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IDENTIFYING CODE NUMBERS FOR SOME MIDDLE GRAPHS

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Abstract. Let $G = (V, E)$ be a simple graph. A subset C of vertices of G is an identifying code of G if for every two vertices x and y the sets $N_G[x] \cap C$ and $N_G[y] \cap C$ are distinct and non-empty. Given a graph G, the smallest size of an identifying code of G is called the identifying code number of G and is denoted by $\gamma^{ID}(G)$. In this paper, we show that for every graph G, the middle graph of G is an identifiable graph. We prove that the identifying code number of the middle graph of G is at most $|V(G)|$. Also, we determine the identifying code number of the middle graph of some graphs. In particular, we determine the identifying code number of the middle graph of a bipartite graph.

1. Introduction

In this paper, we assume that all graphs are finite, simple, and undirected. We will often use the notation $G = (V, E)$ to denote the graph with a non-empty vertex set $V = V(G)$ and edge set $E = E(G)$. The *order* of a graph G is the number of vertices and is denoted by $|G|$. An edge of G with endpoints v and u is denoted vu. For each vertex $x \in V(G)$, the *open neighborhood* of vertex x is denoted by $N_G(x)$ and defined as $N_G(x) = \{y \in V(G) : xy \in E(G)\}$. Also, the *closed neighborhood* of vertex $x \in V(G)$ is $N_G[x] = \{x\} \cup N_G(x)$. A *bipartite graph* G is a graph whose vertex set V can be partitioned into two nonempty subsets X and Y (i.e., $X \cup Y = V$ and $X \cap Y = \emptyset$) such that each edge of G has one endpoint in X and one endpoint in Y. The *symmetric difference* of set A and set B is denoted by $A \Delta B$ and defined as $A \Delta B = (A \setminus B) \cup (B \setminus A)$.

Hamada and Yoshimura in [5] introduced the concept of the middle graph $M(G)$. The *middle graph* of

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G is denoted by $M(G)$ whose vertex set is $V(M(G)) = V(G) \cup V_E(G)$, where $V_E(G) = \{x_e : e \in E(G)\}$. The vertex a is adjacent to vertex b in $M(G)$, if

i) $a = x_e,\ b = x_{e'},\ e = v_iv_j$ and $e' = v_iv_\ell$, or

ii) $a = x_e$ and $e = v_ib$.

A subset S of the vertices of G is a *dominating set* of G, if for every v, either $v \in S$ or there exists $s \in S$ with $v \in N_G(s)$. The *domination number* of G, denoted by $\gamma(G)$, is

$$\gamma(G) = min\{|S| : S\ is\ a\ dominating\ set\ of\ G\}.$$

A subset C of $V(G)$ is an *identifying code* of G, if C is a dominating set of G and for every pair $u,\ v$ of vertices of G, $N_G[u] \cap C \neq N_G[v] \cap C$. The graph G is an *identifiable graph*, if G has an identifying code. For the graph G, the *identifying code number* of G is denoted by $\gamma^{ID}(G)$ and defined by

$$\gamma^{ID}(G) = min\{|C| : C\ is\ a\ identifying\ code\ of\ G\}.$$

Two distinct vertices u and v are called *twins,* if $N_G[u] = N_G[v]$. It is clear that G is an identifiable graph if and only if G is twin-free. Recently, the domination theory is an interesting branch of graph theory. The domination extended to signed domination, 2-rainbow domination, total Roman domination and identifying code. For more information on this subject, the reader can see [1, 9–13]. Karpovsky et al. [8] in 1998 introduced the identifying code concept. Later, several various families of graphs have been studied, such as paths and cycles [3, 4], trees [2], triangular and square grids [7]. Also, the identifying codes have found applications in various fields.

In this paper we study the identifying code number of the middle graphs. In Section 2, it is shown that for each graph G, $M(G)$ is an identifiable graph and the identifying code number of $M(G)$ is at most $|G|$. In Section 3, we prove that the identifying code number of the middle graph of bipartite graph G is equal to $|G|$. In Section 4, we determine the identifying code number of the corona products $C_n \odot K_1$, $C_n \odot K_2$ and $C_n \odot \overline{K_r}$.

2. Preliminaries

In this Section, we will point out some facts that will be needed in the following Sections. We denote the open neighborhood of vertex x in the middle graph $M(G)$ by $N_M(x)$.

Lemma 2.1. *Let G be a graph and C be an identifying code of G. If $N_G[x] \Delta N_G[y] = B$, then $|B \cap C| \geq 1$.*

Proof. If $|B \cap C| = 0$, then $N_G[x] \cap C = N_G[y] \cap C$, which is not true. □

Theorem 2.2. *Let G be a graph of order $n \geq 2$. Then $M(G)$ is an identifiable graph.*

Proof. Let y and z be two arbitrary and distinct vertices in $M(G)$.

If $\{y, z\} \subseteq V(G)$, then $y \notin N_M[z]$. So $N_M[y] \neq N_M[z]$.

Let $\{y, z\} \subseteq V_E(G)$, $y = x_e$ and $z = x_{e'}$, where $e = ab$, $e' = cd$.

If $\{a, b\} \cap \{c, d\} = \emptyset$, then y is not adjacent to z in $M(G)$. Thus $N_M[y] \neq N_M[z]$.

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If $\{a,b\} \cap \{c,d\} \neq \emptyset$, then $|\{a,b\} \cap \{c,d\}| = 1$. Without loss the generality, we can assume that $b = c$. Then $a \in N_M[y] \setminus N_M[z]$. So $N_M[y] \neq N_M[z]$.

Let $y \in V(G)$, $z \in V_E(G)$, where $z = x_{e'}$ and $e' = cd$.

If $y = c$ (or $y = d$), then $d \in N_M[z] \setminus N_M[y]$ (or $c \in N_M[z] \setminus N_M[y]$).

If $y \neq c$ and $y \neq d$, then $d \in N_M[z] \setminus N_M[y]$. However $N_M[y] \neq N_M[z]$. Therefore, $M(G)$ is an identifiable graph. □

Theorem 2.3. *If G is a graph of order $n \geq 2$, then $\gamma^{ID}(M(G)) \leq n$.*

Proof. Let $C = V(G)$. It is clear that C is a dominating set of $M(G)$. We claim that C is an identifying code of $M(G)$. For this purpose, let y and z be two distinct vertices in $V(M(G))$. We consider the following cases:

Case 1) Let $\{y,z\} \subseteq V(G)$. Then $y \in N_M[y] \cap C$ and $y \notin N_M[z] \cap C$. Hence, $N_M[y] \cap C \neq N_M[z] \cap C$.

Case 2) Let $\{y,z\} \subseteq V_E(G)$, $y = x_e$ and $z = x_{e'}$, where $e = ab$, $e' = cd$ with $\{a,b,c,d\} \subseteq V(G)$.

If $\{a,b\} \cap \{c,d\} = \emptyset$, then $d \in N_M[z] \cap C$ and $d \notin N_M[y] \cap C$.

If $\{a,b\} \cap \{c,d\} \neq \emptyset$, for example $a = c$, then $d \in N_M[z] \cap C$ and $d \notin N_M[y] \cap C$.

However $N_M[y] \cap C \neq N_M[z] \cap C$.

Case 3) Let $y \in V(G)$, $z \in V_E(G)$, $z = x'_e$, where $e' = cd$.

If $y = c$ (or $y = d$), then $d \in N_M[z] \cap C$ and $d \notin N_M[y] \cap C$ (or $c \in N_M[z] \cap C$ and $c \notin N_M[y] \cap C$).

If $y \neq c$ and $y \neq d$, then $d \in N_M[z] \cap C$ and $d \notin N_M[y] \cap C$. Thus $N_M[y] \cap C \neq N_M[z] \cap C$.

Hence $C = V(G)$ is an identifying code of $M(G)$. Therefore, $\gamma^{ID}(M(G)) \leq |C| = n$. □

3. Identifying code number of the middle graph of bipartite graphs

Let G be a bipartite graph such that $V(G) = X \cup Y$, $X = \{v_i \mid 1 \leq i \leq r\}$, $Y = \{u_j \mid 1 \leq j \leq s\}$, $X \cap Y = \emptyset$ and $r \geq s$. Also, let $V(M(G)) = V(G) \cup \{x_{ij} \mid v_iu_j \in E(G),\ 1 \leq i \leq r,\ 1 \leq j \leq s\}$ and if $v_iu_j \in E(G)$, then $N_M(x_{ij}) \cap V(G) = \{v_i, u_j\}$.

We introduce the following table consisting of the vertices of $M(G)$.

	v_1	v_2	$\dots$	v_r
u_1	x_{11}	x_{21}	$\dots$	x_{r1}
u_2	x_{12}	x_{22}	$\dots$	x_{r2}
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
u_s	x_{1s}	x_{2s}	$\dots$	x_{rs}

Table 1. Vertices of $M(G)$

Suppose that $C_i = \{x_{ij} \mid v_iu_j \in E(G),\ 1 \leq j \leq s\} \cup \{v_i\}$ and $R_j = \{x_{ij} \mid v_iu_j \in E(G),\ 1 \leq i \leq r\} \cup \{u_j\}$, where $1 \leq i \leq r$ and $1 \leq j \leq s$. By definition of $M(G)$, we have $N_M[v_i] = C_i$, $|C_i| = deg_G(v_i) + 1$, $N_M[u_j] = R_j$, $|R_j| = deg_G(u_j) + 1$ and $N_M[x_{ij}] = C_i \cup R_j$, where $1 \leq i \leq r$ and

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$1 \leq j \leq s$.

For example the table of the vertices of $M(P_7)$ is:

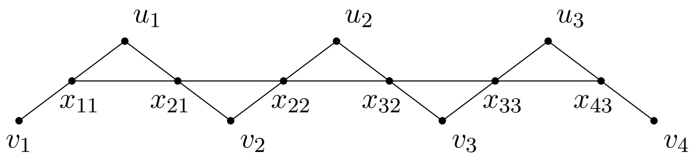

FIGURE 1. $M(P_7)$

	v_1	v_2	v_3	v_4
u_1	x_{11}	x_{21}		
u_2		x_{22}	x_{32}	
u_3			x_{33}	x_{43}

TABLE 2. Vertices of $M(P_7)$

We have, $C_1 = \{v_1, x_{11}\},\ C_2 = \{v_2, x_{21}, x_{22}\},\ C_3 = \{v_3, x_{32}, x_{33}\},\ C_4 = \{v_4, x_{43}\}$,

$$R_1 = \{u_1, x_{11}, x_{21}\},\ R_2 = \{u_2, x_{22}, x_{32}\},\ R_3 = \{u_3, x_{33}, x_{43}\}.$$

Theorem 3.1. *Let G be a bipartite graph. Then $\gamma^{ID}(M(G)) = |G|$.*

Proof. Let C be an identifying code of $M(G)$ such that $\gamma^{ID}(M(G)) = |C|$. By Theorem 2.3, $|C| \leq r+s$. By the above notation, we have the following two Claims:

Claim 1: If $x_{ij} \in C$, for some $1 \leq i \leq r$ and $1 \leq j \leq s$, then $|C_i \cap C| \geq 2$ and $|R_j \cap C| \geq 2$.

For proof of Claim 1, we assume that $|C_i \cap C| = 1$ and $C_i \cap C = \{x_{ij}\}$. Then $N_M[u_j] \cap C = N_M[x_{ij}] \cap C$, which is a contradiction. Similarly, we have $|R_j \cap C| \geq 2$.

Claim 2: If $|C_i| = 2$ for some $1 \leq i \leq r$, then $v_i \in C$.

For proof of Claim 2, if $|C_i| = 2$ and $v_i \notin C$, then $x_{ij} \in C$, for some $1 \leq j \leq s$, and based on Claim 1, it is not true. So $v_i \in C$. Similarly, if $|R_j| = 2$ for some $1 \leq j \leq s$, then $u_j \in C$.

On the contrary, let $|C| < r+s$. We have the following three cases:

Case 1: Let $X \subseteq C$. Since C is a dominating set of $M(G)$, $|R_j \cap C| \geq 1$, for every $j,\ 1 \leq j \leq s$. Thus $|C| \geq r+s$. It contradicts this fact that $|C| < r+s$.

Case 2: Let $X \cap C = \emptyset$. Since C is a dominating set of $M(G)$, for every $i,\ 1 \leq i \leq r$, we have $|(C_i \setminus X) \cap C| \geq 1$.

Suppose there exists $i,\ 1 \leq i \leq r$ such that $|C_i| = 2$. Then based on the above Claim 2, it is not true. Thus for every $i,\ 1 \leq i \leq r$, we have $|C_i| \geq 3$. By the above Claim 1, $|C| \geq 2r \geq r+s$, which is not true.

Case 3: Let $|X \cap C| = k \geq 1$ and $X \cap C = \{v_{i\ell} \mid 1 \leq \ell \leq k\}$. Then for every h, $h \notin \{i\ell \mid 1 \leq \ell \leq k\}$, by the Claim 2, $|C_h| \geq 3$. We have three following subcases:

Subcase 1: Suppose that $Y \subseteq C$. From the dominating set of C and the above Claim 1, we have

$$|C| \geq 2(r-k)+k+s = 2r-k+s = r+s+(r-k) \geq r+s.$$

Which is false.

Subcase 2: Suppose that $Y \cap C = \emptyset$. Since C is a dominating set of $M(G)$, for every j, $1 \leq j \leq s$, we have $|(R_j \setminus Y) \cap C| \geq 1$.

Suppose there exists j, $1 \leq j \leq s$ such that $|R_j| = 2$. Then based on the above Claim 2, it is not true. Thus $|R_j| \geq 3$ for every j, $1 \leq j \leq s$. By the Claim 1, $|C| \geq 2s+k$ and $|C| \geq 2(r-k)+k$.

If $r-k < s$, then $|C| \geq 2s+k > 2s+r-s = r+s$.

If $r-k \geq s$, then $|C| \geq 2(r-k)+k = 2r-k = r+(r-k) \geq r+s$.

Which is not true.

Subcase 3: Suppose that $|Y \cap C| = t \geq 1$ and $Y \cap C = \{v_{j\ell} \mid 1 \leq \ell \leq t\}$. Then for every h, $h \notin \{j\ell \mid 1 \leq \ell \leq t\}$, by the Claim 2, $|R_h| \geq 3$. By the Claim 1, $|C| \geq 2(s-t)+t+k$ and $|C| \geq 2(r-k)+t+k$.

If $r-k < s-t$, then $|C| \geq 2(s-t)+t+k = 2s-t+k = s+(s-t)+k > s+(r-k)+k = r+s$.

If $r-k \geq s-t$, then $|C| \geq 2(r-k)+t+k = 2r-k+t = r+(r-k)+t \geq r+(s-t)+t = r+s$.

Which is a contradiction. Therefore, $|C| = r+s$. □

Corollary 3.2. *If T is a tree of order n, then $\gamma^{ID}(M(T)) = n$.*

Proof. Every tree is a bipartite graph. By Theorem 3.1, $\gamma^{ID}(M(T)) = n$. □

Corollary 3.3. *For each $n \geq 3$, $\gamma^{ID}(M(C_n)) = \gamma^{ID}(M(P_n)) = n$.*

Proof. By Corollary 3.2, $\gamma^{ID}(M(P_n)) = n$.

If n is even, then C_n is a bipartite graph. By Theorem 3.1, $\gamma^{ID}(M(C_n)) = n$.

Let n be odd, $V(C_n) = \{v_i | 1 \leq i \leq n\}$ and $V(M(C_n)) = V(C_n) \cup \{u_i | 1 \leq i \leq n\}$ such that $N_M(u_n) \cap V(C_n) = \{v_1, v_n\}$ and for every $i, 1 \leq i \leq n-1$, $N_M(u_i) \cap V(C_n) = \{v_i, v_{i+1}\}$ (See Figure 2).

Let C be an identifying code of $M(C_n)$ with $|C| = \gamma^{ID}(M(C_n))$. By Theorem 2.3, $|C| \leq n$. Since $N_M[v_n] \Delta N_M[u_n] = \{v_1, u_1\}$, by Lemma 2.1, $v_1 \in C$ or $u_1 \in C$. Also, for every $i, 1 \leq i \leq n-1$, we have $N_M[v_i] \Delta N_M[u_i] = \{v_{i+1}, u_{i+1}\}$. By Lemma 2.1, $v_{i+1} \in C$ or $u_{i+1} \in C$, for every $i, 1 \leq i \leq n-1$. Hence $|C| \geq n$. Therefore, $|C| = n$ and so $\gamma^{ID}(M(C_n)) = n$.

□

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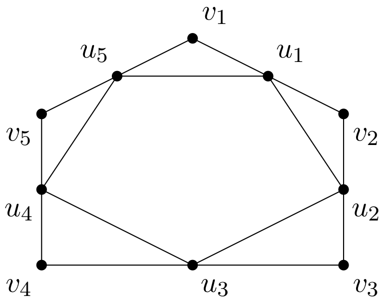

FIGURE 2. $M(C_5)$

4. Identifying code number of the middle graph of the corona product

In this Section, we determine the identifying code number of the corona products $C_n \odot K_1$, $C_n \odot K_2$ and $C_n \odot \overline{K_r}$.

Definition 4.1. *The corona product of two graphs G and H, written as $G \odot H$, is made by taking one copy of G and for each vertex in G, attaching a whole copy of H. Each vertex in G is connected to all the vertices in its corresponding copy of H.*

Lemma 4.2. *Let G be a connected graph, $V(G) = \{v_i | 1 \leq i \leq n\}$ and $V(G \odot K_1) = V(G) \cup \{u_i | 1 \leq i \leq n\}$. If C is an identifying code of $M(G \odot K_1)$, then for every $i, 1 \leq i \leq n$, $u_i \in C$.*

Proof. Let $X = \{x_i | 1 \leq i \leq n\} \subseteq V(M(G \odot K_1))$ such that $N_M(x_i) \cap V(G \odot K_1) = \{u_i, v_i\}$. Since for every $i, 1 \leq i \leq n$, $N_M[v_i] \Delta N_M[x_i] = \{u_i\}$, by Lemma 2.1, $u_i \in C$. □

Theorem 4.3. *If $G \cong C_n \odot K_1$ and $n \geq 3$, then $\gamma^{ID}(M(G)) = 2n$. (See Figure 3)*

Proof. Let $V(G) = \{v_i | 1 \leq i \leq n\} \cup \{u_i | 1 \leq i \leq n\}$, where $deg_G(u_i) = 1$ and for every $i, 1 \leq i \leq n$, $deg_G(v_i) = 3$.
Also let $V(M(G)) = V(G) \cup \{x_i | 1 \leq i \leq n\} \cup \{y_i | 1 \leq i \leq n\}$ and for every $i, 1 \leq i \leq n$, $N_M(x_i) \cap V(G) = \{u_i, v_i\}$ and $N_M(y_i) \cap V(G) = \{v_i, v_{i+1}\}$ and notice that $v_{n+1} = v_1$.
Let C be an identifying code of $M(G)$ with minimum cardinality. By Theorem 2.3, $|C| \leq 2n$.
By Lemma 4.2, $u_i \in C$ for every $i, 1 \leq i \leq n$. So $|C| \geq n$. For every $i, 1 \leq i \leq n$, we have $N_M[v_i] \Delta N_M[y_i] = \{v_{i+1}, y_{i+1}, x_{i+1}\}$.
By Lemma 2.1, for every $i, 1 \leq i \leq n$, $|\{v_{i+1}, y_{i+1}, x_{i+1}\} \cap C| \geq 1$ and $\{v_{i+1}, y_{i+1}, x_{i+1}\}$ are pairwise disjoint and also disjoint from the already forced set $\{u_1, \ldots, u_n\}$. So $|C| \geq n + n = 2n$. Therefore, $\gamma^{ID}(M(G)) = |C| = 2n$.

□

Theorem 4.4. *If $G \cong C_n \odot K_2$ and $n \geq 3$, then $\gamma^{ID}(M(G)) = 3n$.*

Proof. Let $V(G) = \{v_i | 1 \leq i \leq n\} \cup \{u_{i_1}, u_{i_2} | 1 \leq i \leq n\}$, where u_{i_1} and u_{i_2} are vertices of $i-th$ copy K_2 in $C_n \odot K_2$. Also let $V(M(G)) = V(G) \cup \{x_{i_1}, x_{i_2} | 1 \leq i \leq n\} \cup \{y_i | 1 \leq i \leq n\} \cup \{z_i | 1 \leq i \leq n\}$ and for

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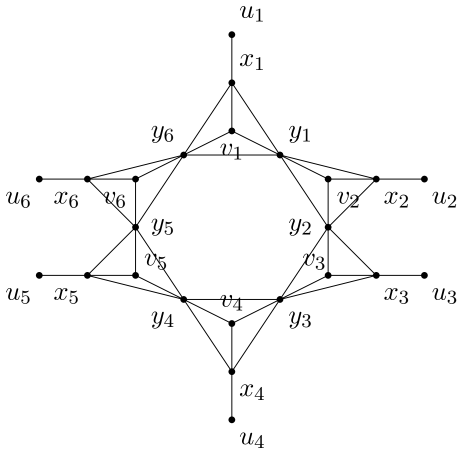

FIGURE 3. $M(C_6 \odot K_1)$

every $i, 1 \leq i \leq n$, $N_M(x_{i_1}) \cap V(G) = \{u_{i_1}, v_i\}$, $N_M(x_{i_2}) \cap V(G) = \{u_{i_2}, v_i\}$, $N_M(z_i) \cap V(G) = \{u_{i_1}, u_{i_2}\}$ and $N_M(y_i) \cap V(G) = \{v_i, v_{i+1}\}$. (See Figure 4)

Let C be an identifying code of $M(G)$ with $|C| = \gamma^{ID}(M(G))$.

Let for every $i, 1 \leq i \leq n$, $A_i = \{x_{i_1}, u_{i_1}, z_i, u_{i_2}, x_{i_2}, v_i, y_i\}$. We have $N_M[v_i] \Delta N_M[x_{i_1}] = \{u_{i_1}, z_i\}$ and $N_M[v_i] \Delta N_M[x_{i_2}] = \{u_{i_2}, z_i\}$. By Lemma 2.1, $|\{u_{i_1}, z_i\} \cap C| \geq 1$ and $|\{u_{i_2}, z_i\} \cap C| \geq 1$. If $|A_i \cap C| = 1$, for some $i, 1 \leq i \leq n$, then $|A_i \cap C| = \{z_i\}$ and so $N_M[x_{i_1}] \cap C = N_M[x_{i_2}] \cap C$. It is contradicts with this fact that C is identifying code of $M(G)$. Thus for every $i, 1 \leq i \leq n$, $|A_i \cap C| \geq 2$ and $|\{u_{i_1}, u_{i_2}, z_i\} \cap C| \geq 2$. Let for some $i, 1 \leq i \leq n$, $|A_i \cap C| = 2$. If $A_i \cap C = \{u_{i_1}, z_i\}$ or $A_i \cap C = \{u_{i_2}, z_i\}$, then $N_M[u_{i_1}] \cap C = N_M[z_i] \cap C$ or $N_M[u_{i_2}] \cap C = N_M[z_i] \cap C$. Which is a contradiction. Thus if $|A_i \cap C| = 2$, then $A_i \cap C = \{u_{i_1}, u_{i_2}\}$.

Suppose that there is $i, 1 \leq i \leq n$, $|A_i \cap C| = 2$. Then $A_i \cap C = \{u_{i_1}, u_{i_2}\}$. Without loss the generality, we can assume that $|A_1 \cap C| = 2$. So $A_1 \cap C = \{u_{1_1}, u_{1_2}\}$. If $y_n \notin C$, then v_1 is not dominated by C. Thus $y_n \in C$ and so $|A_n \cap C| \geq 3$.

If $|A_2 \cap C| = 2$, then v_2 is not dominated by C. So $|A_2 \cap C| \geq 3$.

If $|A_3 \cap C| = 2$, then $N_M[y_2] \cap C = N_M[v_2] \cap C$. Which is false. So $|A_3 \cap C| \geq 3$.

If $|A_4 \cap C| = 2$, then $N_M[y_3] \cap C = N_M[v_3] \cap C$. It is not true. So $|A_4 \cap C| \geq 3$. In the same way for every $i, 2 \leq i \leq n-1$, $|A_i \cap C| \geq 3$. Thus $|C| \geq 3(n-1) + 2 = 3n - 1$.

Let $|C| = 3n - 1$. Then for every $i, 2 \leq i \leq n$, $|A_i \cap C| = 3$ and $|A_1 \cap C| = 2$. It is easy to see that $N_M[y_n] \cap C = N_M[v_n] \cap C$. It is not true. Thus for every $i, 1 \leq i \leq n$, $|A_i \cap C| \geq 3$ and so $|C| \geq 3n$. By Theorem 2.3, $\gamma^{ID}(M(G)) = 3n$.

□

Theorem 4.5. *If* $G = C_n \odot \overline{K_r}$ *and* $n \geq 3$ *and* $r \geq 2$, *then* $\gamma^{ID}(M(G)) = (r+1)n$.

Proof. Let $V(G) = \{v_i | 1 \leq i \leq n\} \cup \{u_{i_j} | 1 \leq i \leq n,\ 1 \leq j \leq r\}$, where u_{i_j} are vertices of $i - th$ copy $\overline{K_r}$ in $C_n \odot \overline{K_r}$. Also, let $V(M(G)) = V(G) \cup \{x_{i_j} | 1 \leq i \leq n,\ 1 \leq j \leq r\} \cup \{y_i | 1 \leq i \leq n\}$ and for

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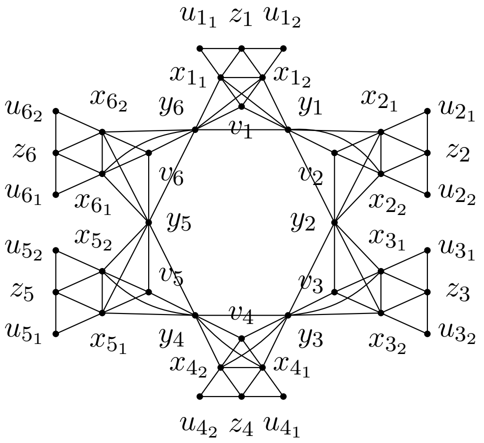

FIGURE 4. $M(C_6 \odot K_2)$

every $i, 1 \leq i \leq n$ and $j, 1 \leq j \leq r$, $N_M(x_{i_j}) \cap V(G) = \{u_{i_j}, v_i\}$ and $N_M(y_i) \cap V(G) = \{v_i, v_{i+1}\}$ (See Figure 5).

By Theorem 2.2, $M(G)$ is an identifiable graph. Let C be an identifying code of $M(G)$ with $|C| = \gamma^{ID}(M(G))$. By Theorem 2.3, $\gamma^{ID}(M(G)) \leq (r+1)n$.

For every $i, 1 \leq i \leq n$ and $j, 1 \leq j \leq r$, $N_M[v_i] \Delta N_M[x_{i_j}] = \{u_{i_j}\}$. By Lemma 2.1, $u_{i_j} \in C$.

Let for every $i, 1 \leq i \leq n$, $A_i = \{u_{i_j}, x_{i_j} | 1 \leq j \leq r\} \cup \{v_i, y_i\}$. Then $|A_i \cap C| \geq r$. If for every $i, 1 \leq i \leq n$, $|A_i \cap C| \geq (r+1)$, then $|C| = \gamma^{ID}(M(G)) = (r+1)n$. So we can assume that there is $i, 1 \leq i \leq n$, $|A_i \cap C| = r$. Without loss the generality, we can assume that $|A_1 \cap C| = r$. So $|A_1 \cap C| = \{u_{1_j} | 1 \leq j \leq r\}$.

If $y_n \notin C$, then v_1 is not dominated by C. So $y_n \in C$ and so $|A_n \cap C| \geq (r+1)$.

If $|A_2 \cap C| = r$, then v_2 is not dominated by C. So $|A_2 \cap C| \geq (r+1)$. If $|A_3 \cap C| = r$, then $N_M[y_2] \cap C = N_M[v_2] \cap C$. Which is false. So $|A_3 \cap C| \geq (r+1)$.

If $|A_4 \cap C| = r$, then $N_M[y_3] \cap C = N_M[v_3] \cap C$. It is not true. So $|A_4 \cap C| \geq (r+1)$. In the same way for every $i, 2 \leq i \leq n$, $|A_i \cap C| \geq (r+1)$. Thus $|C| \geq (r+1)(n-1) + r = (r+1)n - 1$.

Let $|C| = (r+1)n - 1$. Then for every $i, 2 \leq i \leq n$, $|A_i \cap C| = (r+1)$ and $|A_1 \cap C| = r$. It is clear that $N_M[y_n] \cap C = N_M[v_n] \cap C$. It is not true. Therefore, $\gamma^{ID}(M(G)) = (r+1)n$.

□

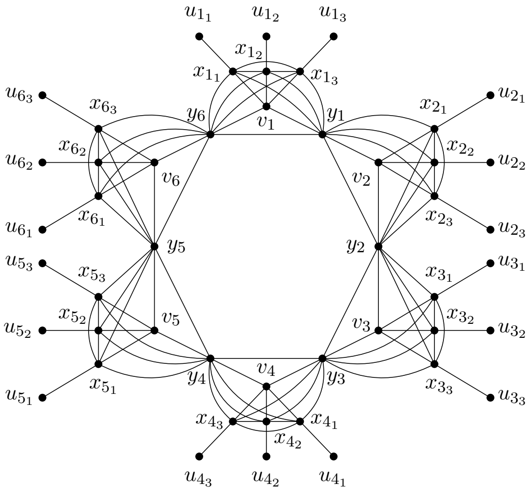

FIGURE 5. $M(C_6 \odot \overline{K_3})$

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