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GEOMETRY OF VARIABLE-EXPONENT BOCHNER-LEBESGUE SPACES: DENTABILITY, RADON-NIKODYM PROPERTY, AND UNIFORM CONVEXITY

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Dedicated to Prof. A. T.-M. Lau

Abstract. This paper provides a comprehensive study of geometric properties of variable-exponent Lebesgue-Bochner spaces $L^{p(\cdot)}(E, X)$. We establish that $L^{p(\cdot)}(E, X)$ has the Radon-Nikodym property (RNP) if and only if X does, under the condition $p_m > 1$. We investigate dentability of specific subsets, showing that while the unit ball inherits dentability from X, the set of simple functions may remain nondentable even when the space possesses RNP. The perseverance of uniform convexity and smoothness are shown when both X and $L^{p(\cdot)}(E)$ have these properties. We introduce quantitative dentability moduli and relate the dentability modulus of $L^{p(\cdot)}(E, X)$ to that of X, showing the relevance to the measure of E, and the oscillation of $p(\cdot)$. The paper reveals how variable exponents create hybrid geometries mixing L^1 and L^p behaviors, with implications for PDE theory, optimization, and geometric analysis.

1. Introduction

Variable-exponent Lebesgue spaces $L^{p(\cdot)}$ and Sobolev spaces $W^{k,p(\cdot)}$ have emerged as essential tools in modern analysis, with applications ranging from electrorheological fluids [10] and image processing [2] to PDEs with non-standard growth conditions [11]. Unlike classical L^p spaces where p is constant, these spaces allow the exponent p to vary as a measurable function $p : E \to (1, \infty)$, leading to a rich functional-analytic structure that generalizes classical theory while presenting new challenges.

The geometry of Banach spaces plays a crucial role in understanding variational problems, convergence of algorithms, and properties of vector-valued measures. For variable-exponent spaces, the

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interplay between the pointwise behavior of $p(\cdot)$ and the geometry of the target space X creates subtle phenomena that demand careful analysis. Key geometric properties include dentability, the Radon-Nikodym property (RNP), uniform convexity, and smoothness. These properties are fundamental for optimization theory, fixed-point theorems, martingale convergence, and the analysis of PDEs.

This paper presents a systematic investigation of these geometric properties in variable-exponent Bochner-Lebesgue spaces $L^{p(\cdot)}(E, X)$. Our work extends the well-developed theory of scalar variable-exponent spaces [5,8] to the vector-valued setting, combining techniques from Banach space geometry [1,6] with the modular analysis characteristic of Musielak-Orlicz spaces [9].

State of the art and main results. The study of geometric properties of variable-exponent Bochner spaces was initiated in the foundational work of Cheng and Xu [3], who introduced the spaces $L^{p(\cdot)}(E, X)$ and established several key properties. In particular, they showed that if X has the Radon-Nikodym property (RNP) and $p_m > 1$, then $L^{p(\cdot)}(E, X)$ also has RNP. They further proved that uniform convexity (respectively, uniform smoothness) of $L^{p(\cdot)}(E, X)$ follows from the corresponding properties of both X and $L^{p(\cdot)}(E)$ under suitable assumptions on $p(\cdot)$. These results extended the classical theory of Bochner spaces to the variable-exponent setting.

Subsequent research has deepened the understanding of these spaces, yet several important questions remain open. For instance, the dentability of specific subsets—such as the unit ball or the set of simple functions—was not systematically investigated. Moreover, quantitative aspects of dentability, including explicit moduli relating the geometry of $L^{p(\cdot)}(E, X)$ to that of X, were unexplored. The interplay between the measure of E, the oscillation of $p(\cdot)$, and the dentability of the unit ball also required clarification.

The present paper provides a comprehensive analysis that both refines and extends the existing theory. Our main contributions are the following:

Theorem 1.1 (A: RNP characterization)**.** *Let (E, Σ, μ) be a finite measure space without atoms, X a Banach space, and let $p : E \to (1, \infty)$ be a measurable function with $1 < p_m \leq p_S < \infty$. Then $L^{p(\cdot)}(E, X)$ has the Radon-Nikodym property (RNP), if and only if X has RNP.*

This sharpens the earlier result of Cheng and Xu [3] in two ways: first, by establishing that the condition on X is also necessary (their proof only addressed sufficiency); and second, by removing all additional assumptions on $p(\cdot)$ beyond essential boundedness.

Theorem 1.2 (B: Dentability of the unit ball)**.** *Under the assumptions of Theorem 1.1, the closed unit ball $\mathbb{B} = \{f \in L^{p(\cdot)}(E, X) : \|f\|_{p(\cdot)} \leq 1\}$ is dentable if and only if X has RNP.*

Theorem 1.3 (C: Non-dentability of simple functions)**.** *Let $\mathcal{S}_M = \{f \text{ simple} : \|f\|_{p(\cdot)} \leq M\}$. If X lacks RNP, then $\mathcal{S}_M$ is not dentable for any $M > 0$.*

Theorems B and C are new even in the constant-exponent case and reveal subtle structural features: while the unit ball inherits dentability from X, the set of simple functions may fail to be dentable despite the space having RNP.

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Theorem 1.4 (D: Uniform convexity and smoothness)**.** *Let (E, Σ, μ) be a σ-finite measure space, and let X be a Banach space, and $p : E \to (1, \infty)$ measurable with $1 < p_m \leq p_S < \infty$.*

1. *$L^{p(\cdot)}(E, X)$ is uniformly convex, if and only if both X and $L^{p(\cdot)}(E)$ are uniformly convex.*
2. *If, in addition, X is reflexive and $p(\cdot)$ is log-Hölder continuous, then $L^{p(\cdot)}(E, X)$ is uniformly smooth if and only if both X and $L^{p(\cdot)}(E)$ are uniformly smooth.*

Theorem D completes the transfer theory initiated in [3] by providing the converse implications and by clarifying the precise assumptions needed for uniform smoothness.

Proposition 1.5 (E: Quantitative dentability modulus)**.** *Let X have RNP with dentability modulus $\delta_X(\varepsilon)$. Then for the unit ball $\mathbb{B}$ of $L^{p(\cdot)}(E, X)$,*

$$\operatorname{dent}(\mathbb{B}, \varepsilon) \geq C \cdot \delta_X(c\varepsilon) \cdot \min\left\{\mu(E)^{1/p_S - 1/p_m}, \mu(E)^{1/p_m - 1/p_S}\right\},$$

where $C, c > 0$ are constants depending only on $p(\cdot)$.

Proposition E is entirely new and provides a quantitative link between the dentability of X and that of $L^{p(\cdot)}(E, X)$, highlighting the role of the measure of E and the oscillation of $p(\cdot)$. A partial result confirming this estimate for constant exponents is proved in Section 6.

We prove Theorems A–D in Sections 4 and 5. Proposition E is motivated by our analysis in Section 6, where we also prove a result for constant exponents.

Structure of the paper. Section 2 collects necessary preliminaries on variable-exponent spaces, vector measures, dentability, and RNP. Section 3 discusses dentability and its geometric meaning, including examples and counterexamples. Section 4 contains the proofs of our main results on RNP and dentability. Section 5 proves the transfer theorems for uniform convexity and smoothness. Section 6 introduces quantitative dentability moduli and states our main proposition. Section 7 discusses applications to PDEs and optimization. Finally, Section 8 summarizes our findings and suggests directions for future research.

2. Preliminaries

2.1. Variable-exponent Bochner-Lebesgue spaces.

Let (E, Σ, μ) be a σ-finite measure space and X a Banach space. A measurable function $p : E \to (1, \infty)$ is called an *exponent function.* We define

$$p_m := \operatorname{ess\,inf}_{x \in E} p(x), \quad p_S := \operatorname{ess\,sup}_{x \in E} p(x),$$

and assume throughout that

$$1 < p_m \leq p_S < \infty.$$

The *variable-exponent Bochner-Lebesgue space* $L^{p(\cdot)}(E, X)$ consists of all strongly measurable functions $f : E \to X$ such that

$$\rho(f) := \int_E \|f(x)\|_X^{p(x)} \, d\mu(x) < \infty.$$

$\rho(f)$ is called the modular. This space is equipped with the Luxemburg norm

$$\|f\|_{p(\cdot)} := \inf\left\{\lambda > 0 : \rho(f/\lambda) \leq 1\right\}.$$

DOI: https://dx.doi.org/10.30504/jims.2026.567288.1310

When $X = \mathbb{R}$, we write $L^{p(\cdot)}(E) := L^{p(\cdot)}(E, \mathbb{R})$.

The following properties of the modular and norm are fundamental (see [5]):

Lemma 2.1. *For* $f, g \in L^{p(\cdot)}(E, X)$

1. $\rho(f) \leq 1$ *if and only if* $\|f\|_{p(\cdot)} \leq 1$*;*
2. *If* $\|f\|_{p(\cdot)} \leq 1$*, then* $\rho(f) \leq \|f\|_{p(\cdot)}$*;*
3. *If* $\|f\|_{p(\cdot)} > 1$*, then* $\rho(f) \geq \|f\|_{p(\cdot)}$*;*
4. *For any* $f \neq 0$*,* $\rho(f/\|f\|_{p(\cdot)}) = 1$*;*
5. *The norm is absolutely continuous:* $\|f_n\|_{p(\cdot)} \to 0$ *if and only if* $\rho(f_n) \to 0$*.*

2.2. **Radon-Nikodym property and dentability.** Let Y be a Banach space. A vector measure $F : \Sigma \to Y$ is said to have *bounded variation* if

$$\operatorname{Var}(F) := \sup \left\{ \sum_{i=1}^{n} \|F(A_i)\|_Y : \{A_i\} \text{ finite partition of } E \right\} < \infty.$$

Definition 2.2 (Radon-Nikodym Property)**.** A Banach space Y has the *Radon-Nikodym property (RNP)* if for every finite measure space $(\Omega, \mathcal{F}, \nu)$ and every ν-continuous vector measure $F : \mathcal{F} \to Y$ of bounded variation, there exists $g \in L^1(\nu, Y)$ such that

$$F(A) = \int_A g \, d\nu \quad \text{for all } A \in \mathcal{F}.$$

The RNP admits several equivalent characterizations. For our purposes, the most relevant is via dentability.

Definition 2.3 (Dentability)**.** A bounded subset $A \subset Y$ is *dentable*, if for every $\varepsilon > 0$ there exists $y \in A$ such that

$$y \notin \overline{\operatorname{conv}}\big(A \setminus B_\varepsilon(y)\big).$$

Such a point y is called an *ε-dent point.*

Theorem 2.4 ([4])**.** *A Banach space Y has RNP if and only if every bounded subset of Y is dentable.*

We also recall the Krein-Milman property (KMP): Y has KMP if every closed bounded convex set is the closed convex hull of its extreme points. It is known that RNP implies KMP, but the converse fails in general.

Example 2.5. The space c_0 has KMP but lacks RNP. The space $L^1[0, 1]$ lacks both RNP and KMP.

2.3. **Uniform convexity and smoothness.**

Definition 2.6 (Uniform Convexity)**.** A Banach space Y is *uniformly convex*, if for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that for all $x, y \in Y$ with $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \varepsilon$,

$$\left\| \frac{x + y}{2} \right\| \leq 1 - \delta(\varepsilon).$$

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Definition 2.7 (Uniform Smoothness)**.** Y is *uniformly smooth* if

$$\lim_{\tau\to 0}\frac{\rho_Y(\tau)}{\tau}=0,$$

where the modulus of smoothness $\rho_Y(\tau)$ is defined by

$$\rho_Y(\tau)=\sup\left\{\frac{\|x+\tau y\|+\|x-\tau y\|}{2}-1:\|x\|=\|y\|=1\right\}.$$

Uniform convexity implies reflexivity (Milman-Pettis theorem) and RNP. Uniform smoothness is dual to uniform convexity: Y is uniformly smooth iff Y^* is uniformly convex.

For scalar variable-exponent spaces, we have:

Theorem 2.8 ([5])**.** *If $p:E\to(1,\infty)$ is log-Hölder continuous and $1<p_m\le p_S<\infty$, then $L^{p(\cdot)}(E)$ is uniformly convex and uniformly smooth.*

2.4. **Constant function embedding.** A crucial tool is the embedding of X into $L^{p(\cdot)}(E,X)$ via constant functions:

Lemma 2.9. *For $a\in X$, define $\hat{a}:E\to X$ by $\hat{a}(x)=a$ for all $x\in E$. Then*

$$c_1\|a\|_X\le\|\hat{a}\|_{p(\cdot)}\le c_2\|a\|_X,$$

where $c_1=\mu(E)^{-1/p_S}$ and $c_2=\mu(E)^{-1/p_m}$.

Proof. We have $\|\hat{a}\|_{p(\cdot)}=\|a\|_X\cdot\lambda_0$, where λ_0 satisfies $\int_E\lambda_0^{-p(x)}d\mu(x)=1$. From

$$\lambda_0^{-p_S}\mu(E)\le\int_E\lambda_0^{-p(x)}d\mu(x)\le\lambda_0^{-p_m}\mu(E),$$

we obtain $\mu(E)^{-1/p_S}\le\lambda_0\le\mu(E)^{-1/p_m}$, which gives the desired inequalities. □

3. Dentability and geometric interpretation

Dentability measures the extent to which a set has "sharp" points that can be isolated from the rest of the set by removing a small ball. In finite dimensions, all bounded sets are dentable, but in infinite dimensions, this fails dramatically.

3.1. **Geometric meaning of dentability.** Consider a bounded set $A\subset Y$. For $\varepsilon>0$, imagine removing from A all points within distance ε of some point y. If y can be chosen so that the convex hull of the remaining points does not come back to y, then y is an ε-dent point. A set is dentable if it admits ε-dent points for every $\varepsilon>0$.

Geometrically, dentable sets have "exposed" points that can be "dent-ed" away from the set. This is a weaker condition than strict convexity or uniform convexity, which requires that the entire boundary be curved away from the interior.

DOI: https://dx.doi.org/10.30504/jims.2026.567288.1310

3.2. Examples and counterexamples.

Example 3.1 (Dentable Spaces)**.** Finite-dimensional spaces, reflexive spaces, and separable dual spaces have RNP , and hence all bounded subsets are dentable. In particular, $L^p(E)$ for $1 < p < \infty$ has RNP.

Example 3.2 (Non-dentable Unit Ball)**.** The unit ball of $L^1[0, 1]$ is not dentable. Indeed, for any $f \in L^1[0, 1]$ with $\|f\|_1 = 1$ and any $\varepsilon > 0$, one can construct $g_1, \dots, g_n \in L^1[0, 1]$ with $\|g_i\|_1 \le 1$, $\|f - g_i\|_1 \ge \varepsilon$, such that f is a convex combination of the g_i's. This shows $f \in \operatorname{conv}(\mathbb{B} \setminus B_\varepsilon(f))$.

3.3. Dentability in variable-exponent spaces.

The dentability picture of $L^{p(\cdot)}(E, X)$ is nuanced.

Proposition 3.3. *Let* $1 < p_m \le p_S < \infty$.

1. *If* X *has RNP, then every bounded subset of* $L^{p(\cdot)}(E, X)$ *is dentable.*
2. *If* X *lacks RNP, then:*
   The unit ball of $L^{p(\cdot)}(E, X)$ *is not dentable.*
   The set of simple functions $\mathcal{S}_M$ *is not dentable for any* $M > 0$.

The second part is surprising: even though simple functions are dense in $L^{p(\cdot)}(E, X)$, the set $\mathcal{S}_M$ can be non-dentable when X lacks RNP. This occurs because $\mathcal{S}_M$ contains an isometric copy of bounded subsets of X via constant functions. If X has a non-dentable bounded set C, then $\hat{C} = \{\hat{c} : c \in C\} \subset \mathcal{S}_M$ is also non-dentable.

3.4. The role of $p_m > 1$.

The condition $p_m > 1$ is essential. When $p_m = 1$, the space $L^{p(\cdot)}(E, X)$ may contain an isomorphic copy of $L^1(E, X)$, which lacks RNP even if X has . Consider $p(x) = 1 + x$ on $[0, 1]$,

Example 3.4. For $p(x) = 1 + x$, we have $p_m = 1$, $p_S = 2$. The space $L^{1+x}[0, 1]$ contains a subspace isomorphic to $L^1[0, \frac{1}{2}]$ (functions supported on $[0, \frac{1}{2}]$). Since $L^1[0, \frac{1}{2}]$ lacks RNP, $L^{1+x}[0, 1]$ cannot have RNP, regardless of X.

This illustrates the hybrid geometry of variable-exponent spaces: where $p(x)$ is close to 1, the local behavior resembles L^1 (non-dentable); where $p(x)$ is large, it resembles L^p with $p > 1$ (dentable). The global property is determined by the most restrictive local behavior.

4. Proof of main results on RNP and dentability

4.1. Proof of Theorem A (RNP characterization).

Proof.

Necessity (⇒): Assume $L^{p(\cdot)}(E, X)$ has RNP. By Lemma 2.9, X embeds isometrically into $L^{p(\cdot)}(E, X)$ via constant functions. Since RNP is inherited by closed subspaces [6, Chapter III], X must have RNP.

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Sufficiency (⇐): Assume X has RNP. We show that every bounded subset of $L^{p(\cdot)}(E,X)$ is dentable. Let $A \subset L^{p(\cdot)}(E,X)$ be bounded with $\operatorname{diam}(A) \leq M$. Fix $\varepsilon > 0$. Since simple functions are dense, there exists a finite $\varepsilon/3$-net $\{f_1, \ldots, f_n\} \subset A$.

For each $x \in E$, define $A(x) = \{f(x) : f \in A\} \subset X$. Since A is bounded in $L^{p(\cdot)}(E,X)$, there exists $C > 0$ such that $\|f(x)\|_X \leq C$ for μ-a.e. x and all $f \in A$ (by the embedding $L^{p(\cdot)} \hookrightarrow L^{p_m}$). Thus $A(x)$ is a bounded subset of X, hence dentable by RNP of X.

By measurable selection (using the fact that X is separable if it has RNP [1]), we can choose $y_x \in A(x)$ such that

$$y_x \notin \overline{\operatorname{conv}}\big(A(x) \setminus B_{\varepsilon/2}(y_x)\big) \quad \mu\text{-a.e.}$$

Select $f_0 \in A$ with $f_0(x) = y_x$ μ-a.e. (This is possible by the Kuratowski-Ryll-Nardzewski theorem). We claim $f_0 \notin \overline{\operatorname{conv}}(A \setminus B_\varepsilon(f_0))$.

Suppose, for contradiction, that $f_0 \in \overline{\operatorname{conv}}(A \setminus B_\varepsilon(f_0))$. Then there exist $g_k \in \operatorname{conv}(A \setminus B_\varepsilon(f_0))$ with $g_k \to f_0$ in $L^{p(\cdot)}(E,X)$. Write $g_k = \sum_{j=1}^{m_k} \lambda_{k,j} h_{k,j}$ with $h_{k,j} \in A \setminus B_\varepsilon(f_0)$ and $\lambda_{k,j} \geq 0$, $\sum_j \lambda_{k,j} = 1$.

For each x, $g_k(x) \in \operatorname{conv}(A(x) \setminus B_\varepsilon(f_0(x)))$, since $\|h_{k,j}(x) - f_0(x)\|_X \geq \varepsilon$ for μ-a.e. x. Passing to a subsequence, $g_k(x) \to f_0(x)$ in X for μ-a.e. x. Hence

$$f_0(x) \in \overline{\operatorname{conv}}\big(A(x) \setminus B_\varepsilon(f_0(x))\big) \quad \mu\text{-a.e.},$$

contradicting the choice of $f_0(x)$. Therefore $f_0 \notin \overline{\operatorname{conv}}(A \setminus B_\varepsilon(f_0))$, so A is dentable. Since A was arbitrary, $L^{p(\cdot)}(E,X)$ has RNP. □

4.2. Proofs of Theorems B and C.

Proof of Theorem 1.2. If X has RNP, then by Theorem 1.1, $L^{p(\cdot)}(E,X)$ has RNP, so all bounded sets—including $\mathbb{B}$—are dentable. Conversely, if $\mathbb{B}$ is dentable, then $L^{p(\cdot)}(E,X)$ has RNP (since dentability of the unit ball implies RNP [4]), and Theorem 1.1 forces X to have RNP. □

Proof of Theorem 1.3. Since X lacks RNP, there exists a bounded non-dentable set $C \subset X$. Let $\hat{C} = \{\hat{c} : c \in C\} \subset \mathcal{S}_M$ for sufficiently large M. By Lemma 2.9, $\hat{C}$ is isometric to C, hence non-dentable. Therefore, $\mathcal{S}_M$ contains a non-dentable subset, so it cannot be dentable. □

4.3. **Remarks on the proof technique.** The proof of Theorem 1.1 uses the dentability characterization of RNP and a pointwise selection argument. The key is that dentability in X implies pointwise dentability of the set $\{f(x) : f \in A\}$ for almost every x, which can be lifted to dentability in $L^{p(\cdot)}(E,X)$ via the Luxemburg norm. The condition $p_m > 1$ ensures that the norm is "locally equivalent" to an L^{p_m} norm, providing the necessary boundedness for the pointwise argument.

5. Uniform convexity and smoothness

We now establish transfer theorems for uniform convexity and smoothness.

DOI: https://dx.doi.org/10.30504/jims.2026.567288.1310

5.1. Preliminary lemmas.

Lemma 5.1 (Pointwise Estimate). *Let X be uniformly convex with modulus δ_X. For any $u, v \in X$,*

$$\left\| \frac{u+v}{2} \right\|_X \leq \frac{\|u\|_X + \|v\|_X}{2} \left(1 - \delta_X \left(\frac{\|u-v\|_X}{\max(\|u\|_X, \|v\|_X)} \right) \right),$$

with the convention $\delta_X(0) = 0$.

Lemma 5.2 (Norm vs. and Modular). *For $f, g \in L^{p(\cdot)}(E, X)$ with $\|f\|_{p(\cdot)} = \|g\|_{p(\cdot)} = 1$, we have*

$$\left\| \frac{f+g}{2} \right\|_{p(\cdot)} \leq 1 - \delta \quad \textit{iff} \quad \rho\left(\frac{f+g}{2(1-\delta)} \right) \leq 1.$$

5.2. Proof of Theorem D.

Proof of Theorem 1.4. We prove both parts separately.

Part (1): Uniform convexity.

Necessity: Suppose $L^{p(\cdot)}(E, X)$ is uniformly convex. For $u, v \in X$ with $\|u\|_X = \|v\|_X = 1$, consider constant functions $f(x) = u$, $g(x) = v$. Then $\|f\|_{p(\cdot)} = \|u\|_X \|1\|_{p(\cdot)} = \|1\|_{p(\cdot)}$, similarly for g, and $\|f - g\|_{p(\cdot)} = \|u - v\|_X \|1\|_{p(\cdot)}$. Uniform convexity of $L^{p(\cdot)}(E, X)$ gives

$$\left\| \frac{f+g}{2} \right\|_{p(\cdot)} \leq (1-\delta)\|1\|_{p(\cdot)}$$

for some $\delta = \delta(\|u-v\|_X \|1\|_{p(\cdot)}) > 0$. But $\|(f+g)/2\|_{p(\cdot)} = \|(u+v)/2\|_X \|1\|_{p(\cdot)}$, so $\|(u+v)/2\|_X \leq 1-\delta$. Hence X is uniformly convex. Taking $X = \mathbb{R}$ shows $L^{p(\cdot)}(E)$ is uniformly convex.

Sufficiency: Assume X and $L^{p(\cdot)}(E)$ are uniformly convex. Let $f, g \in L^{p(\cdot)}(E, X)$ with $\|f\|_{p(\cdot)} = \|g\|_{p(\cdot)} = 1$ and $\|f - g\|_{p(\cdot)} \geq \varepsilon$. Define $\phi(x) = \|f(x)\|_X$, $\psi(x) = \|g(x)\|_X$. Then $\phi, \psi \in L^{p(\cdot)}(E)$ with $\|\phi\|_{p(\cdot)} = \|f\|_{p(\cdot)} = 1$, similarly $\|\psi\|_{p(\cdot)} = 1$, and $\|\phi - \psi\|_{p(\cdot)} \leq \|f - g\|_{p(\cdot)} \geq \varepsilon$. By uniform convexity of $L^{p(\cdot)}(E)$,

$$\left\| \frac{\phi+\psi}{2} \right\|_{p(\cdot)} \leq 1 - \delta_L(\varepsilon)$$

for some $\delta_L(\varepsilon) > 0$.

Let $A_\eta = \{x \in E : \|f(x) - g(x)\|_X \geq \eta\}$. For $x \in A_\eta$, by uniform convexity of X,

$$\left\| \frac{f(x)+g(x)}{2} \right\|_X \leq \frac{\phi(x)+\psi(x)}{2} \left(1 - \delta_X \left(\frac{\eta}{\max(\phi(x), \psi(x))} \right) \right).$$

For $x \notin A_\eta$, we simply have $\|(f(x) + g(x))/2\|_X \leq (\phi(x) + \psi(x))/2$.

Now compute $\rho((f + g)/2)$. Using the inequalities above and the convexity of $t \mapsto t^{p(x)}$, we obtain after integration (details omitted, see [5, Theorem 3.4.9] for the scalar case):

$$\rho\left(\frac{f+g}{2} \right) \leq 1 - \delta$$

for some $\delta > 0$ depending on ε, δ_L, δ_X, and η. By Lemma 5.2, this implies $\|(f + g)/2\|_{p(\cdot)} \leq 1 - \delta'$. Hence $L^{p(\cdot)}(E, X)$ is uniformly convex.

Part (2): Uniform smoothness.

DOI: https://dx.doi.org/10.30504/jims.2026.567288.1310

We use duality: a Banach space is uniformly smooth if and only if its dual is uniformly convex. Under the assumptions, we have $[L^{p(\cdot)}(E,X)]^* \cong L^{p'(\cdot)}(E,X^*)$ [5, Theorem 3.6.1]. Now $L^{p(\cdot)}(E,X)$ is uniformly smooth if and only if $L^{p'(\cdot)}(E,X^*)$ is uniformly convex. By part (1), the latter holds iff both X^* and $L^{p'(\cdot)}(E)$ are uniformly convex. But X^* uniformly convex is equivalent to X uniformly smooth (since X is reflexive), and $L^{p'(\cdot)}(E)$ uniformly convex is equivalent to $L^{p(\cdot)}(E)$ uniformly smooth by scalar duality. This yields the equivalence. ☐

5.3. **Examples and applications.**

Example 5.3 (Hilbert Space Valued Functions)**.** If $X = H$ is a Hilbert space and $p(\cdot)$ is log-Hölder continuous with $1 < p_m \le p_S < \infty$, then $L^{p(\cdot)}(E,H)$ is both uniformly convex and uniformly smooth.

Example 5.4 (ℓ^p Valued Functions)**.** Let $X = \ell^q$ with $1 < q < \infty$. Then X is uniformly convex and smooth. If $p(\cdot)$ satisfies the conditions, then $L^{p(\cdot)}(E,\ell^q)$ inherits these properties.

Remark 5.5. *The log-Hölder condition on $p(\cdot)$ is essential for uniform smoothness, as it ensures boundedness of the Hardy-Littlewood maximal operator and hence good duality properties [5].*

6. **Quantitative dentability moduli**

To measure dentability quantitatively, we introduce dentability moduli and formulate a precise proposition.

6.1. **Dentability modulus.**

Definition 6.1 (Dentability Modulus)**.** For a bounded set A in a Banach space Y and $\varepsilon > 0$, we define

$$\operatorname{dent}(A,\varepsilon) := \sup\left\{ r \ge 0 : \exists y \in A,\ y \notin \overline{\operatorname{conv}}(A \setminus B_r(y)) \right\}.$$

For the unit ball $\mathbb{B}_Y$, we write $\operatorname{dent}_Y(\varepsilon) := \operatorname{dent}(\mathbb{B}_Y,\varepsilon)$.

Thus A is dentable if and only if $\operatorname{dent}(A,\varepsilon) > 0$ for all $\varepsilon > 0$. The modulus measures how "deeply" one can dent the set.

Example 6.2. For a Hilbert space H, $\operatorname{dent}_H(\varepsilon) \ge 1 - \sqrt{1-(\varepsilon/2)^2} \approx \varepsilon^2/8$ for small ε.

6.2. **proposition for variable-exponent spaces.** Based on our analysis, we propose proposition 1.5. The factor $\Delta(p) := \min\{\mu(E)^{1/p_S - 1/p_m}, \mu(E)^{1/p_m - 1/p_S}\}$ captures the interplay between measure and exponent variability:

If $p(\cdot)$ is constant ($p_m = p_S$), then $\Delta(p) = 1$.
If $\mu(E)$ is small, then $\Delta(p)$ is small, reflecting that small domains limit dentability.
If $p_S - p_m$ is large, $\Delta(p)$ is small, indicating high exponent variability reduces dentability.

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6.3. **Evidence for the proposition.** Theorem 1.1 proves the qualitative version: $\operatorname{dent}(\mathbb{B},\varepsilon) > 0$ for all $\varepsilon > 0$ if and only if X has RNP and $p_m > 1$. The proposition provides a quantitative refinement.

Proposition 6.3. *If X is uniformly convex with modulus δ_X, and $p(\cdot)$ is constant $p > 1$, then*

$$\operatorname{dent}(\mathbb{B},\varepsilon) \geq C_p \delta_X(c_p \varepsilon),$$

where C_p, c_p depend only on p.

Proof. For constant p, $L^p(E,X)$ is uniformly convex with modulus $\delta(\varepsilon) \geq C_p \delta_X(c_p\varepsilon)$ [3]. Uniform convexity implies $\operatorname{dent}(\mathbb{B},\varepsilon) \geq \delta(\varepsilon)$. ☐

The variable-exponent case is more delicate due to the non-homogeneity of the Luxemburg norm.

6.4. **Implications.** If proved, proposition 1.5 would provide:

Quantitative convergence rates for algorithms in variable-exponent spaces.
Stability estimates for PDEs with variable growth.
A tool to compare dentability across different exponent functions.

7. Applications to PDEs and optimization

7.1. **PDEs with variable growth.** Consider the $p(x)$-Laplacian equation:

$$-\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = f(x,u) \quad \text{in } \Omega,$$

with boundary conditions. The natural energy space is $W^{1,p(\cdot)}(\Omega)$, a subspace of $L^{p(\cdot)}(\Omega)$. For the vector-valued case ($u : \Omega \to X$), the space is $W^{1,p(\cdot)}(\Omega,X) \subset L^{p(\cdot)}(\Omega,X)$.

Proposition 7.1. *If X has RNP and $p_m > 1$, then minimizers of variational problems in $W^{1,p(\cdot)}(\Omega,X)$ exist and satisfy appropriate Euler-Lagrange equations.*

Proof. RNP ensures that bounded sequences have weakly convergent subsequences (via dentability and the James theorem). Combined with the modular geometry of $L^{p(\cdot)}$, this yields the existence of minimizers. ☐

7.2. **Optimization in variable-exponent spaces.** Consider the problem: minimize $\Phi(f) = \int_E \phi(x,f(x))\,d\mu(x)$ over $f \in L^{p(\cdot)}(E,X)$ with $\|f\|_{p(\cdot)} \leq 1$.

Proposition 7.2. *If X has RNP and $\phi(x,\cdot)$ is convex, then a minimizer exists. Moreover, if X is uniformly convex and $p(\cdot)$ satisfies the conditions of Theorem 1.4, then the minimizer is unique.*

Dentability of the unit ball ensures that the minimization problem is well-posed, while uniform convexity guarantees uniqueness and stability.

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7.3. **Fixed-point theory.** Many fixed-point theorems for nonexpansive mappings require dentability of the domain [1]. Our results imply:

Corollary 7.3. *Let $T : \mathbb{B} \to \mathbb{B}$ be a nonexpansive mapping on the unit ball of $L^{p(\cdot)}(E, X)$. If X has RNP and $p_m > 1$, then T has a fixed point.*

This follows from the fact that dentable sets have the fixed-point property for nonexpansive mappings [7].

8. Conclusion and future directions

8.1. **Summary of results.** This paper has established fundamental geometric properties of variable-exponent Bochner-Lebesgue spaces $L^{p(\cdot)}(E, X)$:

1. **RNP characterization**: $L^{p(\cdot)}(E, X)$ has RNP if and only if X does, provided $p_m > 1$ (Theorem 1.1).
2. **Dentability of specific sets**: The unit ball is dentable if and only if X has RNP; simple functions may be non-dentable even when the space has RNP (Theorems 1.2, 1.3).
3. **Uniform convexity/smoothness transfer**: These properties transfer if and only if both X and $L^{p(\cdot)}(E)$ have them (Theorem 1.4).
4. **Quantitative proposition**: We formulated proposition 1.5 relating dentability moduli to exponent variability and measure.

These results reveal the intricate geometry of variable-exponent spaces, where local exponent behavior interacts with target space geometry to determine global properties.

8.2. **Open problems and future work.**

1. **Prove proposition 1.5**: This requires new techniques to handle the non-homogeneous Luxemburg norm quantitatively.
2. **Extend to $p_m = 1$**: Characterize when $L^{p(\cdot)}(E, X)$ has RNP when $p_m = 1$. This likely involves conditions on how quickly $p(x)$ moves away from 1.
3. **Sobolev spaces**: Study dentability and RNP for $W^{k,p(\cdot)}(E, X)$. The gradient structure adds complexity.
4. **Non-commutative analogs**: Investigate variable-exponent spaces for operator-valued functions, with applications to quantum mechanics.
5. **Numerical implications**: Develop algorithms that leverage the geometric properties, e.g., optimization methods with convergence rates depending on dentability moduli.
6. **Applications to fluid dynamics**: Use these results to study existence and regularity for electrorheological fluid models with variable exponent.

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