Journal of the Iranian Mathematical Society
ISSN (on-line): 2717-1612
J. Iran. Math. Soc. 6 (2025), no. 2, 173-183

ON DEFERRED STATISTICAL CONVERGENCE IN $\mathscr{A}$- METRIC SPACES

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Abstract. In this study, we introduce the notions of deferred statistical convergence and deferred strong Cesàro summability in $\mathscr{A}$-metric spaces, which represent some of the most prominent examples of generalized metric spaces that have been extensively investigated in recent developments in functional analysis and summability theory. Then, we conduct a detailed investigation into the relationships among statistical convergence, deferred statistical convergence, and deferred strong Cesàro summability in the context of $\mathscr{A}$-metric spaces. Additionally, we present several inclusion relations among these concepts within the context of $\mathscr{A}$-metric spaces.

1. Introduction

Statistical convergence is a flexible generalization of classical convergence, based on the density of subsets of natural numbers. This concept was introduced independently by Fast [12] and Steinhaus [37] in 1951, and later re-introduced by Schoenberg [35] in 1959. Researchers have been interested in this idea ever since, and it has been thoroughly examined both theoretically and practically in a variety of fields. Some important sources to learn more about these studies are [5–7, 9, 13, 15, 16, 18, 38].

The concept of metric spaces, which was introduced in the early 20th century and is based on the distance function, has become an indispensable structure in fields such as theoretical and applied mathematics, computer science, and engineering. This concept has attracted considerable attention from researchers since it was introduced by Fréchet [14] in 1905. In the following years, when metric spaces proved inadequate in certain situations, generalizations of these spaces were attempted. Even though some problems appeared in the new structures created by generalizing metric spaces, work in this area has continued steadily. For example, 2-metric spaces introduced by Gähler [17], D-metric

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Communicated by Fatemeh Panjeh Ali Beik

MSC(2020): Primary: 40A05; Secondary: 40A35, 40G99.

Keywords: Statistical convergence, deferred Statistical convergence , $\mathscr{A}$-metric space.

Received: 12 August 2025, Accepted: 25 November 2025.

DOI: https://dx.doi.org/10.30504/jims.2025.540372.1276

spaces proposed by Dhage [8], as well as G-metric spaces by Mustafa and Sims [27], and S-metric spaces by Sedghi et al. [36] are all examples of this research (see also reference [25]). One of the studies conducted in this context is the introduction of $\mathscr{A}$-metric spaces by Abbas et al. [2] in 2015 , which represents one of the most prominent generalizations of metric spaces. Initially, these generalized metric spaces were intensely investigated within the framework of fixed-point theory; more recently, they have also been widely studied in the areas of summability and functional analysis. For further studies on this topic, the following references may be consulted: [1, 4, 19, 28, 31–34].

In 1932, Agnew [3] introduced the concept of the deferred Cesàro mean based on the classical Cesàro mean. In 2016, Küçükaslan and Yılmaztürk [26] proposed the concept of deferred statistical convergence by combining it with statistical convergence. Later, in 2019, Et et al. [10, 11] obtained significant results by studying deferred statistical convergence in the context of metric spaces.

Moreover, researchers may refer to [20–24,29,30] for various recent applications related to this topic.

The convergence of sequences in metric spaces is extensively studied by researchers not only in theoretical mathematics but also due to its wide range of applications in fields such as physics, engineering, and data analysis. In recent years, classical convergence concepts have often fallen short when dealing with indeterminate cases. This has led to the emergence of more general and flexible notions, such as deferred statistical convergence. However, the literature offers limited systematic treatment of these approaches within generalized metric spaces. This gap has encouraged further exploration of deferred statistical convergence in the framework of $\mathscr{A}$-metric spaces, which is an important generalization of classical metric spaces that has recently gained attention.

In this context, the notions of statistical convergence, deferred statistical convergence, and deferred strong Cesàro summability have been defined in $\mathscr{A}$-metric spaces, and the relationships among these concepts have been thoroughly explored. Addressing these concepts in tandem contributes to a more profound understanding of convergence theory in $\mathscr{A}$-metric spaces, shedding light on its theoretical framework. Thus, this study aims to develop a new perspective on the analysis of convergence in $\mathscr{A}$-metric spaces, thereby contributing to recent developments in functional analysis and summability theory and filling the existing gap in the literature. Moreover, this approach offers valuable insights for practical applications across various related disciplines.

Although the methods and results in this study are similar to those in [10], the findings are more comprehensive and generalize the previous results.

2. Preliminaries

In this section we present the basic concepts and notations that be utilized throughout our research. Let K be a set of positive integers. The asymptotic (or natural) density of K is given by

$$\delta(K) = \lim_{t} \frac{1}{t} |\{k \leq t : k \in K\}|$$

DOI: https://dx.doi.org/10.30504/jims.2025.540372.1276

[15].

A sequence (y_k) is said to be statistically convergent to ξ, iffor every $\varepsilon > 0$,

$$\lim_{t\to\infty} \frac{1}{t} |\{k \leq t : |y_k - \xi| \geq \varepsilon\}| = 0$$

[15].

Agnew [3] presented the concept of the deferred Cesàro mean for real (or complex) sequences, given by

$$(\mathscr{D}_{r,s} y)_t = \frac{1}{s_t - r_t} \sum_{k=r_t+1}^{s_t} y_k, \quad t = 1, 2, 3, \ldots$$

where $r = (r_t)$ and $s = (s_t)$ are sequences of non-negative integers satisfying

$$r_t < s_t \quad \text{and} \quad \lim_{t\to\infty} s_t = \infty. \tag{2.1}$$

Definition 2.1 ([26])**.** *A real-valued sequence $y = (y_k)$ is said to be deferred statistically convergent to ξ , iffor every $\varepsilon > 0$,*

$$\lim_{t\to\infty} \frac{1}{s_t - r_t} |\{r_t < k \leq s_t : |y_k - \xi| \geq \varepsilon\}| = 0.$$

We write the limit as $\mathscr{D}\mathcal{S}^{r,s}\text{-}\lim y_k = \xi$.

Definition 2.2 ([2])**.** *Let $\mathscr{Y}$ be a nonempty set. A function $\mathscr{A} : \mathscr{Y}^n \to [0, \infty)$ is called an $\mathscr{A}$-metric on $\mathscr{Y}$ if for any $y_i, a \in \mathscr{Y}, i = 1, 2, \ldots, n$ and $n \geq 2$, the following conditions hold;*

- (A1) $\mathscr{A}(y_1, y_2, \ldots, y_{n-1}, y_n) \geq 0$,
- (A2) $\mathscr{A}(y_1, y_2, \ldots, y_{n-1}, y_n) = 0 \Leftrightarrow y_1 = y_2 = \ldots = y_n$,
- (A3) $\mathscr{A}(y_1, y_2, ..., y_{n-1}, y_n) \leq \sum_{j=1}^{n} \mathscr{A}(\underbrace{y_j, y_j, \cdots, y_j}_{n-1}, a)$.

The pair $(\mathscr{Y}, \mathscr{A})$ is called an $\mathscr{A}$-metric space. An $\mathscr{A}$-metric space is a generalized metric space, where the ordinary metric corresponds to $n = 2$ and the S-metric corresponds to $n = 3$, as special cases of the $\mathscr{A}$-metric.

Example 2.3 ([2])**.** *Let $\mathscr{Y} = \mathbb{R}$. A function $\mathscr{A} : \mathscr{Y}^n \to [0, \infty)$ by*

$$\mathscr{A}(y_1, y_2, ..., y_{n-1}, y_n) = \sum_{k=1}^{n} \sum_{k<j} |y_k - y_j|$$

is an $\mathscr{A}$-metric on $\mathscr{Y}$.

Lemma 2.4 ([2])**.** *Let $(\mathscr{Y}, \mathscr{A})$ be an $\mathscr{A}$-metric space. Then $\mathscr{A}(y, y, ..., y, z) = \mathscr{A}(z, z, ..., z, y)$ for each pair of points $y, z \in \mathscr{Y}$.*

Lemma 2.5 ([2])**.** *Let $(\mathscr{Y}, \mathscr{A})$ be an $\mathscr{A}$-metric space. Then for each pair of points y, z in $\mathscr{Y}$, it satisfies the condition:*

$$\mathscr{A}(y, y, ..., y, w) \leq (n-1)\mathscr{A}(y, y, ..., y, z) + \mathscr{A}(z, z, ..., z, w) \text{ and}$$
$$\mathscr{A}(y, y, ..., y, w) \leq (n-1)\mathscr{A}(y, y, ..., y, z) + \mathscr{A}(w, w, ..., w, z).$$

Definition 2.6 ([2]). *An $\mathscr{A}$- metric space $(\mathscr{Y},\mathscr{A})$ is said to be bounded if there exists a real number $r>0$ such that $\mathscr{A}(y,y,...,y,z)\leq r$ for all $y,z\in\mathscr{Y}$. Otherwise, $\mathscr{Y}$ is unbounded.*

Definition 2.7 ([2]). *Let $(\mathscr{Y},\mathscr{A})$ be an $\mathscr{A}$- metric space and (y_k) be a sequence in this space.*

- (i) *We say that the sequence (y_k) converges to a point $\xi\in\mathscr{Y}$, if for every $\varepsilon>0$, there exists a positive integer k_0 such that $\mathscr{A}(y_k,y_k,...,y_k,\xi)<\varepsilon$ for every $k\geq k_0$.*
- (ii) *We also say that the sequence (y_k) is a Cauchy sequence , if for every $\varepsilon>0$, there exists a positive integer m such that $\mathscr{A}(y_k,y_k,...,y_k,y_l)<\varepsilon$ for all $k,l\geq m$.*

Definition 2.8 ([31]). *Let $(\mathscr{Y},\mathscr{A})$ be an $\mathscr{A}$-metric space and let (y_k) be a sequence in this space. The sequence (y_k) is said to be statistically convergent to $\xi\in\mathscr{Y}$ if, for every $\varepsilon>0$,*

$$\lim_{t\to\infty}\frac{1}{t}|\{k\leq t:\mathscr{A}(y_k,y_k,\ldots,y_k,\xi)\geq\varepsilon\}|=0.$$

We denote this by $y_k\xrightarrow{\mathscr{A}S}\xi$ or $\mathcal{S}_{\mathscr{A}}-\lim y_k=\xi$. We also denote by $\mathcal{S}_{\mathscr{A}}$ the set of all statistically convergent sequences in an $\mathscr{A}$-metric space $(\mathscr{Y},\mathscr{A})$.

Throughout the study, (r_t) and (s_t) are sequences of non-negative integers that satisfy condition (2.1).

3. Main results

In this section, we define deferred statistical convergence and deferred strong Cesàro summability in an $\mathscr{A}$-metric space. In addition, we show several inclusion relationships between these notions in the context of $\mathscr{A}$-metric spaces.

Definition 3.1. *Let $(\mathscr{Y},\mathscr{A})$ be an $\mathscr{A}$-metric space. Consider two sequences (r_t) and (s_t) of non-negative integers satisfying condition (2.1). A sequence $y=(y_k)\in\mathscr{Y}$ is said to be $\mathscr{D}\mathcal{S}^{r,s}_{\mathscr{A}}$-convergent (or* deferred $\mathscr{A}$-statistically convergent*) to ξ, if there exists a real number $\xi\in\mathscr{Y}$ such that*

$$\lim_{t\to\infty}\frac{1}{s_t-r_t}|\{r_t<k\leq s_t:\mathscr{A}(y_k,y_k,\cdots,y_k,\xi)\geq\varepsilon\}|=0.$$

We denote this by $\mathscr{D}\mathcal{S}^{r,s}_{\mathscr{A}}\text{-}\lim y_k=\xi$ or $y_k\to\xi\,(\mathscr{D}\mathcal{S}^{r,s}_{\mathscr{A}})$. We also denote by $\mathscr{D}\mathcal{S}^{r,s}_{\mathscr{A}}$ the set of all $\mathscr{D}\mathcal{S}^{r,s}_{\mathscr{A}}$-statistically convergent sequences in an $\mathscr{A}$-metric space $(\mathscr{Y},\mathscr{A})$.

Remark 3.2. *If we take $s_t=t$ and $r_t=0$, then in an $\mathscr{A}$-metric space $(\mathscr{Y},\mathscr{A})$, deferred statistical convergence reduces to the usual statistical convergence in $\mathscr{A}$-metric spaces [31]. Also, if we take $s_t=k_t$ and $r_t=k_{t-1}$ (for any lacunary sequence of nonnegative integers $\{k_t\}$ with $k_t-k_{t-1}\to\infty$ as $t\to\infty$), then deferred statistical convergence reduces to lacunary statistical convergence in $\mathscr{A}$-metric spaces [32].*

Definition 3.3. *Let $(\mathscr{Y},\mathscr{A})$ be an $\mathscr{A}$-metric space. Consider two sequences (r_t) and (s_t) of non-negative integers satisfying condition (2.1). A sequence $y=(y_k)\in\mathscr{Y}$ is said to be strongly*

DOI: https://dx.doi.org/10.30504/jims.2025.540372.1276

$DW^{r,s}_{\mathscr{A}}$–summable (or deferred strongly $\mathscr{A}$-Cesàro summable) to ξ if there exists a real number $\xi \in \mathscr{Y}$ such that

$$\lim_{t\to\infty} \frac{1}{s_t - r_t} \sum_{k=r_t+1}^{s_t} \mathscr{A}(y_k, y_k, \cdots, y_k, \xi) = 0.$$

We denote this by $\mathscr{D}\mathcal{W}^{r,s}_{\mathscr{A}} - \lim y_k = \xi$ or $y_k \to \xi\,(\mathscr{D}\mathcal{W}^{r,s}_{\mathscr{A}})$. We also denote by $\mathscr{D}\mathcal{W}^{r,s}_{\mathscr{A}}$ the set of all $\mathscr{D}\mathcal{W}^{r,s}_{\mathscr{A}}$-summable sequences in an $\mathscr{A}$-metric space $(\mathscr{Y}, \mathscr{A})$.

Theorem 3.4. *Let $(\mathscr{Y}, \mathscr{A})$ be an $\mathscr{A}$-metric space, and let (y_k) be a sequence in this space. Consider two sequences (r_t) and (s_t) of non-negative integers satisfying condition (2.1). If the sequence (y_k) is deferred strongly $\mathscr{A}$-Cesàro summable to ξ then it is also deferred $\mathscr{A}$-statistically convergent to ξ in the $\mathscr{A}$-metric space $(\mathscr{Y}, \mathscr{A})$.*

Proof. Let $y_k \to \xi\left(\mathscr{D}\mathcal{W}^{\mathscr{A}}_{r,s}\right)$, and $\varepsilon > 0$ be arbitrary. Then we have the inequality:

$$\begin{aligned}
\frac{1}{s_t - r_t} \sum_{r_t+1}^{s_t} \mathscr{A}(y_k, y_k, \cdots, y_k, \xi) &= \frac{1}{s_t - r_t} \sum_{\substack{k=r_t+1 \\ \mathscr{A}(y_k, y_k \cdots, y_k, \xi) \geq \varepsilon}}^{s_t} \mathscr{A}(y_k, y_k, \ldots, y_k, \xi) \\
&\quad + \frac{1}{s_t - r_t} \sum_{\substack{k=r_t+1 \\ \mathscr{A}(y_k, y_k \cdots, y_k, \xi) < \varepsilon}}^{s_t} \mathscr{A}(y_k, y_k, \ldots, y_k, \xi) \\
&\geq \frac{1}{s_t - r_t} \sum_{\substack{k=r_t+1 \\ \mathscr{A}(y_k, y_k \cdots, y_k, \xi) \geq \varepsilon}}^{s_t} \mathscr{A}(y_k, y_k, \ldots, y_k, \xi) \\
&\geq \frac{1}{s_t - r_t} \mid \{r_t < k \leq s_t : \mathscr{A}(y_k, y_k, \ldots, y_k, \xi) \geq \varepsilon\} \mid \varepsilon.
\end{aligned}$$

Taking the limit as $t \to \infty$, we obtain:

$$\lim_{t\to\infty} \frac{1}{s_t - r_t} \mid \{t \leq k : \mathscr{A}(y_k, y_k, \ldots, y_k, \xi) \geq \varepsilon\} \mid = 0.$$

This completes the proof.

□

The converse of Theorem 3.4 is generally not true. Consider the $\mathscr{A}$-metric defined in Example 2.3, where $\mathscr{Y} = \mathbb{R}$ and $n \geq 2$, given by

$$\mathscr{A} : \underbrace{\mathscr{Y} \times \mathscr{Y} \times \cdots \times \mathscr{Y}}_{n \text{ times}} \to [0, \infty)$$

and

$$\mathscr{A}(a_1, a_2, \ldots, a_n) = \sum_{k=1}^{n} \sum_{k<j} |a_k - a_j|.$$

Consider the following sequence $y = (y_k)$ similar to [26].

DOI: https://dx.doi.org/10.30504/jims.2025.540372.1276

$$y_k := \begin{cases} k^2, & \text{if } \lfloor \sqrt{s_t} \rfloor - p_0 < k \leq \lfloor \sqrt{s_t} \rfloor \text{ for } t = 1, 2, 3, \ldots \\ 0, & \text{otherwise} \end{cases}$$

Here s_t is a monotone increasing sequence and p_0 is a fixed positive integer, and r_t is a sequence satisfying:

$$0 < r_t \leq \lfloor \sqrt{s_t} \rfloor - p_0.$$

Then for $\varepsilon > 0$, we get

$$\frac{1}{s_t - r_t} \left| \{ k : r_t < k \leq s_t,\, \mathscr{A}(y_k, y_k, \cdots, y_k, 0) \geq \varepsilon \} \right| = \frac{p_0}{s_t - r_t} \to 0 \quad \text{as } t \to \infty,$$

which implies that $y_k \to 0\,(\mathscr{D}\mathcal{S}^{r,s}_{\mathscr{A}})$. However,

$$\frac{1}{s_t - r_t} \sum_{k=r_t+1}^{s_t} \mathscr{A}(y_k, y_k, \cdots, y_k, 0) \geq \frac{p_0 \left(\lfloor \sqrt{s_t} \rfloor - p_0 \right)^2}{s_t - r_t} \to p_0 \quad \text{as } t \to \infty,$$

$p_0 \neq 0$. So $y_k \not\to 0\,(\mathscr{D}\mathcal{W}^{r,s}_{\mathscr{A}})$.

Theorem 3.5. *Let $(\mathscr{Y}, \mathscr{A})$ be an $\mathscr{A}$-metric space. Suppose that (y_k) is a bounded sequence in this space, and let (r_t) and (s_t) be sequences of non-negative integers satisfying condition (2.1), If the sequence $y = (y_k)$ is deferred $\mathscr{A}$-statistically convergent to ξ, then it is also deferred strongly $\mathscr{A}$-Cesàro summable to ξ in the $\mathscr{A}$-metric space $(\mathscr{Y}, \mathscr{A})$.*

Proof. Let $y_k \to \xi \left(\mathscr{D}\mathcal{S}^{\mathscr{A}}_{r,s} \right)$, and let $\varepsilon > 0$ be arbitrary. Then there exists a point $\xi \in \mathscr{Y}$ such that the following condition is satisfied:

$$\lim_{t \to \infty} \frac{1}{s_t - r_t} \mid \{ t \leq k : \mathscr{A}(y_k, y_k, \ldots, y_k, \xi) \geq \varepsilon \} \mid = 0,$$

Since $y = (y_k)$ is a bounded sequence in an $\mathscr{A}$-metric space $(\mathscr{Y}, \mathscr{A})$, a positive real number $M > 0$ such that $\mathscr{A}(y_k, y_k, \cdots, y_k, \xi) < M$ for all $k \in \mathbb{N}$. Therefore, we have:

$$\begin{aligned} \frac{1}{s_t - r_t} \sum_{k=r_t+1}^{s_t} \mathscr{A}(y_k, y_k, \cdots, y_k, \xi) &= \frac{1}{s_t - r_t} \sum_{\substack{k=r_t+1 \\ \mathscr{A}(y_k, y_k \cdots, y_k, \xi) \geq \varepsilon}}^{s_t} \mathscr{A}(y_k, y_k, \ldots, y_k, \xi) \\ &\quad + \frac{1}{s_t - r_t} \sum_{\substack{k=r_t+1 \\ \mathscr{A}(y_k, y_k \cdots, y_k, \xi) < \varepsilon}}^{s_t} \mathscr{A}(y_k, y_k, \ldots, y_k, \xi) \\ &\leq \frac{M}{s_t - r_t} \mid \{ r_t < k \leq s_t : \mathscr{A}(y_k, y_k, \ldots, y_k, \xi) \geq \varepsilon \} \mid + \varepsilon. \end{aligned}$$

Taking the limit as $t \to \infty$, we obtain:

$$\lim_{t \to \infty} \frac{1}{s_t - r_t} \mid \{ k \leq t : \mathscr{A}(y_k, y_k, \ldots, y_k, \xi) \geq \varepsilon \} \mid = 0.$$

This completes the proof.

□

Theorem 3.6. *Let $(\mathscr{Y}, \mathscr{A})$ be an $\mathscr{A}$-metric space. Consider two sequences (r_t) and (s_t) of non-negative integers satisfying condition (2.1). If $\lim_{t\to\infty} \inf \frac{s_t}{r_t} > 1$, then $\mathcal{S}_{\mathscr{A}} \subset \mathscr{D}\mathcal{S}^{r,s}_{\mathscr{A}}$.*

Proof. Assume that $\lim_{t\to\infty} inf \frac{s_t}{r_t} > 1$. Then there exists a constant $\sigma > 0$ such that for all sufficiently large t, $\frac{s_t}{r_t} \geq 1+\sigma$. This implies

$$\frac{s_t - r_t}{s_t} \geq \frac{\sigma}{1+\sigma}$$

Suppose that $y_k \xrightarrow{\mathscr{A}\mathcal{S}} \xi$. Then for every $\varepsilon > 0$ and for all sufficiently large t, we have:

$$\begin{aligned}\frac{1}{s_t}\left|\{k \leq s_t : \mathscr{A}(y_k, y_k, \ldots, y_k, \xi) \geq \varepsilon\}\right| &\geq \frac{1}{s_t}\left|\{r_t < k \leq s_t : \mathscr{A}(y_k, y_k, \ldots, y_k, \xi) \geq \varepsilon\}\right| \\ &\geq \frac{\sigma}{1+\sigma} \cdot \frac{1}{s_t - r_t}\left|\{r_t < k \leq s_t : \mathscr{A}(y_k, y_k, \ldots, y_k, \xi) \geq \varepsilon\}\right|.\end{aligned}$$

This completes the proof. □

Theorem 3.7. *Let $(\mathscr{Y}, \mathscr{A})$ be an $\mathscr{A}$-metric space. Consider two sequences (r_t) and (s_t) of non-negative integers satisfying condition (2.1). If $\lim_{t\to\infty} \inf \frac{s_t - r_t}{t} > 0$ and $s_t < t$, then $\mathcal{S}_{\mathscr{A}} \subseteq \mathscr{D}\mathcal{S}^{r,s}_{\mathscr{A}}$.*

Proof. Let $y = (y_k)$ be a sequence in an $\mathscr{A}$-metric space $(\mathscr{Y}, \mathscr{A})$. Assume that $y = (y_k)$ is statistically convergent to $\xi \in \mathscr{Y}$, that is,

$$y \in \mathcal{S}_{\mathscr{A}} \quad \text{and} \quad \lim_{t\to\infty} \frac{1}{t}\Big|\{k \leq t : \mathscr{A}(y_k, y_k, \ldots, y_k, \xi) \geq \varepsilon\}\Big| = 0 \quad \text{for all } \varepsilon > 0.$$

Suppose further that $\lim_{t\to\infty} \inf \frac{s_t - r_t}{t} > 0$ and $s_t < t$. Then for each $\varepsilon > 0$, we write

$$\{k \leq t : \mathscr{A}(y_k, y_k, \ldots, y_k, \xi) \geq \varepsilon\} \supseteq \{r_t < k \leq s_t : \mathscr{A}(y_k, y_k, \ldots, y_k, \xi) \geq \varepsilon\}.$$

Therefore, we obtain the inequality below:

$$\frac{1}{t}\left|\{k \leq t : \mathscr{A}(y_k, y_k, \ldots, y_k, \xi) \geq \varepsilon\}\right| \geq \frac{1}{t}\left|\{r_t < k \leq s_t : \mathscr{A}(y_k, y_k, \ldots, y_k, \xi) \geq \varepsilon\}\right|.$$

We can restate the right-hand side as:

$$= \frac{s_t - r_t}{t} \cdot \frac{1}{s_t - r_t}\left|\{r_t < k \leq s_t : \mathscr{A}(y_k, y_k, \ldots, y_k, \xi) \geq \varepsilon\}\right|.$$

Therefore, $y_k \to \xi \left(\mathscr{D}\mathcal{S}^{r,s}_{\mathscr{A}}\right)$, i.e., $y \in \mathscr{D}\mathcal{S}^{r,s}_{\mathscr{A}}$. Thus we have $\mathcal{S}_{\mathscr{A}} \subseteq \mathscr{D}\mathcal{S}^{r,s}_{\mathscr{A}}$. □

Theorem 3.8. *Let $(\mathscr{Y}, \mathscr{A})$ be an $\mathscr{A}$-metric space and let $(r_t), (s_t), (p_t)$ and (q_t) be four sequences of non-negative integers satisfying condition (2.1), such that*

$$p_t < r_t < s_t < q_t \tag{3.1}$$

for all $t \in \mathbb{N}$.

(i) *If*

$$\lim_{t\to\infty} \frac{s_t - r_t}{q_t - p_t} = m > 0, \tag{3.2}$$

then $\mathscr{D}\mathcal{S}^{p,q}_{\mathscr{A}} \subseteq \mathscr{D}\mathcal{S}^{r,s}_{\mathscr{A}}$.

DOI: https://dx.doi.org/10.30504/jims.2025.540372.1276

(ii) *If*

$$\lim_{t\to\infty}\frac{q_t-p_t}{s_t-r_t}=1, \tag{3.3}$$

then $\mathscr{DS}_{\mathscr{A}}^{r,s}\subseteq\mathscr{DS}_{\mathscr{A}}^{p,q}$.

Proof. (i) Suppose condition (3.2) is satisfied. Let (y_k) be a sequence in an $\mathscr{A}$-metric space $(\mathscr{Y},\mathscr{A})$ such that $y_k\to\xi\,(\mathscr{DS}_{\mathscr{A}}^{p,q})$, $\xi\in\mathscr{Y}$. For arbitrary $\varepsilon>0$, we get

$$\{p_t<k\le q_t:\mathscr{A}(y_k,y_k,\ldots,y_k,\xi)\ge\varepsilon\}\supseteq\{r_t<k\le s_t:\mathscr{A}(y_k,y_k,\ldots,y_k,\xi)\ge\varepsilon\},$$

and hence

$$\begin{aligned}&\frac{1}{q_t-p_t}\left|\{p_t<k\le q_t:\mathscr{A}(y_k,y_k,\ldots,y_k,\xi)\ge\varepsilon\}\right|\\&\ge\frac{s_t-r_t}{q_t-p_t}\cdot\frac{1}{s_t-r_t}\left|\{r_t<k\le s_t:\mathscr{A}(y_k,y_k,\ldots,y_k,\xi)\ge\varepsilon\}\right|.\end{aligned}$$

Therefore, $\mathscr{DS}_{\mathscr{A}}^{p,q}\subseteq\mathscr{DS}_{\mathscr{A}}^{r,s}$.

(ii) Assume that condition (3.3) holds. Let (y_k) be a sequence in an $\mathscr{A}$-metric space $(\mathscr{Y},\mathscr{A})$ such that $y_k\to\xi\,(\mathscr{DS}_{\mathscr{A}}^{r,s})$, $\xi\in\mathscr{Y}$. Given $\varepsilon>0$, we obtain

$$\begin{aligned}&\frac{1}{q_t-p_t}\left|\{p_t<k\le q_t:\mathscr{A}(y_k,y_k,\ldots,y_k,\xi)\ge\varepsilon\}\right|\\&\le\frac{1}{q_t-p_t}\left|\{p_t<k\le r_t:\mathscr{A}(y_k,y_k,\ldots,y_k,\xi)\ge\varepsilon\}\right|\\&\quad+\frac{1}{q_t-p_t}\left|\{s_t<k\le q_t:\mathscr{A}(y_k,y_k,\ldots,y_k,\xi)\ge\varepsilon\}\right|\\&\quad+\frac{1}{q_t-p_t}\left|\{r_t<k\le s_t:\mathscr{A}(y_k,y_k,\ldots,y_k,\xi)\ge\varepsilon\}\right|\\&\le\frac{r_t-p_t+q_t-s_t}{q_t-p_t}+\frac{1}{q_t-p_t}\left|\{r_t<k\le s_t:\mathscr{A}(y_k,y_k,\ldots,y_k,\xi)\ge\varepsilon\}\right|\\&=\frac{(q_t-p_t)-(s_t-r_t)}{q_t-p_t}+\frac{1}{q_t-p_t}\left|\{r_t<k\le s_t:\mathscr{A}(y_k,y_k,\ldots,y_k,\xi)\ge\varepsilon\}\right|\\&\le\left(\frac{q_t-p_t}{s_t-r_t}-1\right)+\frac{1}{s_t-r_t}\left|\{r_t<k\le s_t:\mathscr{A}(y_k,y_k,\ldots,y_k,\xi)\ge\varepsilon\}\right|\end{aligned}$$

Taking the limit as $t\to\infty$, we have:

$$\lim_{t\to\infty}\frac{1}{(q_t-p_t)}\left|\{p_t<k\le q_t:\mathscr{A}(y_k,y_k,\ldots,y_k,\xi)\ge\varepsilon\}\right|=0.$$

Therefore, $\mathscr{DS}_{\mathscr{A}}^{r,s}\subseteq\mathscr{DS}_{\mathscr{A}}^{p,q}$.

□

We omit the proof of the following theorem because it can be proven in a similar way using standard methods.

Theorem 3.9. *$(\mathscr{Y},\mathscr{A})$ be an $\mathscr{A}$-metric space and let $(r_t),(s_t),(p_t)$, and (q_t) be sequences of non-negative integers satisfying* (2.1) *and* (3.1).

(i) *If condition* (3.2) *holds, then* $\mathscr{DW}_{\mathscr{A}}^{p,q}\subset\mathscr{DW}_{\mathscr{A}}^{r,s}$.

DOI: https://dx.doi.org/10.30504/jims.2025.540372.1276

(ii) *If condition* (3.3) *holds and* $(\mathscr{Y}, \mathscr{A})$ *is bounded, then* $\mathscr{D}\mathcal{W}_{\mathscr{A}}^{r,s} \subset \mathscr{D}\mathcal{W}_{\mathscr{A}}^{p,q}$.

Theorem 3.10. *Let* $(\mathscr{Y}, \mathscr{A})$ *be an* $\mathscr{A}$*-metric space and let* $(r_t), (s_t), (p_t)$*, and* (q_t) *be sequences of non-negative integers satisfying* (2.1) *and* (3.1)*. Then:*

(i) *Let condition* (3.2) *hold. If a sequence is strongly* $\mathscr{D}\mathcal{W}_{\mathscr{A}}^{p,q}$*-summable to* ξ*, then it is* $\mathscr{D}\mathcal{S}_{\mathscr{A}}^{r,s}$*-convergent to* ξ*.*

(ii) *Given that* (3.3) *holds, assume that* $(\mathscr{Y}, \mathscr{A})$ *is bounded. If a sequence is* $\mathscr{D}\mathcal{S}_{\mathscr{A}}^{r,s}$*-convergent to* ξ*, then it is strongly* $\mathscr{D}\mathcal{W}_{\mathscr{A}}^{p,q}$*-summable to* ξ*.*

Proof. (i) Omitted.

(ii) Assume that $\mathscr{D}\mathcal{S}_{\mathscr{A}}^{r,s} - \lim y_k = \xi$, and $(\mathscr{Y}, \mathscr{A})$ is bounded. From the boundedness of $(\mathscr{Y}, \mathscr{A})$, there exists an $M > 0$ such that $\mathscr{A}(y_k, y_k, \ldots, y_k, \xi) < M$ for all k. Then for every $\varepsilon > 0$, we may write:

$$\begin{aligned}
&\frac{1}{q_t - p_t} \sum_{k=p_t+1}^{q_t} \mathscr{A}(y_k, y_k, \ldots, y_k, \xi) \\
&\qquad = \frac{1}{q_t - p_t} \sum_{k=s_t-r_t+1}^{q_t-p_t} \mathscr{A}(y_k, y_k, \ldots, y_k, \xi) + \frac{1}{q_t - p_t} \sum_{k=r_t+1}^{s_t} \mathscr{A}(y_k, y_k, \ldots, y_k, \xi) \\
&\qquad \leq \left(\frac{(q_t - p_t) - (s_t - r_t)}{q_t - p_t} \right) M + \frac{1}{q_t - p_t} \sum_{k=r_t+1}^{s_t} \mathscr{A}(y_k, y_k, \ldots, y_k, \xi) \\
&\qquad \leq \left(\frac{q_t - p_t}{s_t - r_t} - 1 \right) M \\
&\qquad\quad + \frac{1}{s_t - r_t} \left(\sum_{\substack{k=r_t+1 \\ \mathscr{A}(y_k, y_k, \ldots, y_k, \xi) \geq \varepsilon}}^{s_t} \mathscr{A}(y_k, y_k, \ldots, y_k, \xi) + \sum_{\substack{k=r_t+1 \\ \mathscr{A}(y_k, y_k, \ldots, y_k, \xi) < \varepsilon}}^{s_t} \mathscr{A}(y_k, y_k, \ldots, y_k, \xi) \right) \\
&\qquad \leq \left(\frac{q_t - p_t}{s_t - r_t} - 1 \right) M + \frac{M}{s_t - r_t} \left| \{ k : r_t < k \leq s_t,\ \mathscr{A}(y_k, y_k, \ldots, y_k, \xi) \geq \varepsilon \} \right| + \frac{q_t - p_t}{s_t - r_t} \varepsilon.
\end{aligned}$$

This concludes the proof. □

4. Conclusions

In this paper, the concepts of deferred statistical convergence and deferred strong Cesàro summability are defined within the framework of $\mathscr{A}$-metric spaces, and the relationships between these concepts are analyzed in detail. The findings provide significant insight into understanding the interplay among statistical convergence, deferred statistical convergence, and deferred strong Cesàro summability in $\mathscr{A}$-metric spaces. Suppose $n = 2$ is taken in the definitions of the concepts, such as deferred statistical convergence and deferred strong Cesàro summability presented here. In that case, these concepts are reduced to the corresponding definitions in ordinary metric spaces. On the other hand, if we take $s_t = t$ and $r_t = 0$, in the definitions given in this study, they coincide with the notions of statistical convergence and strong Cesàro summability in $\mathscr{A}$-metric spaces. Therefore, compared to the results presented in studies [10,31], these findings offer a more flexible and comprehensive framework. In this context, they can have both theoretical and practical impact across various fields where the notions of convergence and metric are applied, such as mathematics, physics, engineering, and system stability. Furthermore, in future studies, the deferred statistical convergence of double sequences of order α in $\mathscr{A}$-metric spaces can be investigated, potentially contributing novel and original insights to the literature.

Acknowledgments

The author sincerely thanks the reviewers for carefully examining this paper and providing their valuable comments.

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DOI: https://dx.doi.org/10.30504/jims.2025.540372.1276