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ON $\mathcal{I}_2$-CONVERGENCE OF COMPLEX UNCERTAIN DOUBLE SEQUENCES

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Abstract. In the context of the uncertainty theory of Liu, we extend the notion of $\mathcal{I}$-convergence of complex uncertain sequences for a single sequence to that of $\mathcal{I}_2$-convergence for double sequences. We compare (uniformly) almost surely $\mathcal{I}_2$-convergence with convergence in measure, in mean, and in distribution. In each case, we either show a deductive relation or provide a counterexample.

1. Introduction

Uncertainty theory of Liu [18] discusses circumstances of unavailability of samples and uncertainty in which conventional statistical methods are inadequate. Liu investigates how an uncertainty theory based on the degree of confidence could replace probability theory, which is based on the frequency of events. As in classical probability theory, several types of convergence for uncertain sequences exist, such as convergence in measure, convergence in mean, convergence in distribution, and almost sure convergence. Later, You [25] added uniformly almost sure convergence to this list. Peng [21] extended the theory from real to complex uncertain variables, and Chen et al. [1] investigated the convergence of complex uncertain sequences.

On the other hand, the characteristics of various types of sequence convergences significantly influence the field of mathematical analysis. Statistical convergence, an extension of the usual idea of convergence for single sequence. An important generalization of this concept, called '$\mathcal{I}$-convergence', was introduced by Kostyrko et al. [17] based on the concept of ideal($\mathcal{I}$) which is defined as follows: Let X be a non-empty set. A family of subsets $\mathcal{I} \subset P(X)$ is called an ideal on X if and only if

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(i) for each $A, B \in \mathcal{I} \implies A \cup B \in \mathcal{I}$;

(ii) for each $A \in \mathcal{I}$ and $B \subset A \implies B \in \mathcal{I}$.

An ideal $\mathcal{I}$ is called non-trivial if $\mathcal{I} \neq \{\phi\}$ and $X \notin \mathcal{I}$.
A non-trivial ideal $\mathcal{I}$ is called an admissible ideal in X if and only if $\{\{x\} : x \in X\} \subset \mathcal{I}$.
For example (i) $\mathcal{I}^f :=$ The set of all finite subsets of $\mathbb{N}$ forms a non-trivial admissible ideal.
(ii) $\mathcal{I}^d :=$ The set of all subsets of $\mathbb{N}$ whose natural density is zero forms a non-trivial admissible ideal.

The concept of statistical convergence for single sequence was further extended to statistical convergence in double sequences explored independently by Mursaleen and Edely [20], Tripathy [24] in the year 2003 based on the notion of two-dimensional analogue of natural density.
Let $K \subseteq \mathbb{N} \times \mathbb{N}$ be a two dimensional set of positive integers and let $K(n, m)$ be the numbers of (i, j) in K such that $i \leq n, j \leq m$. If the sequence $\left(\frac{K(n,m)}{n \cdot m}\right)_{n,m \in \mathbb{N}}$ has a limit in Pringsheim's sense then we say that K has double natural density and is denoted by

$$d_2(K) = \lim_{\substack{m \to \infty \\ n \to \infty}} \frac{K(n, m)}{n \cdot m}.$$

In these airticle, the authors also introduced the notion of statistical convergence of double sequences. A double sequence $(x_{mn})_{m,n \in \mathbb{N}}$ of real numbers is said to be statistically convergent to $\ell \in \mathbb{R}$ if for every $\varepsilon > 0$,

$$d_2\left(A(\varepsilon)\right) = 0, \text{ where } A(\varepsilon) = \left\{(m, n) \in \mathbb{N} \times \mathbb{N} : |x_{mn} - \ell| \geq \varepsilon\right\}.$$

Then this study further studied by Sahiner et al. [22] and Móricz [19]. Dems [8] introduced the notion of $\mathcal{I}_2$-convergence of double sequences based on the notion of two-dimensional analogue of idel($\mathcal{I}_2$). A nontrivial ideal $\mathcal{I}_2$ of $\mathbb{N} \times \mathbb{N}$ is called strongly admissible (also admissible ideal) when $\{x\} \times \mathbb{N}$ and $\mathbb{N} \times \{x\}$ belongs to $\mathcal{I}_2$ for each $x \in \mathbb{N}$.
Let $\mathcal{I}_2^0 = \{B \subset \mathbb{N} \times \mathbb{N} : (\exists\ m(B) \in \mathbb{N}), (i, j \geq m(B) \Rightarrow (i, j) \notin B)\}$. Then $\mathcal{I}_2^0$ is a nontrivial strongly admissible ideal and clearly an ideal $\mathcal{I}_2$ is strongly admissible if and only if $\mathcal{I}_2^0 \subset \mathcal{I}_2$.
The authors also defined the concept $\mathcal{I}_2$-convergent, A double sequence $(x_{mn})_{m,n \in \mathbb{N}}$ of real numbers is said to be $\mathcal{I}_2$-convergent to $\ell \in \mathbb{R}$ if for every $\varepsilon > 0$,

$$\{(m, n) \in \mathbb{N} \times \mathbb{N} : |x_{mn} - \ell| \geq \varepsilon\} \in \mathcal{I}_2.$$

In uncertainty theory, the notion of statistical convergence of complex uncertain sequences was introduced by Tripathy et al. [23] and further studied by several authors [6, 7, 11–15]. Das et al. [2] introduced the notion of statistical convergence for complex uncertain double sequences; see also [3–5, 9, 16].

In this article, we extend the notion of $\mathcal{I}$-convergence of complex uncertain sequences from single sequences to that of $\mathcal{I}_2$-convergence for double sequences. We define the notions of (uniformly) almost sure $\mathcal{I}_2$-convergence, as well as $\mathcal{I}_2$-convergence in measure, in mean, and in distribution, and study the relationships among them. In each case, we either establish a deductive relationship or rule it out by providing a counterexample.

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2. Definitions and Preliminaries

To develop the notion of $\mathcal{I}_2$-convergence of complex uncertain double sequence, we first recall some basic definitions from uncertainty theory. In uncertainty theory, Liu [18] introduced the notion of uncertain measure which is defined as follows: Let $\mathcal{L}$ be a σ-algebra on a nonempty set Γ. A set function $\mathcal{M}$ on Γ is called an uncertain measure if it satisfies the following axioms:
Axiom 1 (Normality): $\mathcal{M}\{\Gamma\} = 1$;
Axiom 2 (Duality): $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any $\Lambda \in \mathcal{L}$;
Axiom 3 (Subadditivity): For every countable sequence of $\{\Lambda_j\} \in \mathcal{L}$,

$$\mathcal{M}\{\bigcup_{j=1}^{\infty} \Lambda_j\} \leq \sum_{j=1}^{\infty} \mathcal{M}\{\Lambda_j\}.$$

The triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space, and each element Λ in $\mathcal{L}$ is called an event. To obtain an uncertain measure of compound event, a product uncertain measure is defined by Liu as:

$$\mathcal{M}\{\prod_{k=1}^{\infty} \Lambda_k\} = \bigwedge_{k=1}^{\infty} \mathcal{M}\{\Lambda_k\}.$$

An uncertain measure $\mathcal{M}$ is called continuous if for any sequence of events Λ_k with $k \to \infty$, we have $\mathcal{M}\left\{\lim_{k\to\infty} \Lambda_k\right\} = \lim_{k\to\infty} \mathcal{M}\{\Lambda_k\}$, which is explore by Gao [10].
Peng [21] defined the notion of complex uncertain variables in the year 2012. A variable $\zeta = \xi + i\eta$ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of complex numbers is a complex uncertain variable if and only if ξ and η are uncertain variables, where ξ and η are the real and imaginary parts of ζ, respectively. Peng also introduced various types of definitions, out of which we have included some definitions here which will be used throughout the article.

Let $\zeta = \xi + i\eta$ be a complex uncertain variable, where ξ and η are real and imaginary part of ζ, respectively. Then the complex uncertainty distribution of ζ is a function from $\mathbb{C}$ to $[0, 1]$ defined by $\Phi(z) = \mathcal{M}\{\xi \leq s, \eta \leq t\}$ for any complex number $z = s + it$.

Let $\zeta = \xi + i\eta$ be a complex uncertain variable. If the expected value of ξ and η i.e., $E[\xi]$ and $E[\eta]$ exists, then the expected value of ζ is defined by

$$E[\zeta] = E[\xi] + iE[\eta].$$

Recently, Halder et al. [11] introduced the notion of various types of $\mathcal{I}$-convergence of complex uncertain sequences such as

A complex uncertain sequence (ζ_n) is said to be $\mathcal{I}$-convergent almost surely to ζ, if for every $\varepsilon > 0$, there exists an event Λ with $\mathcal{M}\{\Lambda\} = 1$ such that

$$\{n \in \mathbb{N} : \|\zeta_n(\gamma) - \zeta(\gamma)\| \geq \varepsilon\} \in \mathcal{I},\ for\ every\ \gamma \in \Lambda.$$

A complex uncertain sequence (ζ_n) is said to be

(i) $\mathcal{I}$-convergent in measure to ζ if for every $\varepsilon, \delta > 0$

$$\{n \in \mathbb{N} : \mathcal{M}\left(\|\zeta_n - \zeta\| \geq \varepsilon\right) \geq \delta\} \in \mathcal{I}$$

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(ii) $\mathcal{I}$-convergent in mean to ζ if for every $\varepsilon > 0$,

$$\{n \in \mathbb{N} : E\left[\|\zeta_n - \zeta\|\right] \geq \varepsilon\} \in \mathcal{I}.$$

(iii) $\mathcal{I}$-convergent in distribution to ζ if for every $\varepsilon > 0$,

$$\{n \in \mathbb{N} : \|\Phi_n(z) - \Phi(z)\| \geq \varepsilon\} \in \mathcal{I}$$

for all z at which $\Phi(z)$ is continuous, where $\Phi, \Phi_1, \Phi_2, ...$ be the complex uncertainty distributions of complex uncertain variables $\zeta, \zeta_1, \zeta_2, ...$, respectively.

A complex uncertain sequence (ζ_n) is said to be $\mathcal{I}$-convergent uniformly almost surely to ζ, if for every $\varepsilon, \delta > 0$, there exists a sequence of events (Λ_n),

$$\{n \in \mathbb{N} : |\mathcal{M}(\Lambda_n)| \geq \varepsilon\} \in \mathcal{I}$$

such that $\{n \in \mathbb{N} : \|\zeta_n(\gamma) - \zeta(\gamma)\| \geq \delta\} \in \mathcal{I}$ for all $\gamma \in \Gamma \setminus \Lambda_n$.

$\mathcal{I}_2$-convergent almost surely of complex uncertain double sequence first explore by Kişi and Gürdal in the year 2023 which is defined as follows:
A complex uncertain double sequence (ζ_{mn}) is said to be $\mathcal{I}_2$-convergent almost surely to ζ, if for every $\varepsilon > 0$, there exists an event Λ with $\mathcal{M}\{\Lambda\} = 1$ such that

$$\{(m, n) \in \mathbb{N} \times \mathbb{N} : \|\zeta_{mn}(\gamma) - \zeta(\gamma)\| \geq \varepsilon\} \in \mathcal{I}_2 \ for\ every\ \gamma \in \Lambda.$$

Throughout the paper, we consider $\mathcal{I}_2$ to be a non-trivial admissible ideal of $\mathbb{N} \times \mathbb{N}$.

3. **Main Results**

In this section, we introduce various modes of $\mathcal{I}_2$-convergence of complex uncertain double sequences. These concepts extend classical convergence notions to the framework of uncertainty theory under the ideal $\mathcal{I}_2$ and play a crucial role in understanding the interrelationships among different types of convergence.

Definition 3.1. *Let (ζ_{mn}) be a complex uncertain double sequence and let ζ be a complex uncertain variable defined on an uncertainty space. Then (ζ_{mn}) is said to be*

(i) *$\mathcal{I}_2$-convergent in measure to ζ if for every $\varepsilon, \delta > 0$,*

$$\{(m, n) \in \mathbb{N} \times \mathbb{N} : \mathcal{M}\left(\|\zeta_{mn} - \zeta\| \geq \varepsilon\right) \geq \delta\} \in \mathcal{I}_2.$$

(ii) *$\mathcal{I}_2$-convergent in mean to ζ if for every $\varepsilon > 0$,*

$$\{(m, n) \in \mathbb{N} \times \mathbb{N} : E\left[\|\zeta_{mn} - \zeta\|\right] \geq \varepsilon\} \in \mathcal{I}_2.$$

(iii) *$\mathcal{I}_2$-convergent in distribution to ζ if for every $\varepsilon > 0$,*

$$\{(m, n) \in \mathbb{N} \times \mathbb{N} : \|\Phi_{mn}(z) - \Phi(z)\| \geq \varepsilon\} \in \mathcal{I}_2$$

for all z at which $\Phi(z)$ is continuous, where Φ_{mn}, Φ be the complex uncertainty distributions of complex uncertain variables ζ_{mn} and ζ, respectively.

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Definition 3.2. *A complex uncertain double sequence (ζ_{mn}) is said to be $\mathcal{I}_2$-convergent uniformly almost surely to ζ if for every $\varepsilon, \delta > 0$, there exists a sequence of events (Λ_{mn}),*

$$\{(m,n) \in \mathbb{N} \times \mathbb{N} : |\mathcal{M}(\Lambda_{mn})| \geq \varepsilon\} \in \mathcal{I}_2,$$

$$\text{such that } \{(m,n) \in \mathbb{N} \times \mathbb{N} : \|\zeta_{mn}(\gamma) - \zeta(\gamma)\| \geq \delta\} \in \mathcal{I}_2 \text{ for all } \gamma \in \Gamma \setminus \Lambda_{mn}.$$

Theorem 3.3. *If a complex uncertain double sequence (ζ_{mn}) is $\mathcal{I}_2$-convergent in mean to ζ, then it is $\mathcal{I}_2$-convergent in measure to ζ.*

Proof. Let the complex uncertain sequence (ζ_{mn}) be $\mathcal{I}_2$-convergent in mean to ζ, then for every $\delta > 0$,

$$\{(m,n) \in \mathbb{N} \times \mathbb{N} : E\left[\|\zeta_{mn} - \zeta\|\right] \geq \delta\} \in \mathcal{I}_2.$$

Using Markov inequality we can see that for given $\varepsilon \geq 1$, we have

$$\mathcal{M}\left\{\|\zeta_{mn} - \zeta\| \geq \varepsilon\right\} \leq \frac{E\left[\|\zeta_{mn} - \zeta\|\right]}{\varepsilon} \leq E\left[\|\zeta_{mn} - \zeta\|\right].$$

Therefore $\{(m,n) \in \mathbb{N} \times \mathbb{N} : \mathcal{M}\left(\|\zeta_{mn} - \zeta\| \geq \varepsilon\right) \geq \delta\}$

$$\subseteq \{(m,n) \in \mathbb{N} \times \mathbb{N} : E\left[\|\zeta_{mn} - \zeta\|\right] \geq \delta\} \in \mathcal{I}_2.$$

Thus $\{(m,n) \in \mathbb{N} \times \mathbb{N} : \mathcal{M}\left(\|\zeta_{mn} - \zeta\| \geq \varepsilon\right) \geq \delta\} \in \mathcal{I}_2$.

Hence sequence (ζ_{mn}) is $\mathcal{I}_2$-convergent in measure to ζ. □

But the converse is not true in general. For example, we consider the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2, \cdots\}$ with power set and $\mathcal{M}\{\Gamma\} = 1$, $\mathcal{M}\{\Phi\} = 0$ and

$$\mathcal{M}\{\Lambda\} = \begin{cases} \sup\limits_{\gamma_{m+n} \in \Lambda} \frac{1}{m+n}, & if \sup\limits_{\gamma_{m+n} \in \Lambda} \frac{1}{m+n} < \frac{1}{2} \\ 1 - \sup\limits_{\gamma_{m+n} \in \Lambda^c} \frac{1}{m+n}, & if \sup\limits_{\gamma_{m+n} \in \Lambda^c} \frac{1}{m+n} < \frac{1}{2} \\ \frac{1}{2}, & \text{otherwise.} \end{cases} \quad for\ m,n = 1,2,3,\cdots.$$

Also we defined the complex uncertain variables ζ_n by

$$\zeta_{mn}(\gamma) = \begin{cases} i(m+n), & if\ \gamma = \gamma_{m+n} \\ 0, & \text{otherwise} \end{cases} \quad for\ m,n = 1,2,3,\cdots$$

and $\zeta \equiv 0$. Take $\mathcal{I}_2$=$\mathcal{I}_2^d$.

Then for every $\varepsilon, \delta > 0$ and the values of m, n such that $(m+n) \geq 3$, we obtain

$\{(m,n) \in \mathbb{N} \times \mathbb{N} : \mathcal{M}\left(\|\zeta_{mn} - \zeta\| \geq \varepsilon\right) \geq \delta\}$

$$= \{(m,n) \in \mathbb{N} \times \mathbb{N} : \mathcal{M}\left(\gamma : \|\zeta_{mn}(\gamma) - \zeta(\gamma)\| \geq \varepsilon\right) \geq \delta\}$$

$$= \{(m,n) \in \mathbb{N} \times \mathbb{N} : \mathcal{M}\left(\gamma_{m+n}\right) \geq \delta\} = \left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{m+n} \geq \delta\right\} \in \mathcal{I}_2.$$

Therefore sequence the (ζ_{mn}) is $\mathcal{I}_2$-convergent in measure to ζ.

However, the complex uncertainty distribution of $\|\zeta_{mn} - \zeta\|$ is as follows

$$\Phi_{mn}(s) = \begin{cases} 0, & \text{if } s < 0 \\ 1 - \frac{1}{m+n}, & \text{if } 0 \leq s < m+n \\ 1, & \text{if } s \geq m+n. \end{cases}$$

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for the values of m, n such that $(m+n) \geq 3$.

So for the values of m, n such that $(m+n) \geq 3$, we have

$$\{(m,n) \in \mathbb{N} \times \mathbb{N} : E\left[\|\zeta_{mn} - \zeta\|\right] \geq \varepsilon\}$$
$$= \left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \left[\int_0^\infty (1 - \Phi_{mn}(x))\, ds\right] \geq \varepsilon\right\}$$
$$= \left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \left[\int_0^{m+n} \left(1 - (1 - \tfrac{1}{m+n})\right) ds\right] \geq \varepsilon\right\} \notin \mathcal{I}_2.$$

Hence the sequence (ζ_{mn}) is not $\mathcal{I}_2$-convergent in mean to ζ.

Theorem 3.4. *The complex uncertain double sequence (ζ_{mn}) where $\zeta_{mn} = \xi_{mn} + i\eta_{mn}$ is $\mathcal{I}_2$-convergent in measure to $\zeta = \xi + i\eta$ if and only if the uncertain sequence (ξ_{mn}) and (η_{mn}) are $\mathcal{I}_2$-convergent in measure to ξ and η, respectively.*

Proof. Let the uncertain double sequence (ξ_{mn}) and (η_{mn}) are $\mathcal{I}_2$-convergent in measure to ξ and η, respectively, then for every $\varepsilon, \delta > 0$, we have

$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \mathcal{M}\left(|\xi_{mn} - \xi| \geq \frac{\varepsilon}{\sqrt{2}}\right) \geq \frac{\delta}{2}\right\} \in \mathcal{I}_2$$
$$\text{and} \left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \mathcal{M}\left(|\eta_{mn} - \eta| \geq \frac{\varepsilon}{\sqrt{2}}\right) \geq \frac{\delta}{2}\right\} \in \mathcal{I}_2.$$

Note that $\|\zeta_{mn} - \zeta\| = \sqrt{|\xi_{mn} - \xi|^2 + |\eta_{mn} - \eta|^2}$.

Thus we have $\{\|\zeta_{mn} - \zeta\| \geq \varepsilon\} \subset \left\{|\xi_{mn} - \xi| \geq \frac{\varepsilon}{\sqrt{2}}\right\} \cup \left\{|\eta_{mn} - \eta| \geq \frac{\varepsilon}{\sqrt{2}}\right\}$

So $\mathcal{M}\{\|\zeta_{mn} - \zeta\| \geq \varepsilon\} \leq \mathcal{M}\left\{|\xi_{mn} - \xi| \geq \frac{\varepsilon}{\sqrt{2}}\right\} + \mathcal{M}\left\{|\eta_{mn} - \eta| \geq \frac{\varepsilon}{\sqrt{2}}\right\}$.

Therefore $\{(m,n) \in \mathbb{N} \times \mathbb{N} : \mathcal{M}\{\|\zeta_{mn} - \zeta\| \geq \varepsilon\} \geq \delta\}$

$$\subseteq \left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \mathcal{M}\left(|\xi_{mn} - \xi| \geq \tfrac{\varepsilon}{\sqrt{2}}\right) \geq \tfrac{\delta}{2}\right\}$$
$$\cup \left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \mathcal{M}\left(|\eta_{mn} - \eta| \geq \tfrac{\varepsilon}{\sqrt{2}}\right) \geq \tfrac{\delta}{2}\right\} \in \mathcal{I}_2.$$

Hence the double sequence (ζ_{mn}) is $\mathcal{I}_2$-convergent in measure to ζ.

Conversely, let the double sequence (ζ_{mn}) is $\mathcal{I}_2$-convergent in measure to ζ, then for every $\varepsilon, \delta > 0$,

$$\{(m,n) \in \mathbb{N} \times \mathbb{N} : \mathcal{M}(\|\zeta_{mn} - \zeta\| \geq \varepsilon) \geq \delta\} \in \mathcal{I}_2.$$

Note that $|\xi_{mn} - \xi| \leq |(\xi_{mn} - \xi) + i(\eta_{mn} - \eta)| = |(\xi_{mn} + i\eta_{mn}) - (\xi + i\eta)| = \|\zeta_{mn} - \zeta\|$.

Thus we have $\{|\xi_{mn} - \xi| \geq \varepsilon\} \subseteq \{\|\zeta_m - \zeta\| \geq \varepsilon\}$

So $\mathcal{M}\{|\xi_{mn} - \xi| \geq \varepsilon\} \leq \mathcal{M}\{\|\zeta_{mn} - \zeta\| \geq \varepsilon\}$.

Therefore $\{(m,n) \in \mathbb{N} \times \mathbb{N} : \mathcal{M}(|\xi_{mn} - \xi| \geq \varepsilon) \geq \delta\}$

$$\subseteq \{(m,n) \in \mathbb{N} \times \mathbb{N} : \mathcal{M}\{\|\zeta_{mn} - \zeta\| \geq \varepsilon\} \geq \delta\} \in \mathcal{I}_2.$$

Hence $\{(m,n) \in \mathbb{N} \times \mathbb{N} : \mathcal{M}(|\xi_{mn} - \xi| \geq \varepsilon) \geq \delta\} \in \mathcal{I}_2$.

Similarly $\{(m,n) \in \mathbb{N} \times \mathbb{N} : \mathcal{M}(|\eta_{mn} - \eta| \geq \varepsilon) \geq \delta\} \in \mathcal{I}_2$.

This completes the proof. □

Proposition 3.5. *Assume that a complex uncertain double sequence (ζ_{mn}) with real part (ξ_{mn}) and imaginary part (η_{mn}) are $\mathcal{I}_2$-convergent in measure to ξ and η, respectively. Then the complex uncertain double sequence (ζ_{mn}) is $\mathcal{I}_2$-convergent in distribution to $\zeta = \xi + i\eta$.*

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Proof. Let $z = s + it$ be a given continuity point of the complex uncertainty distribution Φ. Additionally, for any $\alpha > s, \beta > t$, we have

$$\begin{aligned}&\{\xi_{mn} \le s, \eta_{mn} \le t\}\\ &\quad = \{\xi_{mn} \le s, \eta_{mn} \le t, \xi \le \alpha, \eta \le \beta\} \cup \{\xi_{mn} \le s, \eta_{mn} \le t, \xi > \alpha, \eta > \beta\}\\ &\qquad \cup \{\xi_{mn} \le s, \eta_{mn} \le t, \xi \le \alpha, \eta > \beta\} \cup \{\xi_{mn} \le s, \eta_{mn} \le t, \xi > \alpha, \eta \le \beta\}\\ &\quad \subset \{\xi \le \alpha, \eta \le \beta\} \cup \{|\xi_{mn} - \xi| \ge \alpha - s\} \cup \{|\eta_{mn} - \eta| \ge \beta - t\}.\end{aligned}$$

Then from third axiom of uncertain measure, we have

$$\Phi_{mn}(z) = \Phi_{mn}(s + it) \le \Phi(\alpha + i\beta) + \mathcal{M}\{|\xi_{mn} - \xi| \ge \alpha - s\} + \mathcal{M}\{|\eta_{mn} - \eta| \ge \beta - t\}.$$

Since (ξ_{mn}) and (η_{mn}) are $\mathcal{I}_2$-convergent in measure to ξ and η, respectively, so for any small $\delta > 0$, we have

$$\{(m,n) \in \mathbb{N} \times \mathbb{N} : \mathcal{M}(|\xi_{mn} - \xi| \ge \alpha - s) \ge \delta\} \in \mathcal{I}_2$$
$$\text{and } \{(m,n) \in \mathbb{N} \times \mathbb{N} : \mathcal{M}(|\eta_{mn} - \eta| \ge \beta - t) \ge \delta\} \in \mathcal{I}_2.$$

Thus we obtain $\mathcal{I}_2 - \limsup_{m,n\to\infty} \Phi_{mn}(z) \le \Phi(\alpha + i\beta)$ for any $\alpha > s, \beta > t$. Letting $\alpha + i\beta \to s + it$, we get

$$\mathcal{I}_2 - \limsup_{m,n\to\infty} \Phi_{mn}(z) \le \Phi(z). \tag{3.1}$$

Additionally, for any $\gamma < s, \kappa < t$, we have

$$\begin{aligned}&\{\xi \le \gamma, \eta \le \kappa\}\\ &\quad = \{\xi_{mn} \le s, \eta_{mn} \le t, \xi \le \gamma, \eta \le \kappa\} \cup \{\xi_{mn} > s, \eta_{mn} > t, \xi \le \gamma, \eta \le \kappa\}\\ &\qquad \cup \{\xi_{mn} > s, \eta_{mn} \le t, \xi \le \gamma, \eta \le \kappa\} \cup \{\xi_{mn} \le s, \eta_{mn} > t, \xi \le \gamma, \eta \le \kappa\}\\ &\quad \subset \{\xi_{mn} \le s, \eta_{mn} \le t\} \cup \{|\xi_{mn} - \xi| \ge s - \gamma\} \cup \{|\eta_{mn} - \eta| \ge t - \kappa\}.\end{aligned}$$

Then from the subadditivity axiom of uncertain measure, we have

$\Phi(\gamma + i\kappa) \le \Phi_{mn}(s + it) + \mathcal{M}\{|\xi_{mn} - \xi| \ge s - \gamma\} + \mathcal{M}\{|\eta_{mn} - \eta| \ge t - \kappa\}.$

Since (ξ_{mn}) and (η_{mn}) are $\mathcal{I}_2$-convergent in measure to ξ and η, respectively, so for any small $\delta > 0$, we have

$$\{(m,n) \in \mathbb{N} \times \mathbb{N} : \mathcal{M}(|\xi_m - \xi| \ge s - \gamma) \ge \delta\} \in \mathcal{I}_2$$
$$\text{and } \{(m,n) \in \mathbb{N} \times \mathbb{N} : \mathcal{M}(|\eta_{mn} - \eta| \ge t - \kappa) \ge \delta\} \in \mathcal{I}_2.$$

Thus we obtain $\Phi(\gamma + i\kappa) \le \liminf_{m,n\to\infty} \Phi_{mn}(s + it)$ for any $\gamma < s, \kappa < t$. Letting $\gamma + i\kappa \to s + it$, we get

$$\Phi(z) \le \mathcal{I}_2 - \liminf_{m,n\to\infty} \Phi_{mn}(z). \tag{3.2}$$

It follows from (3.1) and (3.2) that the sequence (ζ_{mn}) is $\mathcal{I}_2$-convergent in distribution to ζ. □

But the converse is not true in general. For example, we consider the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2\}$ with $\mathcal{M}(\gamma_1) = \mathcal{M}(\gamma_2) = \frac{1}{2}$. Now a complex uncertain variable can be define as

$$\zeta(\gamma) = \begin{cases} i, & if\ \gamma = \gamma_1 \\ -i, & if\ \gamma = \gamma_2. \end{cases}$$

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Additionally, we define $\zeta_{mn} = -\zeta$ for $m, n = 1, 2, \cdots$. Take $\mathcal{I}_2$=$\mathcal{I}_2^d$.
Thus, the sequence (ζ_{mn}) and ζ exhibit the same distribution as follows:

$$\Phi_{mn}(z) = \Phi_{mn}(s+it) = \begin{cases} 0, \ if \ s < 0, -\infty < t < +\infty \\ 0, \ if \ s \geq 0, t < -1 \\ \frac{1}{2}, \ if \ s \geq 0, -1 \leq t < 1 \\ 1, \ if \ s \geq 0, t \geq 1. \end{cases}$$

So (ζ_{mn}) is $\mathcal{I}_2$-convergent in distribution to ζ.
Now, for a given $\varepsilon, \delta > 0$, we have

$$\{(m,n) \in \mathbb{N}\times\mathbb{N} : \mathcal{M}\left(\|\zeta_{mn} - \zeta\| \geq \varepsilon\right) \geq \delta\} \notin \mathcal{I}_2.$$

Thus the sequence (ζ_{mn}) is not $\mathcal{I}_2$-convergent in measure to ζ.

Proposition 3.6. *A complex uncertain double sequence* (ζ_{mn}) *is*

(i) $\mathcal{I}_2$*-convergent almost surely to* ζ *if and only if for every* $\varepsilon, \delta > 0$,

$$\left\{(k,t) \in \mathbb{N}\times\mathbb{N} : \mathcal{M}\left(\bigcap_{k,t=1}^{\infty}\bigcup_{m=k}^{\infty}\bigcup_{n=t}^{\infty}\{\|\zeta_{mn} - \zeta\| \geq \varepsilon\}\right) \geq \delta\right\} \in \mathcal{I}_2.$$

(ii) $\mathcal{I}_2$*-convergent uniformly almost surely to* ζ *if and only if for every* $\varepsilon, \delta > 0$,

$$\left\{(k,t) \in \mathbb{N}\times\mathbb{N} : \mathcal{M}\left(\bigcup_{m=k}^{\infty}\bigcup_{n=t}^{\infty}\{\|\zeta_{mn} - \zeta\| \geq \varepsilon\}\right) \geq \delta\right\} \in \mathcal{I}_2.$$

Proof. (i) Let (ζ_{mn}) be $\mathcal{I}_2$-convergent almost surely to ζ, then for every $\varepsilon > 0$,

$$\{(m,n) \in \mathbb{N}\times\mathbb{N} : \|\zeta_{mn}(\gamma) - \zeta(\gamma)\| \geq \varepsilon\} \in \mathcal{I}_2,$$

for every $\gamma \in \Lambda$ with $\mathcal{M}\{\Lambda\} = 1$.
Also, for every ε greater than 0, $\exists\ k, t$ such that $\|\zeta_{mn} - \zeta\| < \varepsilon$ where $m > k, n > t$ and for any $\gamma \in \Lambda$, $i.e.$, equivalent to

$$\left\{(k,t) \in \mathbb{N}\times\mathbb{N} : \mathcal{M}\left(\bigcup_{k,t=1}^{\infty}\bigcap_{m=k}^{\infty}\bigcap_{n=t}^{\infty}\|\zeta_{mn} - \zeta\| < \varepsilon\right) \geq 1\right\} \in \mathcal{I}_2.$$

It follows from the duality axiom of uncertain measure that

$$\left\{(k,t) \in \mathbb{N}\times\mathbb{N} : \mathcal{M}\left(\bigcap_{k,t=1}^{\infty}\bigcup_{m=k}^{\infty}\bigcup_{n=t}^{\infty}\|\zeta_{mn} - \zeta\| \geq \varepsilon\right) \geq \delta\right\} \in \mathcal{I}_2.$$

(ii) Let (ζ_{mn}) be $\mathcal{I}_2$-convergent uniformly almost surely to ζ, then for every $\varepsilon, \delta > 0$, $\exists$ a sequence of events (Λ_{mn}), we have

$$\{(m,n) \in \mathbb{N}\times\mathbb{N} : |\mathcal{M}(\Lambda_{mn})| \geq \varepsilon\} \in \mathcal{I}_2$$

$therefore \{(m,n) \in \mathbb{N}\times\mathbb{N} : \|\zeta_{mn}(\gamma) - \zeta(\gamma)\| \geq \delta\} \in \mathcal{I}_2$ for all $\gamma \in \Gamma \setminus \Lambda_{mn}$.
Then for every small positive value of ε, there exists $k, t > 0$ such that $\|\zeta_{mn} - \zeta\| < \varepsilon$ where $m \geq$

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$k, n \geq t$ and for all $\gamma \in \Gamma \setminus \Lambda_{mn}$.

This can be rewritten as $\bigcup_{m=k}^{\infty} \bigcup_{n=t}^{\infty} \{\|\zeta_{mn} - \zeta\| \geq \varepsilon\} \subset \Lambda_{mn}$.

By subadditivity property of uncertain measures, we derive the conclusion that:

$$\mathcal{M}\left(\bigcup_{m=k}^{\infty} \bigcup_{n=t}^{\infty} \{\|\zeta_{mn} - \zeta\| \geq \varepsilon\}\right) \leq \mathcal{M}\left(\Lambda_{mn}\right).$$

Then for every $\delta > 0$, we have

$$\left\{(k,t) \in \mathbb{N} \times \mathbb{N} : \mathcal{M}\left(\bigcup_{m=k}^{\infty} \bigcup_{n=t}^{\infty} \{\|\zeta_{mn} - \zeta\| \geq \varepsilon\}\right) \geq \delta\right\}$$
$$\subseteq \{(m,n) \in \mathbb{N} \times \mathbb{N} : |\mathcal{M}(\Lambda_{mn})| \geq \delta\} \in \mathcal{I}_2.$$

To prove converse part, let us take for any $\varepsilon, \delta > 0$,

$$\left\{(k,t) \in \mathbb{N} \times \mathbb{N} : \mathcal{M}\left(\bigcup_{m=k}^{\infty} \bigcup_{n=t}^{\infty} \{\|\zeta_{mn} - \zeta\| \geq \varepsilon\}\right) \geq \delta\right\} \in \mathcal{I}_2,$$

i.e., $\left\{(k,t) \in \mathbb{N} \times \mathbb{N} : \mathcal{M}\left(\bigcup_{m=k}^{\infty} \bigcup_{n=t}^{\infty} \{\|\zeta_{mn} - \zeta\| \geq \varepsilon\}\right) < \delta\right\} \in \mathcal{F}\left(\mathcal{I}_2\right).$

Then for given $\delta > 0$ and in particular if $\varepsilon = \frac{1}{g}(g \geq 1)$, there exists k_g and t_g such that

$$\mathcal{M}\left(\bigcup_{m=k_g}^{\infty} \bigcup_{n=t_g}^{\infty} \left\{\|\zeta_{mn} - \zeta\| \geq \tfrac{1}{g}\right\}\right) < \tfrac{\delta}{2^g}.$$

If we take $\Lambda_{kt} = \bigcup_{g=1}^{\infty} \bigcup_{m=k_g}^{\infty} \bigcup_{n=t_g}^{\infty} \left\{\|\zeta_{mn} - \zeta\| \geq \frac{1}{g}\right\}$, then

$$\mathcal{M}\left(\Lambda_{kt}\right) \leq \sum_{g=1}^{\infty} \mathcal{M}\left(\bigcup_{m=k_g}^{\infty} \bigcup_{n=t_g}^{\infty} \left\{\|\zeta_{mn} - \zeta\| \geq \frac{1}{g}\right\}\right) < \sum_{g=1}^{\infty} \frac{\delta}{2^g} = \delta.$$

So for every $\delta > 0$, we have

$$\{(m,n) \in \mathbb{N} \times \mathbb{N} : |\mathcal{M}(\Lambda_{mn})| \geq \delta\} \in \mathcal{I}_2.$$

Furthermore, for every $\varepsilon > 0$ and $m \geq k_g$, $n \geq t_g$ $(g \geq 1)$, we have

$$\{(m,n) \in \mathbb{N} \times \mathbb{N} : \|\zeta_{mn}(\gamma) - \zeta(\gamma)\| \geq \varepsilon\} \in \mathcal{I}_2 \text{ for all } \gamma \in \Gamma \setminus \Lambda_{mn}.$$ □

Theorem 3.7. *If a complex uncertain double sequence* (ζ_{mn}) *is* $\mathcal{I}_2$*-convergent uniformly almost surely to* ζ*, then it is* $\mathcal{I}_2$*-convergent almost surely to* ζ*.*

Proof. Let (ζ_{mn}) is $\mathcal{I}_2$-convergent uniformly almost surely to ζ, then from proposition 3.5, we can say

$$\left\{(k,t) \in \mathbb{N} \times \mathbb{N} : \mathcal{M}\left(\bigcup_{m=k}^{\infty} \bigcup_{n=t}^{\infty} \{\|\zeta_{mn} - \zeta\| \geq \varepsilon\}\right) \geq \delta\right\} \in \mathcal{I}_2.$$

Since $\mathcal{M}\left(\bigcap_{k,t=1}^{\infty} \bigcup_{m=k}^{\infty} \bigcup_{n=t}^{\infty} \{\|\zeta_{mn} - \zeta\| \geq \varepsilon\}\right) \leq \mathcal{M}\left(\bigcup_{m=k}^{\infty} \bigcup_{n=t}^{\infty} \{\|\zeta_{mn} - \zeta\| \geq \varepsilon\}\right)$,

so we have $\left\{(k,t) \in \mathbb{N} \times \mathbb{N} : \mathcal{M}\left(\bigcap_{k,t=1}^{\infty} \bigcup_{m=k}^{\infty} \bigcup_{n=t}^{\infty} \{\|\zeta_{mn} - \zeta\| \geq \varepsilon\}\right) \geq \delta\right\}$

$$\subseteq \left\{(k,t) \in \mathbb{N} \times \mathbb{N} : \mathcal{M}\left(\bigcup_{m=k}^{\infty} \bigcup_{n=t}^{\infty} \{\|\zeta_{mn} - \zeta\| \geq \varepsilon\}\right) \geq \delta\right\}.$$

Thus we get $\left\{(k,t) \in \mathbb{N}\times\mathbb{N} : \mathcal{M}\left(\bigcap_{k,t=1}^{\infty}\bigcup_{m=k}^{\infty}\bigcup_{n=t}^{\infty}\{\|\zeta_{mn}-\zeta\| \geq \varepsilon\}\right) \geq \delta\right\} \in \mathcal{I}_2$.
By proposition 3.6, (ζ_{mn}) is $\mathcal{I}_2$-convergent uniformly almost surely to ζ. □

But the converse is not true in general. For example, we consider the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2, ...\}$ with power set and $\mathcal{M}\{\Gamma\} = 1$, $\mathcal{M}\{\Phi\} = 0$ and

$$\mathcal{M}\{\Lambda\} = \begin{cases} \sup\limits_{\gamma_{m+n}\in\Lambda} \frac{(m+n)\beta_{mn}}{2(m+n)+1}, & if \sup\limits_{\gamma_{m+n}\in\Lambda} \frac{(m+n)\beta_{mn}}{2(m+n)+1} < \frac{1}{2} \\ 1 - \sup\limits_{\gamma_{m+n}\in\Lambda^c} \frac{(m+n)\beta_{mn}}{2(m+n)+1}, & if \sup\limits_{\gamma_{m+n}\in\Lambda^c} \frac{(m+n)\beta_{mn}}{2(m+n)+1} < \frac{1}{2} \\ \frac{1}{2}, & otherwise, \end{cases}$$

where $\beta_{mn} = \begin{cases} n, & \text{if m is odd} \\ 0, & \text{if m is even} \end{cases} \quad for\ m = 1, 2, 3, \cdots.$
Also, the complex uncertain variables are defined by

$$\zeta_{mn}(\gamma) = \begin{cases} (m+n+1)i, & if\ \gamma = \gamma_{m+n} \\ 0, & otherwise \end{cases} \quad for\ m, n = 1, 2, 3, \cdots$$

and $\zeta \equiv 0$. Take $\mathcal{I}_2$=$\mathcal{I}_2^d$.
For any $\varepsilon > 0$ and there exists an event Λ with $\mathcal{M}\{\Lambda\} = 1$, we have

$$\{(m,n) \in \mathbb{N}\times\mathbb{N} : \|\zeta_{mn}(\gamma) - \zeta(\gamma)\| \geq \varepsilon\} = \{(m,n) \in \mathbb{N}\times\mathbb{N} : \|\zeta_{mn}(\gamma)\| \geq \varepsilon\} \in \mathcal{I}_2.$$

Also for every $\varepsilon > 0$, we have

$$\mathcal{M}\left(\bigcup_{m=k}^{\infty}\bigcup_{n=t}^{\infty}\{\gamma\in\Gamma : \|\zeta_{mn}(\gamma) - \zeta(\gamma)\| \geq \varepsilon\}\right) = \mathcal{M}\left(\bigcup_{m=k}^{\infty}\bigcup_{n=t}^{\infty}\{\gamma_{m+n}\}\right).$$

Then for every $\delta > 0$,

$$\left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \mathcal{M}\left(\bigcup_{m=k}^{\infty}\bigcup_{n=t}^{\infty}\{\gamma\in\Gamma : \|\zeta_{mn}(\gamma) - \zeta(\gamma)\| \geq \varepsilon\}\right) \geq \delta\right\}$$
$$= \left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \mathcal{M}\left(\bigcup_{m=k}^{\infty}\bigcup_{n=t}^{\infty}\{\gamma_{m+n}\}\right) \geq \delta\right\} \notin \mathcal{I}_2.$$

Hence the sequence (ζ_{mn}) is $\mathcal{I}_2$-convergent almost surely to $\zeta \equiv 0$ but it is not $\mathcal{I}_2$-convergent uniformly almost surely to $\zeta \equiv 0$.

Remark. A complex uncertain double sequence (ζ_{mn}) defined on the continuous uncertainty space is $\mathcal{I}_2$-convergent uniformly almost surely to ζ if it is $\mathcal{I}_2$-convergent almost surely to ζ.

Proof. Let the complex uncertain double sequence (ζ_{mn}) be $\mathcal{I}_2$-convergent almost surely to ζ, then for every $\varepsilon, \delta > 0$, we have

$$\left\{(k,t) \in \mathbb{N}\times\mathbb{N} : \mathcal{M}\left(\bigcap_{k,t=1}^{\infty}\bigcup_{m=k}^{\infty}\bigcup_{n=t}^{\infty}\{\|\zeta_{mn}-\zeta\| \geq \varepsilon\}\right) \geq \delta\right\} \in \mathcal{I}_2.$$

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Let $(p,q) \in \left\{(k,t) \in \mathbb{N}\times\mathbb{N} : \mathcal{M}\left(\bigcup_{m=k}^{\infty}\bigcup_{n=t}^{\infty}\{\|\zeta_{mn}-\zeta\| \geq \varepsilon\}\right) \geq \delta\right\}$, then

$$\mathcal{M}\left(\bigcup_{m=p}^{\infty}\bigcup_{n=q}^{\infty}\{\|\zeta_{mn}-\zeta\| \geq \delta\}\right) \geq \varepsilon$$

this implies $\lim_{p,q\to\infty}\mathcal{M}\left(\bigcup_{m=p}^{\infty}\bigcup_{n=q}^{\infty}\{\|\zeta_{mn}-\zeta\| \geq \delta\}\right) \geq \lim_{p,q\to\infty}\varepsilon = \varepsilon$

therfore $\mathcal{M}\left(\lim_{p,q\to\infty}\bigcup_{m=p}^{\infty}\bigcup_{n=q}^{\infty}\{\|\zeta_{mn}-\zeta\| \geq \delta\}\right) \geq \varepsilon$, since uncertainty space is continuous

so $\mathcal{M}\left(\bigcap_{p,q=1}^{\infty}\bigcup_{m=p}^{\infty}\bigcup_{n=q}^{\infty}\{\|\zeta_{mn}-\zeta\| \geq \delta\}\right) \geq \varepsilon$

therfore $(p,q) \in \left\{(k,t) \in \mathbb{N}\times\mathbb{N} : \mathcal{M}\left(\bigcap_{k,t=1}^{\infty}\bigcup_{m=k}^{\infty}\bigcup_{n=t}^{\infty}\{\|\zeta_k-\zeta\| \geq \delta\}\right) \geq \varepsilon\right\}$.

Thus $\left\{(k,t) \in \mathbb{N}\times\mathbb{N} : \mathcal{M}\left(\bigcup_{m=k}^{\infty}\bigcup_{n=t}^{\infty}\{\|\zeta_{mn}-\zeta\| \geq \delta\}\right) \geq \varepsilon\right\}$

$$\subseteq \left\{(k,t) \in \mathbb{N}\times\mathbb{N} : \mathcal{M}\left(\bigcap_{k,t=1}^{\infty}\bigcup_{m=k}^{\infty}\bigcup_{n=t}^{\infty}\{\gamma \in \Gamma : \|\zeta_k-\zeta\| \geq \delta\}\right) \geq \varepsilon\right\} \in \mathcal{I}_2.$$

Hence the sequence (ζ_{mn}) is $\mathcal{I}_2$-convergent uniformly almost surely to ζ. □

Theorem 3.8. *If a complex uncertain double sequence (ζ_{mn}) is $\mathcal{I}_2$-convergent uniformly almost surely to ζ, then it is $\mathcal{I}_2$-convergent in measure to ζ.*

Proof. Let the complex uncertain double sequence (ζ_{mn}) be $\mathcal{I}_2$-convergent uniformly almost surely to ζ, then for every $\varepsilon, \delta > 0$, we have

$$\left\{(k,t) \in \mathbb{N}\times\mathbb{N} : \mathcal{M}\left(\bigcup_{m=k}^{\infty}\bigcup_{n=t}^{\infty}\{\|\zeta_{mn}-\zeta\| \geq \delta\}\right) \geq \varepsilon\right\} \in \mathcal{I}_2.$$

Since $\mathcal{M}\{\|\zeta_{mn}-\zeta\| \geq \delta\} \leq \mathcal{M}\left(\bigcup_{m=k}^{\infty}\bigcup_{n=t}^{\infty}\{\|\zeta_{mn}-\zeta\| \geq \delta\}\right)$, then for any $\varepsilon > 0$,

$\{(m,n) \in \mathbb{N}\times\mathbb{N} : \mathcal{M}\{\|\zeta_{mn}-\zeta\| \geq \delta\} \geq \varepsilon\}$

$$\subseteq \left\{(k,t) \in \mathbb{N}\times\mathbb{N} : \mathcal{M}\left(\bigcup_{m=k}^{\infty}\bigcup_{n=t}^{\infty}\{\|\zeta_{mn}-\zeta\| \geq \delta\}\right) \geq \varepsilon\right\} \in \mathcal{I}_2.$$

Thus the sequence (ζ_{mn}) is $\mathcal{I}_2$-convergent in measure to ζ. □

But the converse is not true in general. For example, we consider the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2, ...\}$ with power set and $\mathcal{M}\{\Gamma\} = 1$, $\mathcal{M}\{\Phi\} = 0$ and

$$\mathcal{M}\{\Lambda\} = \begin{cases} \sup\limits_{\gamma_{m+n}\in\Lambda}\frac{(m+n)\beta_{mn}}{2(m+n)+1}, & if \ \sup\limits_{\gamma_{m+n}\in\Lambda}\frac{(m+n)\beta_{mn}}{2(m+n)+1} < \frac{1}{2} \\ 1 - \sup\limits_{\gamma_{m+n}\in\Lambda^c}\frac{(m+n)\beta_{mn}}{2(m+n)+1}, & if \ \sup\limits_{\gamma_{m+n}\in\Lambda^c}\frac{(m+n)\beta_{mn}}{2(m+n)+1} < \frac{1}{2} \\ \frac{1}{2}, & otherwise \end{cases}$$

where $\beta_{mn} = \begin{cases} n, \ if \ m = k^2, k \in \mathbb{N} \\ 0, \quad otherwise \end{cases} \quad for \ m = 1, 2, 3, \cdots.$

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Also, the complex uncertain variables are defined by

$$\zeta_{mn}(\gamma) = \begin{cases} (m+n+1)i, & if\ \gamma = \gamma_{m+n} \\ 0, & otherwise \end{cases} \quad for\ m,n = 1,2,3,\cdots$$

and $\zeta \equiv 0$. Take $\mathcal{I}_2$=$\mathcal{I}_2^d$.
It can be shown that the sequence (ζ_{mn}) is $\mathcal{I}_2$-convergent in measure to $\zeta \equiv 0$ but it is not $\mathcal{I}_2$-convergent uniformly almost surely to $\zeta \equiv 0$.

Proposition 3.9. *Suppose (ζ_{mn}) where $\zeta_{mn} = \xi_{mn} + i\eta_{mn}$ be a double sequence of complex uncertain variables and ζ where $\zeta = \xi + i\eta$ be a complex uncertain variables such that $\zeta_{kt} \geq \zeta_{mn} \geq \zeta$ in the sense that*

$$\xi_{kt} \geq \xi_{mn} \geq \xi \quad and \quad \eta_{kt} \geq \eta_{mn} \geq \eta \ \ for \ \ m \geq k, n \geq t.$$

Then (ζ_{mn}) is $\mathcal{I}_2$-convergent uniformly almost surely to ζ if it is $\mathcal{I}_2$-convergent in measure to ζ.

Proof. Let $\xi_{kt} \geq \xi_{mn} \geq \xi$ and $\eta_{kt} \geq \eta_{mn} \geq \eta$ for $m \geq k, n \geq t$, then
$\|\zeta_{mn} - \zeta\| \leq \|\zeta_{kt} - \zeta\|$ for $m \geq k, n \geq t$.
Now for every $\varepsilon > 0$, we have
$\{\|\zeta_{mn} - \zeta\| \geq \varepsilon\} \subseteq \{\|\zeta_{mn} - \zeta\| \geq \varepsilon\}$.
Therefore $\bigcup\limits_{m=k}^{\infty} \bigcup\limits_{n=t}^{\infty} \{\|\zeta_{mn} - \zeta\| \geq \delta\} = \{\|\zeta_{mn} - \zeta\| \geq \delta\}$.
Since the complex uncertain double sequence (ζ_{mn}) is $\mathcal{I}_2$-convergent in measure to ζ, then for every $\varepsilon, \delta > 0$, we have

$$\{(m,n) \in \mathbb{N} \times \mathbb{N} : \mathcal{M}\left(\{\|\zeta_{mn} - \zeta\| \geq \varepsilon\}\right) \geq \delta\} \in \mathcal{I}_2$$
$$\Rightarrow \left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \mathcal{M}\left(\bigcup_{m=k}^{\infty} \bigcup_{n=t}^{\infty} \{\|\zeta_{mn} - \zeta\| \geq \varepsilon\}\right) \geq \delta\right\} \in \mathcal{I}_2.$$

Hence the sequence (ζ_{mn}) is $\mathcal{I}_2$-convergent uniformly almost surely to ζ. □

Corollary 3.10. *Let (ζ_{mn}) be a complex uncertain double sequence defined on a continuous uncertainty space and let ζ be a complex uncertain variable. Then the following statements hold:*

(i) *If (ζ_{mn}) is $\mathcal{I}_2$-convergent uniformly almost surely to ζ, then it is $\mathcal{I}_2$-convergent in distribution to ζ;*
(ii) *If (ζ_{mn}) is $\mathcal{I}_2$-convergent almost surely to ζ, then it is $\mathcal{I}_2$-convergent in measure to ζ;*
(iii) *If (ζ_{mn}) is $\mathcal{I}_2$-convergent almost surely to ζ, then it is $\mathcal{I}_2$-convergent in distribution to ζ;*
(iv) *Suppose (ζ_{mn}), where $\zeta_{mn} = \xi_{mn} + i\eta_{mn}$, and $\zeta = \xi + i\eta$, satisfy*

$$\xi_{kt} \geq \xi_{mn} \geq \xi \quad and \quad \eta_{kt} \geq \eta_{mn} \geq \eta \quad for\ m \geq k,\ n \geq t.$$

If (ζ_{mn}) is $\mathcal{I}_2$-convergent in measure to ζ, then it is $\mathcal{I}_2$-convergent almost surely to ζ.

Proof. The results follow directly from theorems 3.4, 3.5, 3.7, 3.8, and 3.9. □

The various types of $\mathcal{I}_2$-convergence for complex uncertain double sequences discussed above are closely interconnected. The overall relationships among these convergence concepts may be summarized in the following diagram.

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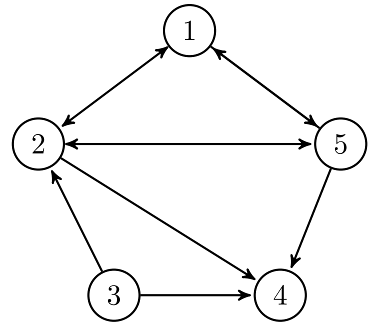
1. $\mathcal{I}_2$-convergence almost surely
2. $\mathcal{I}_2$-convergence in measure
3. $\mathcal{I}_2$-convergence in mean
4. $\mathcal{I}_2$-convergence in distribution
5. $\mathcal{I}_2$-convergence uniformly almost surely



4. Conclusion

This paper has been mainly devoted to the discussion of some newly introduced $\mathcal{I}_2$-convergence concepts of complex uncertain double sequences. We initiate the notion of $\mathcal{I}_2$-convergence almost surely, $\mathcal{I}_2$-convergence in measure, $\mathcal{I}_2$-convergence in mean, $\mathcal{I}_2$-convergence in distribution, $\mathcal{I}_2$-convergence uniformly almost surely of complex uncertain double sequences and include some interesting example related the notion. Also, in this paper, we try to establish the relationships among the above $\mathcal{I}_2$-convergence concepts of complex uncertain double sequences but we see that some of them are not related to each other. This is an open problem for further study and it may attract future researchers in this direction.

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