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# ANALYSIS OF BALANCE IN THE PRODUCTS OF CONJUGATE SKEW GAIN GRAPHS

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Abstract. A conjugate skew gain graph is a graph whose edges are oriented and assigned labels—termed edge skew gains—from the multiplicative group $\mathbb{C}^{\times}$ of nonzero complex numbers, such that reversing the orientation of an edge replaces its label with its complex conjugate. In this article, we define various products of conjugate skew gain graphs such as the cartesian product, the lexicographic product, the strong product, and the tensor product. We characterize the balance in these product graphs in terms of the balance of the constituent conjugate skew gain graphs.

## 1. **Introduction**

In this article, we define four types of products of conjugate skew gain graphs and characterize the balance in the corresponding product graphs. Conjugate skew gain graphs encompass those discrete structures such as graphs, signed graphs, and complex unit gain graphs. To begin with, we provide a few basic definitions and notations used in this paper.

We use the notation $G = (V, E)$, throughout this paper for a graph that is finite and simple [4]. As we require prescribed orientations for the edges, the notation $\overrightarrow{E}$ stands for the collection of oriented edges such that for an edge $uv \in E$, there are two oriented parts $\overrightarrow{uv}$ and $\overrightarrow{vu}$ in $\overrightarrow{E}$.

**Definition 1.1** ( [2]). A *conjugate skew gain graph* (abbreviated from now on as csg) $G^{\varphi} = (G, \varphi)$ is such that the *conjugate skew gain function* $\varphi : \overrightarrow{E} \to \mathbb{C}^{\times}$ satisfies $\varphi(\overrightarrow{uv}) = \overline{\varphi(\overrightarrow{vu})}$, the complex conjugate of $\varphi(\overrightarrow{uv})$.

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For a complex unit gain graph [5], the gains are selected from the unit circle $\mathbb{T} = \{z \in \mathbb{C}^{\times} : |z| = 1\}$ in the complex plane. In the case of a signed graph [9], the signs come from the multiplicative group $\{-1, 1\}$. In the following, we discuss an important recent extension [2] of balance theory from signed graphs and complex unit gain graphs to conjugate skew gain graphs. Given a csg $G^{\varphi}$ and an oriented cycle $\overrightarrow{C}$ in it, the edge skew gain $\varphi(\overrightarrow{C})$ of $\overrightarrow{C}$ is defined as the product of the edge skew gains of those oriented edges in that cycle.

**Definition 1.2** ([2])**.** *An oriented cycle $\overrightarrow{C}$ in a csg $G^{\varphi}$ is said to be balanced if $\varphi(\overrightarrow{C})$ is a positive real number. Moreover, if every oriented cycle in a csg $G^{\varphi}$ satisfies this property, then the csg itself is said to be balanced.*

By defining the negative of a csg $G^{\varphi} = (G, \varphi)$ as the csg $G^{-\varphi} = (G, -\varphi)$ obtained by replacing each edge skew gain by its negative, we say that $G^{\varphi}$ is anti-balanced if $G^{-\varphi}$ is balanced. We also need the definition of an important operation in csgs called switching which transforms a given csg into another with many properties remaining invariant under this operation.

**Definition 1.3.** *If $G^{\varphi}$ is a given csg, then a function $\zeta : V(G^{\varphi}) \to \mathbb{T} \subset \mathbb{C}^{\times}$ is called a switching function, if it provides a csg $G^{\varphi^{\zeta}}$ with edge skew gain function satisfying $\varphi^{\zeta}(\overrightarrow{uv}) = \overline{\zeta(u)}\varphi(\overrightarrow{uv})\zeta(v)$.*

We say that the switching function $\zeta$ switches $G^{\varphi}$ to $G^{\varphi^{\zeta}}$. Generally, two csgs $G^{\varphi_1}$ and $G^{\varphi_2}$ are said to be switching equivalent if there is a switching function $\zeta$ such that $G^{\varphi_1^{\zeta}} = G^{\varphi_2}$. From [2], we select the following result that gives a characterization using switching for the balanced csgs.

**Theorem 1.4** ([2])**.** *A csg $G^{\varphi}$ is balanced if and only if $G^{\varphi}$ and $G^{|\varphi|}$ are switching equivalent, where the notation $|\varphi|$ denotes the function that gives absolute values of those of the conjugate skew gain function $\varphi$.*

## 2. Products of two csgs and their balance

We present here four products of csgs, namely the cartesian, the lexicographic, the tensor and the strong products of two csgs. We omit the arrows above the oriented edges to avoid the cumbersome notations and adopt the convention that the notation $\varphi(uv)$ means that it is the edge skew gain for the oriented edge $\overrightarrow{uv}$ with the tail being the vertex $u$ and the head being the vertex $v$.

Let the two given csgs be $G_1^{\varphi_1}$ and $G_2^{\varphi_2}$ with the underlying graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ respectively. We proceed with the definitions of the products of these two csgs as follows, assuming that the readers know the basic definitions of the products, $G_1 \,\square\, G_2$ (the cartesian product), $G_1[G_2]$ (lexicographic product), $G_1 \otimes G_2$ (tensor product) and $G_1 \boxtimes G_2$ (strong product) of the underlying graphs (see for example [3]).

**Definition 2.1.** *The cartesian product $G_1^{\varphi_1} \,\square\, G_2^{\varphi_2} = (G_1 \,\square\, G_2, \varphi)$ has the edge skew gain function $\varphi$ defined by:*

$$\varphi\big((u_i, v_j)(u_k, v_l)\big) = \begin{cases} \varphi_1(u_i u_k), & \textit{if } v_j = v_l \quad \textit{and} \quad u_i u_k \in E_1, \\ \varphi_2(v_j v_l), & \textit{if } u_i = u_k \quad \textit{and} \quad v_j v_l \in E_2. \end{cases} \tag{2.1}$$

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**Definition 2.2.** *The lexicographic product $G_1^{\varphi_1}[G_2^{\varphi_2}] = (G_1[G_2], \varphi)$ gets the edge skew gain function $\varphi$ defined by:*

$$\varphi\big((u_i, v_j)(u_k, v_l)\big) = \begin{cases} \varphi_1(u_iu_k), & \text{if } u_iu_k \in E_1, \\ \varphi_2(v_jv_l), & \text{if } u_i = u_k \text{ and } v_jv_l \in E_2. \end{cases} \tag{2.2}$$

**Definition 2.3.** *The tensor product $G_1^{\varphi_1} \otimes G_2^{\varphi_2} = (G_1 \otimes G_2, \varphi)$ has the edge skew gain function $\varphi$ given by:*

$$\varphi\big((u_i, v_j)(u_k, v_l)\big) = \varphi_1(u_iu_k) \cdot \varphi_2(v_jv_l), \tag{2.3}$$

*where $u_iu_k \in E_1$ and $v_jv_l \in E_2$.*

**Definition 2.4.** *The strong product $G_1^{\varphi_1} \boxtimes G_2^{\varphi_2} = (G_1 \boxtimes G_2, \varphi)$ gets the edge skew gain function $\varphi$ given by:*

$$\varphi\big((u_i, v_j)(u_k, v_l)\big) = \begin{cases} \varphi_1(u_iu_k), & \text{if } v_j = v_l \text{ and } u_iu_k \in E_1, \\ \varphi_2(v_jv_l), & \text{if } u_i = u_k \text{ and } v_jv_l \in E_2, \\ \varphi_1(u_iu_k) \cdot \varphi_2(v_jv_l), & \text{if } u_iu_k \in E_1 \text{ and } v_jv_l \in E_2. \end{cases} \tag{2.4}$$

We move on now to characterize the balance in these products. First we define subgraphs of a csg.

**Definition 2.5.** *A csg $H^\psi$ is said to be a subgraph of the csg $G^\varphi$ if the underlying graph $H$ is a subgraph of $G$ where for every $uv \in E(H), \psi(uv) = \varphi(uv)$.*

**Remark 2.6.** The csg obtained from the cartesian product of two csgs is a subgraph of the corresponding lexicographic product and the strong product. The tensor product is a subgraph of their strong product.

**Remark 2.7.** Even though the underlying graphs of the tensor product and the strong product are subgraphs of the lexicographic product, they need not be so as conjugate skew gain subgraphs. For example we take two csgs with the same underlying graph $K_2 = uv$ with different skew gain values $z_1$ and $z_2$, then in the tensor (or the strong) product the skew gain value of the edge $\big((u, v)(u, v)\big)$ is $z_1z_2$ but for the lexicographic product it is either $z_1$ or $z_2$.

The important role of a switching function while analysing balance is mentioned in Theorem 1.4 and the Lemma 2.8 given below gives a way of constructing a switching function in the cartesian product of two switched csgs.

**Lemma 2.8.** *Suppose $G_1^{\varphi_1}$ has a switching function $\zeta_1$ and $G_2^{\varphi_2}$ has a switching function $\zeta_2$. Then $\zeta(u, v) := \zeta_1(u)\zeta_2(v)$ defines a switching function on $G_1^{\varphi_1} \square G_2^{\varphi_2}$ and $(G_1^{\varphi_1} \square G_2^{\varphi_2})^\zeta = {G_1^{\varphi_1}}^{\zeta_1} \square {G_2^{\varphi_2}}^{\zeta_2}$.*

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*Proof.* The function $\zeta : V(G_1)\times V(G_2)\to \mathbb{T}$ satisfying $\zeta(u,v):=\zeta_1(u)\zeta_2(v)$ is well defined. As per the definition of the cartesian product and that of the switching of csgs,

$$\begin{aligned}\varphi^{\zeta}\big((u_i,v_j)(u_k,v_l)\big) &= \overline{\zeta}(u_i,v_j)\varphi\big((u_i,v_j)(u_k,v_l)\big)\zeta(u_k,v_l)\\ &= \begin{cases} \overline{\zeta_1}(u_i)\overline{\zeta_2}(v_j)\varphi_1(u_iu_k)\zeta_1(u_k)\zeta_2(v_j), & \text{if } v_j=v_l \text{ and } u_iu_k\in E_1,\\ \overline{\zeta_1}(u_i)\overline{\zeta_2}(v_j)\varphi_2(v_jv_l)\zeta_1(u_i)\zeta_2(v_l), & \text{if } u_i=u_k \text{ and } v_jv_l\in E_2.\end{cases}\end{aligned}$$

Since $|\zeta_1(u_i)|=|\zeta_2(v_j)|=1$, this gives $\varphi^{\zeta}\big((u_i,v_j)(u_k,v_l)\big)$

$$= \begin{cases} \varphi_1^{\zeta_1}(u_iu_k), & \text{if } v_j=v_l \quad \text{and} \quad u_iu_k\in E_1,\\ \varphi_2^{\zeta_2}(v_jv_l), & \text{if}u_i=u_k \quad \text{and} \quad v_jv_l\in E_2.\end{cases}$$

This proves the required result that $(G_1^{\varphi_1}\,\square\, G_2^{\varphi_2})^{\zeta} = {G_1^{\varphi_1}}^{\zeta_1}\,\square\,{G_2^{\varphi_2}}^{\zeta_2}$. ☐

**Theorem 2.9.** *If $G_1^{\varphi_1}=(V_1,E_1,\varphi_1)$ and $G_2^{\varphi_2}=(V_2,E_2,\varphi_2)$ are two csgs, then their cartesian product $G_1^{\varphi_1}\,\square\, G_2^{\varphi_2}$ is balanced if and only if $G_1^{\varphi_1}$ and $G_2^{\varphi_2}$ are both balanced.*

*Proof.* Let $G_1^{\varphi_1}$ and $G_2^{\varphi_2}$ be balanced. Then by Lemma 1.4, there are two switching functions $\zeta_1$ and $\zeta_2$ such that $\varphi_1^{\zeta_1}=|\varphi_1|$ and $\varphi_2^{\zeta_2}=|\varphi_2|$. By Lemma 2.8, $(G_1^{\varphi_1}\,\square\, G_2^{\varphi_2})^{\zeta} = {G_1^{\varphi_1}}^{\zeta_1}\,\square\,{G_2^{\varphi_2}}^{\zeta_2}$. This shows that $G_1^{\varphi_1}\square G_2^{\varphi_2}$ is switching equivalent to $G_1^{|\varphi_1|}\square G_2^{|\varphi_2|}$ and hence balanced. Conversely, note that both $G_1^{\varphi_1}$ and $G_2^{\varphi_2}$ can be considered as subgraphs of $G_1^{\varphi_1}\,\square\, G_2^{\varphi_2}$. If we assume that at least one of $G_1^{\varphi_1}$ or $G_2^{\varphi_2}$ is unbalanced, then each being a subgraph of the product, such an assumption will affect the balance of the cartesian product. ☐

To illustrate various product graphs, we take the csgs $G_1^{\varphi_1}$ and $G_2^{\varphi_2}$ shown in Figure 1 as the factor graphs for all the products that follow.

FIGURE 1. $G_1^{\varphi_1}:=(K_3,\varphi_1)$ and $G_2^{\varphi_2}:=(K_2,\varphi_2)$

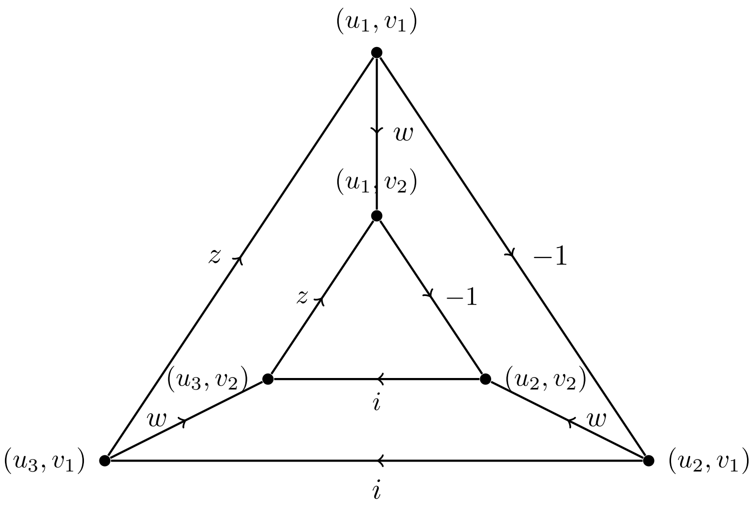

FIGURE 2. The cartesian product, $G_1^{\varphi_1} \Box G_2^{\varphi_2}$ is balanced when $z = ri \in \mathbb{C}, r > 0$ and unbalanced otherwise.

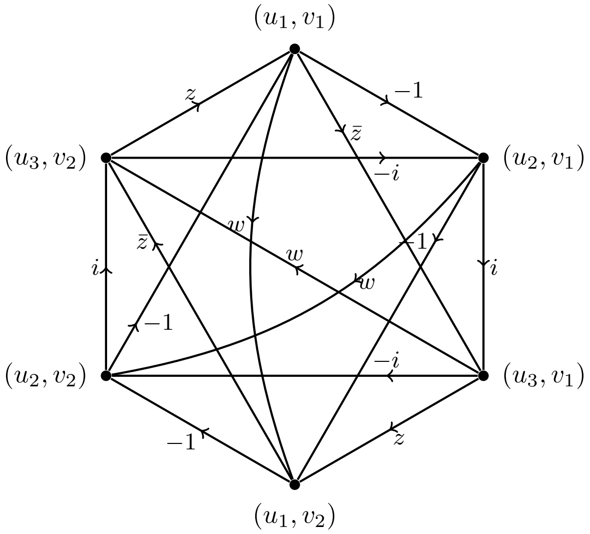

FIGURE 3. The lexicographic product $G_1^{\varphi_1}[G_2^{\varphi_2}]$ is balanced when $z = ri \in \mathbb{C}, r > 0$ and $w$ is a positive real number; otherwise, it is unbalanced.

**Theorem 2.10.** *If $G_1^{\varphi_1} = (V_1, E_1, \varphi_1)$ and $G_2^{\varphi_2} = (V_2, E_2, \varphi_2)$ are two csgs, then their lexicographic product $G_1^{\varphi_1}[G_2^{\varphi_2}] = (G_1[G_2], \varphi)$ is balanced if and only if $G_1^{\varphi_1}$ is balanced and for $G_2^{\varphi_2}$ the edge skew gain function satisfies $\varphi_2 = |\varphi_2|$.*

*Proof.* If $G_1^{\varphi_1}$ is balanced and $\varphi_2$ satisfies the condition $\varphi_2 = |\varphi_2|$, then by Theorem 1.4, it is possible to switch $G_1^{\varphi_1}$ to $G_1^{|\varphi_1|}$ by a switching function $\zeta$, where $\zeta : V_1 \to \mathbb{T}$. Define $\zeta' : V(G_1^{\varphi_1}[G_2^{\varphi_2}]) \to \mathbb{T}$ by $\zeta'(u_i, v_j) = \zeta(u_i)$. Then we claim that $(G_1^{\varphi_1}[G_2^{\varphi_2}])^{\varphi^{\zeta'}} = (G_1^{\varphi_1}[G_2^{\varphi_2}])^{|\varphi|}$. To see this, note that $G_1^{\varphi_1}$ is switching equivalent to $G_1^{{\varphi_1}^{\zeta}}$. Hence the equation of switching ${\varphi_1}^{\zeta}(u_iu_k) = \overline{\zeta}(u_i)\varphi_1(u_iu_k)\zeta(u_k)$ implies

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that $\varphi_1(u_iu_k) = \zeta(u_i)|\varphi_1(u_iu_k)|\overline{\zeta}(u_k)$, since ${\varphi_1}^{\zeta}$ is $|\varphi_1|$. Also, for $\mathbf{uv} = (u_i, v_j)(u_k, v_l) \in E(G_1^{\varphi_1}[G_2^{\varphi_2}])$

$$\varphi^{\zeta'}(\mathbf{uv}) = \overline{\zeta'}(\mathbf{u})\varphi(\mathbf{uv})\zeta'(\mathbf{v}) = \overline{\zeta}(u_i)\varphi(\mathbf{uv})\zeta(u_k)$$

Using the definition of the lexicographic product of csgs (see Equation (2.2)), this gives when $i \neq k$

$$\varphi^{\zeta'}(\mathbf{uv}) = \overline{\zeta}(u_i)\varphi_1(u_iu_k)\zeta(u_k) = \overline{\zeta}(u_i)\zeta(u_i)|\varphi_1(u_iu_k)|\overline{\zeta}(u_k)\zeta(u_k) = |\varphi_1(u_iu_k)|$$

and if $i = k$,

$$\varphi^{\zeta'}(\mathbf{uv}) = \overline{\zeta}(u_i)\varphi_2(v_jv_l)\zeta(u_i) = |\zeta(u_i)|^2|\varphi_2(v_jv_l)| = |\varphi_2(v_jv_l)|,$$

since $\varphi_2$ satisfies the condition $\varphi_2 = |\varphi_2|$. This thus leads to the fact that $\varphi^{\zeta'}(\mathbf{uv}) = |\varphi(\mathbf{uv})|$ for all $\mathbf{uv} \in E(G_1^{\varphi_1}[G_2^{\varphi_2}])$, as required. Conversely, assume that $G_1^{\varphi_1}[G_2^{\varphi_2}]$ is balanced. Then the cartesian product $G_1^{|\varphi_1|} \,\Box\, G_2^{|\varphi_2|}$ is a subgraph of $G_1^{\varphi_1}[G_2^{\varphi_2}]$. As such Theorem 2.9 is applicable; so $G_1^{\varphi_1}$ and $G_2^{\varphi_2}$ must be at least balanced. Now, we claim that $G_2^{\varphi_2}$ cannot have any edge with the edge skew gain other than positive real numbers. On the contrary, if we assume that $G_2^{\varphi_2}$ contains an edge with edge skew gain of not this nature, say $v_jv_l$, then we claim that it would result in an unbalanced triangle in $G_1^{\varphi_1}[G_2^{\varphi_2}]$ as per the definition of the lexicographic product, leading to a contradiction. To prove this claim consider one unbalanced triangle, for example, $(u_i, v_l)(u_k, v_j), (u_k, v_j)(u_k, v_l), (u_k, v_l)(u_i, v_1)$ for which the product of edge skew gains will be $\varphi_1(u_iu_k)\varphi_2(v_jv_l)\varphi_1(u_ku_i) = \varphi_1(u_iu_k)\overline{\varphi_1(u_iu_k)}\varphi_2(v_jv_l) = |\varphi_1(u_iu_k)|^2\varphi_2(v_jv_l)$. Thus this triangle is not balanced. Hence the proof. $\square$

**Lemma 2.11.** *For every cycle $C_m$ in the csg $G_1^{\varphi_1}$, there exist an even cycle $C^*$ in the tensor product $(G_1^{\varphi_1} \otimes G_2^{\varphi_2}, \varphi)$ such that*

$$\varphi(C^*) = \begin{cases} r^m\varphi_1(C_m), & \text{if } m \text{ is even,} \\ \Big(r^m\varphi_1(C_m)\Big)^2, & \text{if } m \text{ is odd,} \end{cases}$$

*for some positive real number $r$.*

*Proof.* First, consider the case when $m$ is even. Let $C_m : u_1u_2\cdots u_mu_1$ be any even cycle in $G_1^{\varphi_1}$, and let $e = v_1v_2 \in E(G_2^{\varphi_2})$ with $\varphi_2(v_1v_2) = z$. We construct the cycle

$$C^* : (u_1, v_1)(u_2, v_2)(u_3, v_1), \cdots (u_m, v_2)(u_1, v_1)$$

in the tensor product $(G_1^{\varphi_1} \otimes G_2^{\varphi_2}, \varphi)$. Then $C^*$ is an even cycle of length $m$ and

$$\varphi(C^*) = \varphi_1(u_1u_2)z\varphi_1(u_2u_3)\overline{z}, \cdots, z\varphi_1(u_mu_1)\overline{z} = r^m\varphi_1(C_m),$$

where $r = |z|$. If $m$ is odd, we instead consider the cycle

$$C^* : (u_1, v_1)(u_2, v_2)(u_3, v_1), \cdots, (u_m, v_1)(u_1, v_2), \cdots, (u_m, v_2)(u_1, v_1),$$

which has length $2m$ in the tensor product. Then, as in the previous case, it is straightforward to compute $\varphi(C^*) = \Big(r^m\varphi_1(C_m)\Big)^2$. $\square$

**Remark 2.12.** It is worthy to note that if the csg $G^\varphi$ is balanced, then by definition $\varphi(C) > 0$ for every cycle $C$ and anti-balanced if $\varphi(C) > 0$ for even cycles and $\varphi(C) < 0$ for odd cycles.

**Remark 2.13.** If $z \in \mathbb{C}$ and $z^2$ is a positive real number, then $z$ must be a nonzero real number.

**Lemma 2.14.** *If $G_1^{\varphi_1} \otimes G_2^{\varphi_2}$ is balanced, then the conjugate skew gain of every cycle $C_m$ in $G_1^{\varphi_1}$ is a real number. Moreover, it is a positive real number if $C_m$ is an even cycle.*

*Proof.* By Lemma 2.11, every cycle $C_m$ in $G_1^{\varphi_1}$ corresponds to an even cycle $C^*$ in $G_1^{\varphi_1} \otimes G_2^{\varphi_2}$. Since $G_1^{\varphi_1} \otimes G_2^{\varphi_2}$ is balanced, the gain $\varphi(C^*)$ is a positive real number. Again, by Lemma 2.11 and Remark 2.13, this implies that $\varphi_1(C_m)$ is a nonzero real number, and it is positive if $m$ is even. □

**Remark 2.15.** The results in Lemma 2.11 and Lemma 2.14 remain valid for cycles $C_m$ in $G_2^{\varphi_2}$ as well.

**Lemma 2.16** ( [7]). *If the tensor product of signed graph $\Sigma = \Sigma_1 \otimes K_2$ of the connected signed graph $\Sigma_1 = (G, \sigma_1)$ with $(K_2, \sigma_2)$ is balanced or anti-balanced, then all the odd cycles in $\Sigma_1$ have the same sign.*

Note that in the above Lemma 2.16, it is necessary that $\Sigma_1$ is connected. For instance, suppose $\Sigma_1$ is a disconnected signed graph consisting of two components: one being a positive odd cycle and a negative one. Then the tensor product $\Sigma = \Sigma_1 \otimes K_2$ is a disconnected graph with two components, each of which is a positive even cycle and hence balanced.

**Lemma 2.17.** *Let $G_1^{\varphi_1} \otimes G_2^{\varphi_2}$ be balanced. Then every odd cycle in $G_1$ (and similarly in $G_2$) has same sign for their conjugate skew gain values.*

*Proof.* Let $C'_{2m+1} = u_1u_2 \cdots u_{2m+1}u_1$ and $C''_{2n+1} = v_1v_2 \cdots v_{2n+1}v_1$, be any two odd cycles in $G_1$ and let $x_1x_2$ be any edge in $G_2$. We consider the following cases.

**Case(i): Cycles are disjoint**

Let $P : u_1w_1w_2 \cdots w_kv_1$ be a minimal connecting path between the cycles. Then we construct a cycle $C^*$ in $G_1^{\varphi_1} \otimes G_2^{\varphi_2}$, given by $C^* : (u_1, x_1)(u_2, x_2) \cdots (u_{2m+1}, x_1)(u_1, x_2)(w_1, x_1)(w_2, x_2) \cdots (w_k, x)(v_1, y)$ $(v_2, x) \cdots (v_{2n+1}, y)(v_1, x)(w_k, y)(w_{k-1}, x) \cdots (w_2, x_1)(w_1, x_2)(u_1, x_1)$, where $(x, y) = (x_1, x_2)$ when $k$ is odd and $(x_2, x_1)$ when $k$ is even. Then

$$\varphi(C^*) = \varphi_1(C'_{2m+1})\varphi_1(C''_{2n+1})|\varphi_1(P)|^2|\varphi_2(x_1x_2)|^{2(m+n+k+2)}.$$

Since $G_1^{\varphi_1} \otimes G_2^{\varphi_2}$ be balanced, $\varphi(C^*) > 0$ and hence $\varphi_1(C'_{2m+1})\varphi_1(C''_{2n+1}) > 0$. Now by Lemma 2.14, $\varphi_1(C'_{2m+1})$ and $\varphi_1(C''_{2n+1})$ are real numbers, and having the same sign.

**Case (ii): Cycles intersect at a single vertex**

In this case we let the cycles intersect at the vertex $u_1 = v_1$ and we construct $C^*$ as in Case(i),

$$C^* : (u_1, x_1)(u_2, x_2) \cdots (u_{2m+1}, x_1)(u_1, x_2)(v_2, x_1)(v_3, x_2) \cdots (v_{2n+1}, x_2)(v_1, x_1).$$

We get $\varphi(C^*) = \varphi_1(C'_{2m+1})\varphi_1(C''_{2n+1})|\varphi_2(x_1x_2)|^{2(m+n+1)}$. Thus $\varphi(C^*) > 0$ implies

$$\varphi(C^*) = \varphi_1(C'_{2m+1})\varphi_1(C''_{2n+1}) > 0.$$

**Case (iii): Cycles intersection is a path of length $k \geq 1$.**

Let $P : aw_1w_2 \cdots w_{k-1}b$, be the intersecting path of the cycles $C'_{2m+1}$ and $C''_{2n+1}$. We denote the paths

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$P_1 : au_1u_2\cdots u_{m'}b$ and $P_2 : av_1v_2\cdots v_{n'}b$ such that $C'_{2m+1} : P_1 \cup P$ and $C''_{2n+1} : P_2 \cup P$. It is noted that $m'$ and $n'$ are of same parity. We construct the cycle $C^*$ in $G_1^{\varphi_1} \otimes G_2^{\varphi_2}$,

$$\begin{aligned} C^* : &(a, x_1)(u_1, x_2)\cdots(u_{m'}, y)(b, x)(w_{k-1}, y)\cdots(w_1, x_1)\\ &(a, x_2)(v_1, x_1)\cdots(v_{n'}, x)(b, y)(w_k, x)\cdots(w_1, x_2)(a, x_1), \end{aligned}$$

where $(x, y) = (x_1, x_2)$ if $m'$ and $n'$ are odd and $(x, y) = (x_2, x_1)$ if $m'$ and $n'$ are even. Then

$$\varphi(C^*) = \varphi_1(C'_{2m+1})\varphi_1(C''_{2n+1})|\varphi_2(x_1x_2)|^{2(m+n+1)}.$$

Hence $\varphi(C^*) > 0$ implies $\varphi_1(C'_{2m+1})\varphi_1(C''_{2n+1}) > 0$.

**Case (iv): Cycles intersect in more than one path.**

By Tutte's Path Theorem [8], there is a sequence of cycles $C'_{2m+1} = C_1, C_2, C_3, \cdots, C_l = C''_{2n+1}$, such that $C_i \cap C_{i+1}$ is a path of length at least one for all $i = 1, 2, \cdots, n-1$. Now, as in the previous case for each $i$, $\varphi_1(C_i)\varphi_1(C_{i+1}) > 0$ and hence $\varphi_1(C'_{2m+1})\varphi_1(C''_{2n+1}) > 0$. □

**Lemma 2.18.** *Let* $(G_1^{\varphi_1} \otimes G_2^{\varphi_2}, \varphi)$ *be balanced, and let* $C'_{2m+1}$ *and* $C''_{2n+1}$ *be any odd cycles in* $G_1^{\varphi_1}$ *and* $G_2^{\varphi_2}$ *respectively. Then* $\varphi_1(C'_{2m+1}).\varphi_2(C''_{2n+1}) > 0$.

*Proof.* If possible suppose $\varphi_1(C'_{2m+1})\varphi_2(C''_{2n+1}) < 0$. Then by Lemma 2.14, without loss of generality we assume $\varphi_1(C'_{2m+1}) > 0$ and $\varphi_2(C''_{2n+1}) < 0$. As such, we can construct an odd cycle $C^*$ of length $l = lcm(2m+1, 2n+1)$ in $(G_1^{\varphi_1} \otimes G_2^{\varphi_2}, \varphi)$ such that $\varphi(C^*) = \varphi_1(C'_{2m+1})^{l/2m+1}\varphi_2(C''_{2n+1})^{l/2n+1} < 0$, a contradiction. □

**Theorem 2.19.** *If* $G_1^{\varphi_1} = (V_1, E_1, \varphi_1)$ *and* $G_2^{\varphi_2} = (V_2, E_2, \varphi_2)$ *are two csgs, then their tensor product* $G_1^{\varphi_1} \otimes G_2^{\varphi_2}$ *is balanced if and only if* $G_1^{\varphi_1}$ *and* $G_2^{\varphi_2}$ *are either both balanced or both anti-balanced.*

*Proof.* First we assume that, $G_1^{\varphi_1}$ and $G_2^{\varphi_2}$ are either both balanced or both anti-balanced. Then there exist switching functions $\zeta_1$ and $\zeta_2$ on $V_1$ and $V_2$ respectively such that $\overline{\zeta_1(u_i)}\varphi_1(u_iu_k)\zeta_1(u_k) = \pm|\varphi_1(u_iu_k)|$, and similarly $\overline{\zeta_2(v_j)}\varphi_2(v_jv_l)\zeta_2(v_l) = \pm|\varphi_2(v_jv_l)|$. Now we define the switching function $\zeta'$ on $V(G_1^{\varphi_1} \otimes G_2^{\varphi_2})$, $\zeta(u_i, v_j) = \zeta_1(u_i)\zeta_2(v_j)$. Then, for $\mathbf{uv} = (u_i, v_j)(u_k, v_l) \in E(G_1^{\varphi_1} \otimes G_2^{\varphi_2})$, and

$$\begin{aligned} \varphi^{\zeta}(\mathbf{uv}) &= \overline{\zeta(\mathbf{u})}\varphi(\mathbf{uv})\zeta(\mathbf{v})\\ &= \overline{\zeta_1(u_i)\zeta_2(v_j)}\varphi_1(u_iu_k)\varphi_2(v_jv_l)\zeta_1(u_k)\zeta_2(v_l)\\ &= |\varphi_1(u_iu_k)||\varphi_2(v_j, v_l)|\\ &= |\varphi(\mathbf{uv})| \end{aligned}$$

and hence $G_1^{\varphi_1} \otimes G_2^{\varphi_2}$ is balanced. Conversely, assume that $G_1^{\varphi_1} \otimes G_2^{\varphi_2}$ is balanced. Then by Lemma 2.14, it is enough to show that all the odd cycles in $G_1$ and $G_2$, if at all any one exists, are of same sign in their conjugate skew gain value. Now by Lemma 2.17 and Lemma 2.18, the conjugate skew gain is either negative for every odd cycle in $G_1$ and $G_2$ or positive for every odd cycle in $G_1$ and $G_2$ and hence the result. □

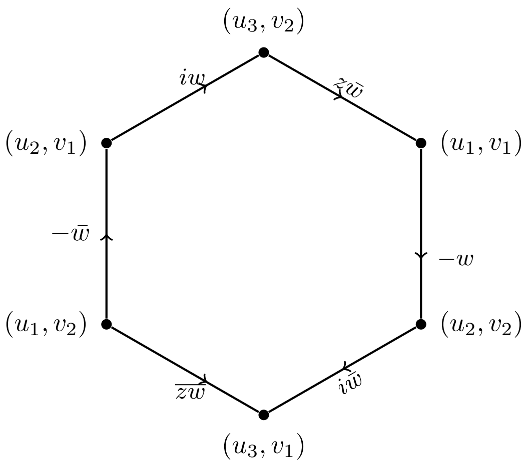

FIGURE 4. The tensor product $G_1^{\varphi_1} \boxtimes G_2^{\varphi_2}$; balanced when $z = ri \in \mathbb{C}, r > 0$ and unbalanced otherwise.

Finally we characterize the balance in the strong product of two csgs.

**Theorem 2.20.** *The strong product* $G_1^{\varphi_1} \boxtimes G_2^{\varphi_2} = (G_1 \boxtimes G_2, \varphi)$ *is balanced if and only if* $G_1^{\varphi_1}$ *and* $G_2^{\varphi_2}$ *are both balanced.*

*Proof.* As already remarked, the cartesian product $G_1^{\varphi_1} \square G_2^{\varphi_2}$ as a subgraph (treated as a csg) of the strong product $G_1^{\varphi_1} \boxtimes G_2^{\varphi_2}$ and the balance of strong product implies the balance of their cartesian product, hence by Theorem 2.9 both $G_1^{\varphi_1}$ and $G_2^{\varphi_2}$ are balanced whenever $G_1^{\varphi_1} \boxtimes G_2^{\varphi_2}$ is. Conversely assume that both $G_1^{\varphi_1}$ and $G_2^{\varphi_2}$ are balanced and hence they are switching equivalent to $G_1^{|\varphi_1|}$ and $G_2^{|\varphi_2|}$ respectively. Consider the corresponding switching functions $\zeta_1$ and $\zeta_2$ and define $\zeta$ on $V(G_1 \boxtimes G_2)$ as $\zeta(u, v) = \zeta_1(u)\zeta_2(v)$.

$$\begin{aligned}
\text{Then } \varphi^{\zeta}\big((u_iv_j)(u_kv_l)\big) &= \overline{\zeta(u_iv_j)}\varphi\big((u_i, v_j)(u_k, v_l)\big)\zeta(u_kv_l) \\
&= \overline{\zeta_1(u_i)\zeta_2(v_j)}\varphi_1(u_iu_k)\varphi_2(v_iv_l)\zeta_1(u_k)\zeta_2(v_l) \\
&= \left[\overline{\zeta_1(u_i)}\varphi_1(u_iu_k)\zeta_1(u_k)\right]\left[\overline{\zeta_1(v_j)}\varphi_2(v_iv_l)\zeta_2(v_l)\right] \\
&= |\varphi_1(u_iu_k)||\varphi_2(v_jv_l)| \\
&= |\varphi\big((u_i, v_j)(u_k, v_l)\big)| \text{ whenever } u_iu_k \in E_1 \text{ and } v_jv_l \in E_2.
\end{aligned}$$

For the cases when $u_iu_k \in E_1, v_j = v_l$ and $u_i = u_k, v_jv_l \in E_2$, these values are $|\varphi_1(u_iu_k)|$ and $|\varphi_2(v_jv_l)|$. i.e., $\varphi^{\zeta} > 0$ and hence $G_1^{\varphi_1} \boxtimes G_2^{\varphi_2}$ is balanced. $\square$

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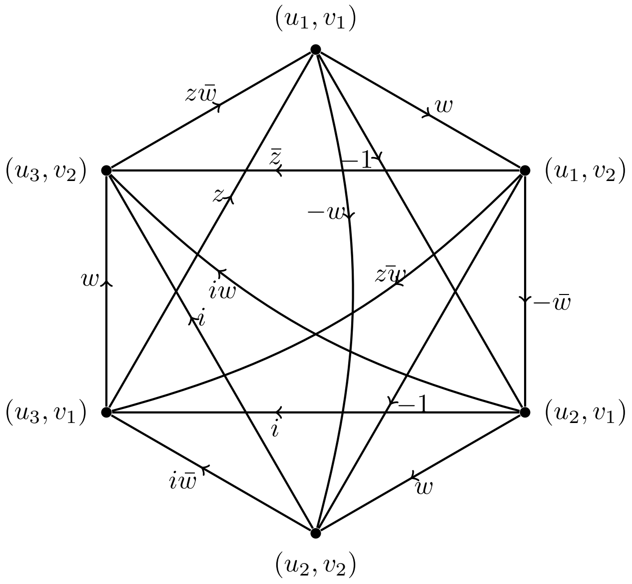

FIGURE 5. Strong product $G_1^{\varphi_1} \boxtimes G_2^{\varphi_2}$; balanced when $z = ri \in \mathbb{C}, r > 0$ and unbalanced otherwise.

## Conclusion

We have characterized the balance in various products of csgs in terms of the balance of their factor csgs. Our discussion focused primarily on four types of products: cartesian, lexicographic, tensor, and strong. There remains scope to study the balance properties of other types of graph products beyond those considered here.

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