldots, a_m, \ldots, a_1, \ldots, a_m, a_1, \ldots, a_r)$ since $p \nmid m$. Thus, $\varphi_g|_N(x) \neq x$. If $g \in C_G(x)$, then $x + g = g + x$ and $-g + x + g = x$. Thus $\varphi_g|_N(x) = -g + x + g = x$, a contradiction. Hence $g \notin C_G(x)$. Since g is an arbitrary element of order p in G, it follows that $C_G(x)$ contains no element of order p. So $\exp[C_G(x)] \neq \exp G$ and G is not an EP group, a contradiction. Hence, we must have $k > (p-1)|A|$. It follows that $k \geq 1 + (p-1)|A| = p - p + 1 + (p-1)|A| = p - (p-1) + (p-1)|A| = p + (p-1)(|A| - 1)$. Therefore $k - (p-1)(|A| - 1) \geq p$.

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(ii) Assume $k \leq 2(p-1)(|A|-1)$. Since G is an EP group, for any $x \in N$ $\exp[C_G(x)] = \exp G$. Therefore, there exists $y \in C_G(x)$ such that $|y| = p$. As above, $\varphi_y|_N = (id_A, \dots, id_A)\sigma_p$ where $\sigma_p \in S_k$ has order p. Further, since $y \in C_G(x)$, we know $\varphi_y|_N(x) = -y + x + y = x$.

From (i), we have $r = k - (p-1)(|A|-1) \geq p$ and consider

$$x_1 = (\underbrace{a_2, \dots, a_2}_{p-1 \text{ times}}, \underbrace{a_3, \dots, a_3}_{p-1 \text{ times}}, \dots, \underbrace{a_m, \dots, a_m}_{p-1 \text{ times}}, \underbrace{a_1, \dots, a_1}_{r \text{ times}}) \in N$$

and

$$x_2 = (\underbrace{a_1, \dots, a_1}_{r \text{ times}}, \underbrace{a_2, \dots, a_2}_{p-1 \text{ times}}, \underbrace{a_3, \dots, a_3}_{p-1 \text{ times}}, \dots, \underbrace{a_m, \dots, a_m}_{p-1 \text{ times}}) \in N.$$

Choose $y_1, y_2 \in G$ with $|y_1| = p$ and $|y_2| = p$ so that $\varphi_{y_1} = (id_A, \dots, id_A)\sigma_1$ fixes x_1 and $\varphi_{y_2} = (id_A, \dots, id_A)\sigma_2$ fixes x_2. An automorphism of order p that fixes x_1 can only permute the a_1. This is also true for x_2. By assumption, $k \leq 2(p-1)(|A|-1)$ which ensures the positions of a_1 in x_1 do not overlap the positions of a_1 in x_2. Thus, we must have that σ_1 and σ_2 are disjoint elements of order p. Therefore σ_1 and σ_2 commute. It follows that y_1 and y_2 commute in G with $\langle y_1 \rangle \cap \langle y_2 \rangle = \{0\}$. So $\langle y_1, y_2 \rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p$ and the Sylow p-subgroup of G is not cyclic, a contradiction. Thus, $k > 2(p-1)(|A|-1)$. □

Continuing the assumption that G has a nontrivial cyclic Sylow p-subgroup, the remaining case is $C_G(M) = \{0\}$ for every nontrivial proper normal subgroup M of G. Our next theorem shows that $|G|$ must be quite large for $M_E(G)$ to be simple under certain divisibility conditions.

Theorem 4.12. *Let G be a finite group having a nontrivial cyclic Sylow p-subgroup for some prime p and $C_G(M) = \{0\}$ for every nontrivial proper normal subgroup M of G. Further assume that G has a proper nontrivial minimal normal subgroup $N = \underbrace{A \times A \times \cdots \times A}_{k \text{ times}}$ where A is a nonabelian simple group and $p \nmid |\mathrm{Out}\, A|$. If $M_E(G)$ is simple, then $|G| \geq 2 \cdot 60^{118}$.*

Proof. Assume $M_E(G)$ is simple. Then G is an EP group by Theorem 3.5. If $k = 1$, then $N = A$ and N is a nonabelian normal simple subgroup of G with $C_G(N) = \{0\}$. By Lemma 4.9, G is not an EP group, a contradiction. So $k > 1$. Since the Sylow p-subgroup of G is cyclic, if $p \mid |A|$, we conclude that $k = 1$, a contradiction. Thus $p \nmid |A|$. By assumption, $p \nmid |\mathrm{Out}\, A|$. So $p \nmid |\mathrm{Aut}\, A|$ by Lemma 4.7. Hence $k > 2(p-1)(|A|-1)$ by Lemma 4.11. Since the nonabelian simple group of smallest order is A_5 with $|A_5| = 60$, we get $k > 2(p-1)(|A|-1) \geq 2(2-1)(60-1) = 118$. Therefore $|G| = [G:N]|N| = [G:N]|A|^k \geq 2 \cdot 60^{118}$. □

We note that the condition $p \nmid |A|$ and $p \nmid |\mathrm{Out}\, A|$ used in the proof above is very common for finite simple groups A. In [4] (pp. 239 - 242), the first example we find of a group for which there is a prime p such that $p||OutA|$, but $p \nmid |A|$ is the Unitary group $A = U_3(32)$ which has order 366,157,135,872. It is possible that there could be Linear groups of the form $L_2(q)$ or $L_3(q)$ for which $p||OutA|$, but $p \nmid |A|$. For such Linear groups, $|A| > 10^6$. In any of these cases, $|A|$ is substantially larger than the 60 we used in computing our lower bound on $|G|$.

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In [1] and [2], it was shown that if G is a finite abelian group, a finite non-perfect group (i.e., G is not equal to its commutator subgroup G'), or a finite characteristically simple group (i.e., G is a direct product of isomorphic simple groups), then $M_E(G)$ is simple if and only if G is a p-group of exponent p for some prime p. Furthermore, $M_E(G) \cong \mathbb{Z}_p$ in these cases.

A goal in [2] was to construct a finite nonabelian, non-characteristically simple, perfect group G and to investigate the simplicity of $M_E(G)$. To that end, two nonabelian, nonisomorphic simple groups S and T were considered where T contained an isomorphic copy of S as a subgroup. For $G = S \times T$, complicated necessary conditions were found on S and T for $M_E(G)$ to be simple, and a conjecture was made that no such groups S and T existed with such restrictions.

In the current work, we found that if G has a cyclic Sylow p-subgroup and $M_E(G)$ is simple, then $G = \mathbb{Z}_p$ or G has very restrictive properties. As in [2], we still conjecture that only p-groups of exponent p yield simple nearrings $M_E(G)$.

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