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CESÀRO HYPERCYCLICITY AND TRANSITIVITY FOR C_0-SEMIGROUPS

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Abstract. In this paper, we introduce the concepts of Cesàro hypercyclicity and Cesàro transitivity for C_0-semigroups. We prove that a C_0-semigroup is Cesàro transitive if and only if it possesses a dense set of Cesàro hypercyclic vectors. Subsequently, we demonstrate that Cesàro transitive C_0-semigroups are hypercyclic. Also, we provide an example of a Cesàro hypercyclic C_0-semigroup that is not hypercyclic. We establish that if a C_0-semigroup contains a Cesàro hypercyclic operator, then the entire semigroup is Cesàro hypercyclic. Furthermore, we characterize the structure of Cesàro hypercyclic vectors. Additionally, we define sequentially Cesàro mixing C_0-semigroups that is a subset of Cesàro hypercyclic C_0-semigroups. We provide certain criteria for sequential Cesàro mixing, and use them to make an example of a Cesàro mixing C_0-semigroup.

1. Introduction

Let X be a complex separable Banach space, and let $B(X)$ denote the algebra of all bounded linear operators on X. An operator $T \in B(X)$ is called hypercyclic, if there exists $x \in X$ such that the orbit $\mathrm{Orb}(T, x) = \{x, Tx, T^2x, \ldots, T^nx, \ldots\}$ is dense in X [3]. For $T \in B(X)$, this property is equivalent to transitivity. An operator $T \in B(X)$ is transitive, if for any nonempty open subsets U and V of X, there exists a nonnegative integer n such that $T^n(U) \cap V \neq \emptyset$ [7]. Imagine U is a small ball around some vector u and V is a small ball around some vector v. The transitivity property says that if you look at the pre-images of V under T^n (i.e.,$(T^n)^{-1}(V)$), these sets are so "spread out" across the space that at least one of them must intersect U. Recent developments in hypercyclicity and related topics can be found in [1] and [2].

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Cesàro hypercyclicity for operators was first introduced by León-Saavedra in [11]. For $n \in \mathbb{N}$, let $M_n(T) = \frac{I+T+T^2+\cdots+T^{n-1}}{n}$ denote the arithmetic mean of the powers of $T \in B(X)$. An operator T is said to be Cesàro hypercyclic if $\{M_n(T)x : n \in \mathbb{N}\}$ is dense in X for some $x \in X$ [4]. It was proved in [11] that Cesàro hypercyclicity of an operator T on X is equivalent to the density of $\{\frac{1}{n}T^n x : n \in \mathbb{N}\}$ in X for some $x \in X$. Orbits of Cesàro-type operators are studied in [10]. The authors of [14] provide an example of a hypercyclic operator that is not Cesàro hypercyclic, and conversely, a Cesàro hypercyclic operator that is not hypercyclic. Cesàro hypercyclic relations are investigated in [6], and subspace Cesàro hypercyclic operators are introduced in [5], where a subspace Cesàro criterion is established.

A vector $x \in X$ is a Cesàro recurrent vector for T, if there exists a strictly increasing sequence $(n_k) \subseteq \mathbb{N}$ such that $M_{n_k}(T)x \to x$ as $k \to \infty$. If the set of Cesàro recurrent vectors of T is dense in X, then T is called a Cesàro recurrent operator [8]. Various properties of Cesàro recurrent operators are established in [8].

C_0-semigroups are fundamental structures in the theory of dynamical systems. A C_0-semigroup $(T_t)_{t\geq 0}$ on X consists of operators $T_t \in B(X)$ satisfying $T_0 = I$, $T_{t+s} = T_t T_s$ for all $t, s \geq 0$, and $\lim_{s\to t} T_s x = T_t x$ for all $x \in X$.

Many concepts defined for single operators have been extended to semigroups in dynamical systems. For instance, a C_0-semigroup $(T_t)_{t\geq 0}$ on X is called hypercyclic, if $\overline{\{T_t x : t \geq 0\}} = X$ for some $x \in X$. Hypercyclicity of $(T_t)_{t\geq 0}$ is equivalent to the condition that for any two non-empty open sets U and V of X, there exists $s \geq 0$ such that $T_s(U) \cap V \neq \emptyset$ [7]. In this case, we say that the C_0-semigroup is transitive. Moreover, $(T_t)_{t\geq 0}$ is called mixing, if for any two non-empty open sets U and V of X, there exists $s \geq 0$ such that $T_t(U) \cap V \neq \emptyset$ for all $t \geq s$ [7].

Recent developments have introduced newer concepts for C_0-semigroups. For example, recurrent C_0-semigroups are introduced and studied in [13]. A C_0-semigroup $(T_t)_{t\geq 0}$ on X is called recurrent, if for any non-empty open set $V \subseteq X$, there exists $s > 0$ such that $T_s(V) \cap V \neq \emptyset$ [13]. Other recent contributions include Faber-hypercyclic semigroups [9] and topologically ergodic C_0-semigroups [12].

In Section 2 of this paper, we define Cesàro hypercyclicity and Cesàro transitivity for C_0-semigroups. We prove that a C_0-semigroup is Cesàro transitive if and only if it has a dense set of Cesàro hypercyclic vectors. Subsequently, we show that Cesàro transitive C_0-semigroups are hypercyclic. We establish that if a C_0-semigroup contains a Cesàro hypercyclic operator, then the entire semigroup is Cesàro hypercyclic. We also characterize the structure of Cesàro hypercyclic vectors. Furthermore, we define sequentially Cesàro mixing semigroups. In Section 3, we provide several criteria for sequential Cesàro mixing and Cesàro hypercyclicity.

2. **Preliminaries**

We begin this section by defining Cesàro hypercyclic C_0-semigroups.

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Definition 2.1. *A C_0-semigroup $(T_t)_{t\geq 0}$ on X is called Cesàro hypercyclic, if there exists $x \in X$ such that the set*

$$\left\{\frac{1}{t}T_t x : t > 0\right\}$$

is dense in X.

Such a vector x is called a Cesàro hypercyclic vector for $(T_t)_{t\geq 0}$. The set of all Cesàro hypercyclic vectors is denoted by $CHC(T_t)_{t\geq 0}$.

Example 2.2. *Consider C_0-semigroup $(T_t)_{t\geq 0}$ on $\mathbb{C}$, where for all $t \geq 0$, T_t is defined with $T_t z = e^{it}z$ for all $z \in \mathbb{C}$. We claim that $x = 1$ is a Cesàro hypercyclic vector for $(T_t)_{t\geq 0}$. According to Definition 2.1, we consider the set:*

$$S = \left\{\frac{1}{t}T_t(1) : t > 0\right\} = \left\{\frac{1}{t}e^{it} : t > 0\right\}.$$

Let $\varepsilon > 0$, and let $w \in \mathbb{C}$. If $w = 0$, then

$$\left|\frac{1}{t}e^{it} - 0\right| = \frac{|e^{it}|}{|t|} = \frac{1}{t}.$$

Note that $\frac{1}{t}e^{it} \in S$, and for large enough t, we have $\frac{1}{t} < \varepsilon$. Hence, for $w = 0$ there exists an element $y := \frac{1}{t}e^{it} \in S$ so that $|y - 0| < \varepsilon$.

Now consider that $w \neq 0$. The element w can be written in the form $w = Re^{it_0}$, with $R > 0$ and $t_0 \in [0, 2\pi)$. Note that for all $n \in \mathbb{N}$ we can replace this t_0 with $t_0 + 2n\pi$. Hence, t_0 can be chosen large enough such that $|\frac{1}{t_0} - R| < \varepsilon$. Therefore,

$$\left|Re^{it_0} - \frac{1}{t_0}e^{it_0}\right| = |R - \frac{1}{t_0}| < \epsilon.$$

Hence, for $w \neq 0$ there exists an element $z := \frac{1}{t_0}e^{it_0} \in S$ so that $|z - 0| < \varepsilon$. Therefore, $(T_t)_{t\geq 0}$ is a Cesàro hypercyclic C_0-semigroup.

The C_0-semigroup that is defined in Example 2.2 is not hypercyclic. Since if $(T_t)_{t\geq 0}$ is a hypercyclic C_0-semigroup, then for all $t > 0$, the operator T_t must be hypercyclic [7, Theorem 6.8]. *But this is impossible since for all $t > 0$, we have $|T_t x| \leq |x|$. Hence, Example 2.2 provides an example of a Cesàro hypercyclic C_0-semigroup that is not hypercyclic.*

The following proposition describes the structure of $CHC(T_t)_{t\geq 0}$.

Proposition 2.3. *Let $(T_t)_{t\geq 0}$ be a Cesàro hypercyclic C_0-semigroup on X, and let $\{B_n\}$ be a countable topology basis for X. Then*

$$CHC(T_t)_{t\geq 0} = \bigcap_{n=1}^{\infty} G_n,$$

where $G_n = \bigcup (\frac{T_t}{t})^{-1}(B_n)$.

Proof. Let $x \in \bigcap_{n=1}^{\infty} G_n$. Then for each $n \in \mathbb{N}$, there exists $t_n > 0$ such that $x \in \left(\frac{T_{t_n}}{t_n}\right)^{-1}(B_n)$, which implies $\frac{T_{t_n}}{t_n}x \in B_n$. Hence, $x \in CHC(T_t)_{t\geq 0}$.

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Conversely, let $x \in CHC(T_t)_{t\geq 0}$. For any $n \in \mathbb{N}$, since B_n is open, there exists $t_n > 0$ such that $\frac{T_{t_n}}{t_n} x \in B_n$. Thus, $x \in \left(\frac{T_{t_n}}{t_n}\right)^{-1}(B_n) \subseteq G_n$. Since n is arbitrary, $x \in \bigcap_{n=1}^{\infty} G_n$. □

Definition 2.4. *A C_0-semigroup $(T_t)_{t\geq 0}$ on X is called Cesàro transitive, if for any non-empty open sets $U, V \subseteq X$, there exists $t_0 > 0$ such that*

$$\frac{1}{t_0} T_{t_0}(U) \cap V \neq \emptyset.$$

This definition says that for any two open sets U and V, there exists some time t_0 such that the average effect of the evolution up to time t_0 maps some point from U into V. We suggest that the reader think of it as "effective transportation", rather than "point to point" mapping. Proposition 2.3 leads to an equivalent condition for Cesàro transitivity, stated in the next theorem.

Theorem 2.5. *A C_0-semigroup $(T_t)_{t\geq 0}$ on X is Cesàro transitive, if and only if $\overline{CHC(T_t)_{t\geq 0}} = X$.*

Proof. Assume $(T_t)_{t\geq 0}$ is Cesàro transitive. Let $\{B_n\}$ be a countable basis for X, and let $U \subseteq X$ be a non-empty open set. By Cesàro transitivity, for each $n \in \mathbb{N}$, there exists $t_n > 0$ such that $\left(\frac{T_{t_n}}{t_n}\right)(U) \cap B_n \neq \emptyset$. Hence, $U \cap \left(\frac{T_{t_n}}{t_n}\right)^{-1}(B_n) \neq \emptyset$, and so $U \cap G_n \neq \emptyset$. Since this holds for all n, we have $U \cap (\bigcap_{n=1}^{\infty} G_n) \neq \emptyset$. By Lemma 2.3, $U \cap CHC(T_t)_{t\geq 0} \neq \emptyset$, implying $\overline{CHC(T_t)_{t\geq 0}} = X$.

Conversely, assume that $\overline{CHC(T_t)_{t\geq 0}} = X$. Let $U, V \subseteq X$ be non-empty open sets. The density of $CHC(T_t)_{t\geq 0}$ implies $U \cap CHC(T_t)_{t\geq 0} \neq \emptyset$. By Lemma 2.3, $U \cap (\bigcap_{n=1}^{\infty} G_n) \neq \emptyset$, so $U \cap G_n \neq \emptyset$ for all $n \in \mathbb{N}$. By definition of G_n, for each n, there exists $t_n > 0$ such that $U \cap \left(\frac{T_{t_n}}{t_n}\right)^{-1}(B_n) \neq \emptyset$, i.e.,

$$\left(\frac{T_{t_n}}{t_n}\right)(U) \cap B_n \neq \emptyset. \tag{2.1}$$

Since $\{B_n\}$ is a basis, there exists $m \in \mathbb{N}$ such that $B_m \subseteq V$. Then by (2.1),

$$\emptyset \neq \left(\frac{T_{t_m}}{t_m}\right)(U) \cap B_m \subseteq \left(\frac{T_{t_m}}{t_m}\right)(U) \cap V,$$

so $(T_t)_{t\geq 0}$ is Cesàro transitive. □

Corollary 2.6. *Every Cesàro transitive C_0-semigroup is Cesàro hypercyclic.*

Proof. If $(T_t)_{t\geq 0}$ is Cesàro transitive, then by Theorem 2.5, $\overline{CHC(T_t)_{t\geq 0}} = X$, so $(T_t)_{t\geq 0}$ is Cesàro hypercyclic. □

By Corollary 2.6, Cesàro transitive C_0-semigroups form a subset of Cesàro hypercyclic C_0-semigroups.

Theorem 2.7. *If a C_0-semigroup $(T_t)_{t\geq 0}$ on X contains a Cesàro hypercyclic operator, then $(T_t)_{t\geq 0}$ is Cesàro hypercyclic.*

Proof. Suppose T_{s_0} is a Cesàro hypercyclic operator in $(T_t)_{t\geq 0}$. Then there exists $x \in X$ such that

$$\overline{\left\{\frac{1}{n} T_{s_0}^n x : n \in \mathbb{N}\right\}} = X.$$

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Since $T_{s_0}^n = T_{ns_0}$ by the semigroup property, we have

$$\overline{\left\{\frac{1}{n}T_{ns_0}x : n \in \mathbb{N}\right\}} = X.$$

First, suppose $s_0 \geq 1$. We claim that $\left\{\frac{1}{ns_0}T_{ns_0}x : n \in \mathbb{N}\right\}$ is dense in X. Let $y \in X$ and $\varepsilon > 0$. By Cesàro hypercyclicity of T_{s_0}, there exists $m \in \mathbb{N}$ such that $\left\|\frac{1}{m}T_{ms_0}x - s_0y\right\| < \varepsilon$. Then

$$\left\|\frac{1}{ms_0}T_{ms_0}x - y\right\| < \frac{\varepsilon}{s_0} \leq \varepsilon.$$

Since y and ε are arbitrary, $\overline{\left\{\frac{1}{ns_0}T_{ns_0}x : n \in \mathbb{N}\right\}} = X$.

Now, suppose $s_0 < 1$. Let $y \in X$ and $\varepsilon > 0$. By Cesàro hypercyclicity of T_{s_0}, there exists $p \in \mathbb{N}$ such that $\left\|\frac{1}{p}T_{ps_0}x - s_0y\right\| < s_0\varepsilon$. Then

$$\left\|\frac{1}{ps_0}T_{ps_0}x - y\right\| < \varepsilon.$$

Hence, $\overline{\left\{\frac{1}{ns_0}T_{ns_0}x : n \in \mathbb{N}\right\}} = X$ in this case as well. Since

$$\left\{\frac{1}{ns_0}T_{ns_0}x : n \in \mathbb{N}\right\} \subseteq \left\{\frac{1}{t}T_tx : t > 0\right\},$$

it follows that $\overline{\left\{\frac{1}{t}T_tx : t > 0\right\}} = X$. □

Definition 2.8. *A C_0-semigroup $(T_t)_{t\geq 0}$ on X is called Cesàro mixing if for any non-empty open sets $U, V \subseteq X$, there exists $t_0 > 0$ such that for all $t \geq t_0$,*

$$\frac{1}{t}T_t(U) \cap V \neq \emptyset.$$

It follows from Definition 2.8 that Cesàro mixing C_0-semigroups are Cesàro transitive, and hence by Corollary 2.6, they are hypercyclic.

3. Criteria for Sequential Cesàro Mixing and Cesàro Hypercyclicity

Definition 3.1. *Let $(T_t)_{t\geq 0}$ be a C_0-semigroup on X, and let (t_n) be a sequence of increasing positive real numbers tending to infinity. Then $(T_t)_{t\geq 0}$ is called sequentially Cesàro mixing with respect to (t_n) if for every non-empty open sets $U, V \subseteq X$, there exists $M \in \mathbb{N}$ such that for all $n \geq M$,*

$$\frac{1}{t_n}T_{t_n}(U) \cap V \neq \emptyset.$$

Clearly, Cesàro mixing C_0-semigroups are sequentially Cesàro mixing (by taking $t_n = n$). Note that sequentially Cesàro mixing C_0-semigroups are Cesàro transitive and hence Cesàro hypercyclic.

Theorem 3.2. *Let $(T_t)_{t\geq 0}$ be a C_0-semigroup on X. Suppose there exist dense subsets X_0 and Y_0 of X, and a sequence (t_n) of increasing positive real numbers tending to infinity such that:*

(1) *$\frac{T_{t_n}}{t_n}x \to 0$ for all $x \in X_0$,*
(2) *There exist mappings $S_{t_n} : Y_0 \to X$ such that for all $y \in Y_0$, $S_{t_n}y \to 0$ and $\frac{T_{t_n}}{t_n}S_{t_n}y \to y$.*

Then $(T_t)_{t\geq 0}$ is sequentially Cesàro mixing. In particular, $(T_t)_{t\geq 0}$ is Cesàro hypercyclic.

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Proof. Let $U, V \subseteq X$ be non-empty open sets. Choose $u \in X_0 \cap U$ and $v \in Y_0 \cap V$. By the assumption,

$$\frac{T_{t_n}}{t_n} u \to 0, \quad S_{t_n} v \to 0, \quad \text{and} \quad \frac{T_{t_n}}{t_n} S_{t_n} v \to v.$$

Let $w_n = u + S_{t_n} v$. Then

$$w_n \to u \quad \text{and} \quad \frac{T_{t_n}}{t_n} w_n = \frac{T_{t_n}}{t_n} u + \frac{T_{t_n}}{t_n} S_{t_n} v \to v.$$

Thus, there exists $M \in \mathbb{N}$ such that for all $n \geq M$,

$$w_n \in U \quad \text{and} \quad \frac{T_{t_n}}{t_n} w_n \in V.$$

Hence, for all $n \geq M$, $\frac{T_{t_n}}{t_n}(U) \cap V \neq \emptyset$, so $(T_t)_{t \geq 0}$ is sequentially Cesàro mixing and therefore Cesàro hypercyclic. □

By Theorem 3.2 we make an example of a sequentially Cesàro mixing C_0-semigroup as follows.

Example 3.3. *Let $X = \ell^2(\mathbb{N})$. We define a C_0-semigroup $(T_t)_{t \geq 0}$ on $\ell^2(\mathbb{N})$ by its action on the standard basis $\{e_n\}_{n=1}^{\infty}$ with*

$$T_t e_n = e^t e_{n-1}, \quad \textit{for all } t > 0,$$

and for all $n \in \mathbb{N}$ with $n \geq 2$ and we define $T_t e_1 = 0$. For $t = 0$, we define $T_0 = I$. Presume $S_t : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ is defined by its action on the standard basis $\{e_n\}_{n=1}^{\infty}$ with

$$S_t e_n = t e^{-t} e_{n+1}, \quad \textit{for all } t \geq 0 \textit{ and for all } n \in \mathbb{N}.$$

Consider X_0 and Y_0 be the set of finite sequences of $\ell^2(\mathbb{N})$. Presume (t_n) is a sequence of increasing positive real numbers tending to infinity. Then for all n,

$$\frac{1}{t_n} T_{t_n} e_n = \frac{1}{t_n} e^{t_n} e_{n-1} \to 0.$$

Also,

$$S_{t_n} e_n = t_n e^{-t_n} e_{n+1} \to 0.$$

Moreover, for all n,

$$\frac{1}{t_n} T_{t_n} S_{t_n} e_n = \frac{1}{t_n} T_{t_n}(t_n e^{-t_n} e_{n+1}) = \frac{1}{t_n}(e^{t_n} t_n e^{-t_n}) e_n \to e_n$$

Therefore, the conditions of Theorem 3.2 are satisfied. Hence, $(T_t)_{t \geq 0}$ is sequentially Cesàro mixing and Cesàro hypercyclic.

Theorem 3.4. *Let $(T_t)_{t \geq 0}$ be a C_0-semigroup on X. Suppose that there exists a dense subset X_0 of X, a sequence (t_n) of increasing positive real numbers tending to infinity, and mappings $S_{t_n} : X_0 \to X$ such that:*

(1) *$S_{t_n} y \to 0$, for all $y \in X_0$,*
(2) *$\frac{T_{t_n}}{t_n} S_{t_n} y \to y$ for all $y \in X_0$,*
(3) *For each $y \in X_0$, there exists a dense subset $X_y \subseteq X$ such that $\frac{T_{t_n}}{t_n} z \to 0$ for all $z \in X_y$.*

Then $(T_t)_{t \geq 0}$ is sequentially Cesàro mixing. In particular, $(T_t)_{t \geq 0}$ is Cesàro hypercyclic.

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Proof. Let $U, V \subseteq X$ be non-empty open sets. Choose $v \in X_0 \cap V$ and let $v_n = S_{t_n} v$. By conditions (1) and (2),

$$v_n \to 0 \quad \text{and} \quad \frac{T_{t_n}}{t_n} v_n = \frac{T_{t_n}}{t_n} S_{t_n} v \to v.$$

By condition (3), for this v, there exists a dense subset $X_v \subseteq X$ such that for all $x \in X_v$,

$$\frac{T_{t_n}}{t_n} x \to 0. \tag{3.1}$$

Choose $u \in U \cap X_v$. Then by (3.1),

$$\frac{T_{t_n}}{t_n} u \to 0. \tag{3.2}$$

Hence,

$$\frac{T_{t_n}}{t_n}(u + v_n) = \frac{T_{t_n}}{t_n} u + \frac{T_{t_n}}{t_n} v_n \to v.$$

Moreover, by condition (1), $u + v_n \to u$. Thus, there exists $M \in \mathbb{N}$ such that for all $n \geq M$,

$$u + v_n \in U \quad \text{and} \quad \frac{T_{t_n}}{t_n}(u + v_n) \in V.$$

Therefore, for all $n \geq M$, $\frac{T_{t_n}}{t_n}(U) \cap V \neq \emptyset$, so $(T_t)_{t \geq 0}$ is sequentially Cesàro mixing and hence Cesàro hypercyclic. □

Theorem 3.5. *Let $(T_t)_{t\geq 0}$ be a C_0-semigroup on X. Suppose there exist dense subsets X_0 and Y_0 of X, and an increasing sequence (t_n) of positive real numbers tending to infinity such that:*

1. *$\frac{T_{t_n}}{t_n} x \to 0$ for all $x \in X_0$,*
2. *For each $z \in Y_0$, there exists a sequence $(z_n) \subseteq X$ such that $z_n \to 0$ and $\frac{T_{t_n}}{t_n} z_n \to z$.*

Then $(T_t)_{t\geq 0}$ is sequentially Cesàro mixing. In particular, $(T_t)_{t\geq 0}$ is Cesàro hypercyclic.

Proof. Let $U, V \subseteq X$ be non-empty open sets. Choose $u \in X_0 \cap U$ and $v \in Y_0 \cap V$. By condition (2), there exists a sequence $(y_n) \subseteq X$ such that $y_n \to 0$ and $\frac{T_{t_n}}{t_n} y_n \to v$. Let $w_n = u + y_n$.

Hence, $w_n \to u$. Also, by condition (1),

$$\frac{T_{t_n}}{t_n} w_n = \frac{T_{t_n}}{t_n} u + \frac{T_{t_n}}{t_n} y_n \to v.$$

Hence, there exists $M \in \mathbb{N}$ such that for all $n \geq M$,

$$w_n \in U \quad \text{and} \quad \frac{T_{t_n}}{t_n} w_n \in V.$$

Therefore, $(T_t)_{t\geq 0}$ is sequentially Cesàro mixing and hence Cesàro hypercyclic. □

Theorem 3.6. *Let $(T_t)_{t\geq 0}$ be a C_0-semigroup on X. Suppose there exist dense subsets X_0 and Y_0 of X, an increasing sequence (t_n) of positive real numbers tending to infinity, and mappings $S_{t_n} : Y_0 \to Y_0$ such that:*

1. *$\left\| \frac{T_{t_n}}{t_n} x \right\| \to 0$ for all $x \in X_0$,*
2. *$\|S_{t_n} y\| \to 0$ for all $y \in Y_0$,*
3. *$\frac{T_{t_n}}{t_n} S_{t_n} = I_{Y_0}$.*

Then $(T_t)_{t\geq 0}$ is sequentially Cesàro mixing. In particular, $(T_t)_{t\geq 0}$ is Cesàro hypercyclic.

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Proof. Let $U, V \subseteq X$ be non-empty open sets. Choose $u \in X_0 \cap U$ and $v \in Y_0 \cap V$. Let $w_n = u + S_{t_n} v$. Then $w_n \to u$ and

$$\frac{T_{t_n}}{t_n} w_n = \frac{T_{t_n}}{t_n} u + \frac{T_{t_n}}{t_n} S_{t_n} v \to v. \tag{3.3}$$

Thus, there exists $M \in \mathbb{N}$ such that for all $n \geq M$,

$$w_n \in U \quad \text{and} \quad \frac{T_{t_n}}{t_n} w_n \in V.$$

Hence, for all $n \geq M$, $\frac{T_{t_n}}{t_n}(U) \cap V \neq \emptyset$, so $(T_t)_{t \geq 0}$ is sequentially Cesàro mixing and therefore Cesàro hypercyclic. □

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