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ADJACENCY SPECTRUM AND ENERGY OF THE EXACT ZERO-DIVISOR GRAPH OF $\mathbb{Z}_n$

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Abstract. This paper investigates the exact zero-divisor graph $E\Gamma(R)$ of a commutative ring, with particular focus on $R = \mathbb{Z}_n$. We describe a partition of the vertex set of $E\Gamma(\mathbb{Z}_n)$ based on the greatest common divisors with n, which provides structural information about its components. The adjacency spectra and energies of $E\Gamma(\mathbb{Z}_{p^{2m+1}})$ and $E\Gamma(\mathbb{Z}_{p^{2m}})$ are computed explicitly, and their asymptotic behaviors are compared. The results reveal clear structural differences between the odd and even power cases.

1. Introduction

The graphical properties of algebraic structures have been extensively studied, beginning with the seminal work of I. Beck in [2]. Let R be a commutative ring with a non-zero identity. A non-zero element $x \in R$ is called a *unit* if there exists $y \in R$ such that $xy = 1$. Conversely, if there exists a non-zero element $y \in R$ such that $xy = 0$, then x is called a *zero-divisor* of R. The set of all zero-divisors of R is denoted by $Z(R)$. Beck introduced the *zero-divisor graph*, in which the vertices are the elements of $Z(R)$, and two distinct vertices x and y are adjacent if and only if $xy = 0$.

Later, Anderson and Livingston [1] refined this idea by defining a subgraph $\Gamma(R)$, whose vertex set is

$$Z(R)^* = Z(R) \setminus \{0\},$$

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the set of all non-zero zero-divisors of R. They investigated how graph-theoretical properties of $\Gamma(R)$ are connected with algebraic structures. Since then, many researchers have introduced and analyzed different kinds of graphs on algebraic structures, thereby expanding this area of study.

For a commutative ring R, the *annihilator* of an element $x \in R$, denoted by $\text{Ann}(x)$, is defined as

$$\text{Ann}(x) = \{y \in R \mid xy = 0\}.$$

The notion of an *exact zero-divisor* was introduced in [7]. Specifically, an element $x \in R$ is called an exact zero-divisor if there exists a non-zero element $y \in R$ such that $\text{Ann}(x) = yR$ and $\text{Ann}(y) = xR$. The set of all exact zero-divisors of R is denoted by $EZ(R)$, and we write

$$EZ(R)^* = EZ(R) \setminus \{0\}$$

for the set of all non-zero exact zero-divisors.

Lalchandani [9] defined a simple graph, called the *exact zero-divisor graph* and denoted by $E\Gamma(R)$, whose vertex set is $EZ(R)^*$. In this graph, two distinct vertices x and y are adjacent if and only if $\text{Ann}(x) = yR$ and $\text{Ann}(y) = xR$. This study was further extended in [10], where several examples of exact zero-divisor graphs for commutative rings were presented. Moreover, in [11], the study was carried forward to reduced rings (i.e., rings in which $x^n = 0 \implies x = 0$). If $x, y \in EZ(R)^*$ are adjacent in $E\Gamma(R)$, then clearly $xy = 0$. Hence, x and y are also adjacent in $\Gamma(R)$. Therefore, whenever $EZ(R)^* \neq \emptyset$, the graph $E\Gamma(R)$ forms a subgraph of $\Gamma(R)$.

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. For two vertices $u, v \in V(G)$, we write $u \sim v$ if they are adjacent in G. The *order* of G is the number of vertices, denoted $|V(G)|$, and the *size* of G is the number of edges, denoted $|E(G)|$. For a vertex $v \in V(G)$, the *degree* of v, written $d_G(v)$ (or simply $d(v)$), is the number of edges incident to it. A graph is *regular* if all vertices have the same degree, and it is r*-regular* if each vertex has degree r.

A *complete graph* is one in which every vertex is adjacent to every other vertex. The complete graph on n vertices is denoted K_n. Its complement, denoted $\overline{K}_n$, is the null graph of order n. In particular, K_n is $(n-1)$-regular, while $\overline{K}_n$ is 0-regular.

A *bipartite graph* G is a graph whose vertex set can be partitioned into two non-empty disjoint subsets A and B such that every edge joins a vertex of A with a vertex of B. A bipartite graph is said to be complete if every vertex of A is adjacent to every vertex of B. If $|A| = m$ and $|B| = n$, the complete bipartite graph is denoted by $K_{m,n}$.

The adjacency matrix of a graph G is the real symmetric matrix $A(G) = [a_{ij}]$ of order n, where $a_{ij} = 1$ if $v_i \sim v_j$ and $a_{ij} = 0$ otherwise. The eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of $A(G)$ are also called the eigenvalues of G. The set of eigenvalues of G, along with their multiplicities, is called the *spectrum* of G. If the distinct eigenvalues are $\lambda_1, \lambda_2, \ldots, \lambda_k$ with multiplicities $m_1, m_2, \ldots, m_k$, then

$$Spec(G) = \begin{Bmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_k \\ m_1 & m_2 & \cdots & m_k \end{Bmatrix}.$$

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The *energy* of G, denoted $E(G)$, is defined as the sum of the absolute values of the eigenvalues of $A(G)$:

$$E(G) = \sum_{i=1}^{k} m_i \, |\lambda_i|.$$

Further details on adjacency spectra can be found in [3].

The ring of integers modulo n is denoted $\mathbb{Z}_n$. Lalchandani [10, Theorem 3.2] proved that for $n \geq 3$, the exact zero-divisor graph $E\Gamma(\mathbb{Z}_n)$ is a disjoint union of $\lfloor n/2 \rfloor$ complete bipartite graphs. Later, in [11, Lemma 2.12 and Lemma 2.14], necessary and sufficient conditions were given for a component g of $E\Gamma(R)$ to be either complete or complete bipartite, depending on whether $\mathrm{Ann}(x) = xR$ for $x \in V(g)$. Furthermore, [11, Theorem 2.15] established that for any ring R (possibly with non-zero nilpotent elements) such that $EZ(R)^* \neq \emptyset$, every component of $E\Gamma(R)$ is either a complete graph or a complete bipartite graph.

Recall that an integer d is called a proper divisor of n, if $1 < d < n$ and $d \mid n$. The greatest common divisor of two integers s and t is denoted by (s, t). Motivated by the above results, in Section 2 we investigate the structure of $E\Gamma(\mathbb{Z}_n)$. We show that the sets $A_{d_1}, A_{d_2}, \ldots, A_{d_k}$ form a partition of the vertex set $V(E\Gamma(\mathbb{Z}_n))$, where $d_1, d_2, \ldots, d_k$ are the proper divisors of n, and

$$A_{d_i} = \{x \in \mathbb{Z}_n \mid (x, n) = d_i\}, \quad 1 \leq i \leq k.$$

In addition, we establish several structural results concerning $E\Gamma(\mathbb{Z}_n)$.

In Section 3, we turn to the adjacency spectrum and the energy of $E\Gamma(\mathbb{Z}_{p^{2m+1}})$ and $E\Gamma(\mathbb{Z}_{p^{2m}})$. By applying decomposition results from Section 2 together with Theorem 3.1, we explicitly determine their spectra and compute their energies. Finally, we compare how these energies grow with respect to both the prime p and the parameter m, highlighting the differences between odd and even powers. This comparison reveals asymptotic properties and structural distinctions between the graphs $E\Gamma(\mathbb{Z}_{p^{2m+1}})$ and $E\Gamma(\mathbb{Z}_{p^{2m}})$.

2. Structure of $E\Gamma(\mathbb{Z}_n)$

As noted in [4, 6], if G_1 and G_2 are any graphs with disjoint vertex sets, then the join $G_1 \vee G_2$ of G_1 and G_2 is the graph obtained from the union of G_1 and G_2 by adding new edges from each vertex of G_1 to every vertex of G_2. The following generalizes the definition of a join graph.

Definition 2.1. *Let $H(V; E)$ be a graph of order k having vertex set $V(H) = \{v_1, v_2, \ldots, v_k\}$, and let $G_i(V_i; E_i)$ be disjoint graphs of order $k_i, 1 \leq i \leq k$. The H-generalized join graph $H[G_1, G_2, \ldots, G_k]$ is the graph formed by replacing each vertex v_i of H by the graph G_i. Then, for each edge $v_i \sim v_j$ in H, connect each vertex of G_i to every vertex of G_j.*

Let $d_1, d_2, \ldots, d_k$ be the distinct proper divisors of n. Let $n = {p_1}^{n_1} {p_2}^{n_2} \cdots {p_r}^{n_r}$ be the prime power factorization of n, where $r, n_1, n_2, \ldots, n_r$ are positive integers and $p_1, p_2, \ldots, p_r$ are distinct prime numbers. Every divisor of n is of the form ${p_1}^{\alpha_1} {p_2}^{\alpha_2} \cdots {p_r}^{\alpha_r}$ for some integers $\alpha_1, \alpha_2, \ldots, \alpha_r$, where

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$0 \leq \alpha_i \leq n_i$ for each $i \in \{1, 2, \ldots, r\}$. So the total number of divisors of n is $\prod_{i=1}^{r}(n_i + 1)$. Since 1 and n are not proper divisor of n, the number k of proper divisors of n is given by:

$$k = \prod_{i=1}^{r}(n_i + 1) - 2. \tag{2.1}$$

Recall from [5, Page No. 270] that for $1 \leq i \leq k$, the sets $A_{d_i} = \{x \in \mathbb{Z}_n : (x, n) = d_i\}$ are pairwise disjoint and from [5, Lemma 2.3, p. 270], $|A_{d_i}| = \varphi\left(\frac{n}{d_i}\right)$, respectively. Furthermore, $A_{d_1} \cup A_{d_2} \cup \ldots \cup A_{d_k}$ is the partition of the vertex set $V(\Gamma(\mathbb{Z}_n))$ of $\Gamma(\mathbb{Z}_n)$.

Definition 2.2 ([8, Defination 2.3]). *If every zero-divisor of R is an exact zero-divisor, then R is an exact zero-divisor ring.*

From [8, Example 2.11], for all $n \geq 2$, $\mathbb{Z}_n$ is an exact zero-divisor ring. Therefore we can conclude that the sets A_{d_i}, for $1 \leq i \leq k$ form a partition of the vertex set $V(E\Gamma(\mathbb{Z}_n))$ of $E\Gamma(\mathbb{Z}_n)$ as follows:

$$V(E\Gamma(\mathbb{Z}_n)) = A_{d_1} \cup A_{d_2} \cup \ldots \cup A_{d_k} \tag{2.2}$$

and

$$|A_{d_i}| = \varphi\left(\frac{n}{d_i}\right) \tag{2.3}$$

Also, we know that any non-zero element a of $\mathbb{Z}_n$ is either a unit or a zero-divisor according as $(a, n) = 1$ or not. The number of vertices in $E\Gamma(\mathbb{Z}_n)$ is $n - \varphi(n) - 1$, where φ is the Euler's totient function.

Lemma 2.3. *Let $x \in A_{d_i}$ and $y \in A_{d_j}$. Then $x \sim y$ if and only if $n = d_i d_j$.*

Proof. Assume first that $n = d_i d_j$. Since $x \in A_{d_i}$, we may write $x = m d_i$ where $\gcd(m, n/d_i) = 1$, and similarly $y = k d_j$ with $\gcd(k, n/d_j) = 1$. The condition $xz \equiv 0 \pmod{n}$ is equivalent to $mz \equiv 0 \pmod{d_j}$, which forces $z \equiv 0 \pmod{d_j}$ because $\gcd(m, d_j) = 1$. Thus $\mathrm{Ann}(x) = d_j\mathbb{Z}_n$, and by symmetry
$Ann(y) = d_i\mathbb{Z}_n$. Moreover, $y = kd_j$ with $\gcd(k, d_i) = 1$ implies $y\mathbb{Z}_n = d_j\mathbb{Z}_n = \mathrm{Ann}(x)$, and similarly $x\mathbb{Z}_n = d_i\mathbb{Z}_n = \mathrm{Ann}(y)$. Hence $\mathrm{Ann}(x) = y\mathbb{Z}_n$ and $\mathrm{Ann}(y) = x\mathbb{Z}_n$, showing that $x \sim y$.

Conversely, suppose that $x \sim y$. Then $\mathrm{Ann}(x) = y\mathbb{Z}_n$ and $\mathrm{Ann}(y) = x\mathbb{Z}_n$. Since $x \in A_{d_i}$ and $y \in A_{d_j}$, we also have $\mathrm{Ann}(x) = (n/d_i)\mathbb{Z}_n$ and $\mathrm{Ann}(y) = (n/d_j)\mathbb{Z}_n$. Comparing sizes, $|y\mathbb{Z}_n| = n/d_j$ while $|(n/d_i)\mathbb{Z}_n| = d_i$, and equality gives $n = d_i d_j$. A similar argument from $x\mathbb{Z}_n = (n/d_j)\mathbb{Z}_n$ leads to the same conclusion. Thus $n = d_i d_j$ is both necessary and sufficient for $x \sim y$. □

The following corollary states that the induced subgraph $E\Gamma(A_{d_i})$ of $E\Gamma(\mathbb{Z}_n)$ on the vertex set A_{d_i} is complete graph or its complement graph.

Corollary 2.4. *The induced subgraph $E\Gamma(A_{d_i})$ of $E\Gamma(\mathbb{Z}_n)$ on the vertex set A_{d_i} is given by*

$$E\Gamma(A_{d_i}) = \begin{cases} K_{\varphi\left(\frac{n}{d_i}\right)}, & \text{if } n = d_i^2, \\ \overline{K}_{\varphi\left(\frac{n}{d_i}\right)}, & \text{otherwise}, \end{cases}$$

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Proof. By (2.3), the number of vertices in $E\Gamma(A_{d_i})$ is $\varphi\left(\frac{n}{d_i}\right)$. The adjacency among these vertices is characterized by Lemma 2.3: when $n = d_i^2$ they form a complete graph, while otherwise they are non-adjacent. □

Recall [6, Defination 3.9.1], an *equitable partition* of vertex set of a graph G, is the partition $V(G) = V_1 \cup V_2 \cup \ldots \cup V_k$ in which every vertex in V_i has the same number of neighbors in V_j, for all $i, j \in \{1, 2, \ldots, k\}$. The next corollary confirms that the partition given by (2.2) is an equitable partition.

Corollary 2.5. *For $i, j \in \{1, 2, \ldots, k\}$ with $i \neq j$, each vertex of A_{d_i} is adjacent to either all or none of the vertices of A_{d_j} in $E\Gamma(\mathbb{Z}_n)$.*

Proof. The claim follows directly from Lemma 2.3 together with Corollary 2.4. □

Now, we define a simple graph Φ_n whose vertices are the proper divisors $d_1, d_2, \ldots, d_k$ of n, where two distinct vertices d_i and d_j are adjacent if and only if $n = d_i d_j$. Therefore, by (2.1),

$$|V(\Phi_n)| = \prod_{i=1}^{r}(n_i + 1) - 2.$$

The graph Φ_n will play a crucial role in the remaining part of the study.

Lemma 2.6. *Let $d_1, d_2, \ldots, d_k$ be the proper divisors of n. Then $E\Gamma(\mathbb{Z}_n)$ can be expressed as a Φ_n-generalized join of the induced subgraphs $E\Gamma(A_{d_1}), E\Gamma(A_{d_2}), \ldots, E\Gamma(A_{d_k})$, namely*

$$E\Gamma(\mathbb{Z}_n) \;=\; \Phi_n\big[E\Gamma(A_{d_1}), E\Gamma(A_{d_2}), \ldots, E\Gamma(A_{d_k})\big].$$

Proof. This follows directly from Lemma 2.3 and Corollary 2.4, which describe the adjacency relations within and between the sets A_{d_i}. □

Now, let p be a prime and $N > 1$ be an integer. The proper divisors of p^N are $\{p, p^2, \ldots, p^{N-1}\}$. From Lemma 2.3, we observe that for $i, j \in \{1, 2, \ldots, N-1\}$, if $x \in A_{p^i}$ and $y \in A_{p^j}$, then adjacency holds if and only if $i + j = N$. Furthermore, by Corollary 2.4, the induced subgraph $E\Gamma(A_{p^i})$ is the complete graph $K_{\varphi(p^{N-i})}$ whenever $2i = N$, and it is the complement of a complete graph $\overline{K}_{\varphi(p^{N-i})}$ otherwise.

Since K_n and $\overline{K}_n$ are $(n-1)$-regular and 0-regular, respectively, we conclude that $E\Gamma(A_{p^i})$ is an r_i-regular graph of order n_i, where

$$n_i = \varphi(p^{N-i}), \quad r_i = \begin{cases} \varphi(p^{N-i}) - 1, & \text{if } 2i = N, \\ 0, & \text{otherwise.} \end{cases} \tag{2.4}$$

Corollary 2.7. *Let p be a prime and $m \geq 1$ be an integer. Then the exact zero-divisor graph of $\mathbb{Z}_{p^{2m+1}}$ is given by*

$$E\Gamma(\mathbb{Z}_{p^{2m+1}}) \;=\; \Phi_{p^{2m+1}}\big[\,\overline{K}_{\varphi(p^{2m})},\, \overline{K}_{\varphi(p^{2m-1})},\, \ldots,\, \overline{K}_{\varphi(p^{m+1})},\, \overline{K}_{\varphi(p^{m})},\, \ldots,\, \overline{K}_{\varphi(p^{2})},\, \overline{K}_{\varphi(p)}\,\big].$$

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Proof. The proper divisors of p^{2m+1} are $\{p, p^2, \ldots, p^m, p^{m+1}, \ldots, p^{2m-1}, p^{2m}\}$.

Applying Lemma 2.3, Corollary 2.4, and Lemma 2.6, it follows that

$$E\Gamma(\mathbb{Z}_{p^{2m+1}}) = \Phi_{p^{2m+1}}\big[\overline{K}_{\varphi(p^{2m})}, \overline{K}_{\varphi(p^{2m-1})}, \ldots, \overline{K}_{\varphi(p^{m+1})}, \overline{K}_{\varphi(p^m)}, \ldots, \overline{K}_{\varphi(p^2)}, \overline{K}_{\varphi(p)}\big],$$

as required. □

Corollary 2.8. *Let p be a prime and $m \geq 2$ be an integer. Then the exact zero-divisor graph of $\mathbb{Z}_{p^{2m}}$ is given by*

$$E\Gamma(\mathbb{Z}_{p^{2m}}) = \Phi_{p^{2m}}\big[\overline{K}_{\varphi(p^{2m-1})}, \ldots, \overline{K}_{\varphi(p^{m+1})}, K_{\varphi(p^m)}, \overline{K}_{\varphi(p^{m-1})}, \ldots, \overline{K}_{\varphi(p)}\big].$$

Proof. The proper divisors of p^{2m} are $\{p, p^2, \ldots, p^{m-1}, p^m, p^{m+1}, \ldots, p^{2m-1}\}$.

By Lemma 2.3, Corollary 2.4, and Lemma 2.6, we obtain

$$E\Gamma(\mathbb{Z}_{p^{2m}}) = \Phi_{p^{2m}}\big[\overline{K}_{\varphi(p^{2m-1})}, \ldots, \overline{K}_{\varphi(p^{m+1})}, K_{\varphi(p^m)}, \overline{K}_{\varphi(p^{m-1})}, \ldots, \overline{K}_{\varphi(p)}\big].$$

□

Thus, by Corollaries 2.7 and 2.8, we conclude that $E\Gamma(\mathbb{Z}_{p^{2m+1}})$ and $E\Gamma(\mathbb{Z}_{p^{2m}})$ can be expressed as the $\Phi_{p^{2m+1}}$-generalized join and $\Phi_{p^{2m}}$-generalized join, respectively, of r_i-regular graphs of order n_i, where r_i and n_i are given by Equation (2.4).

3. Adjacency spectrum and energy of $E\Gamma(\mathbb{Z}_n)$

The following theorem was proved by Cardoso et al. [4, Theorem 5], in which the adjacency spectrum of a generalized join graph $H[G_1, G_2, \ldots, G_k]$ is expressed in terms of the adjacency spectrum of the graphs G_i and the spectrum of $k \times k$ matrix $C_A(H)$.

Theorem 3.1 ([4]). *Let H be a graph with $V(h) = \{1, 2, ..., k\}$, and let G_i, $1 \leq i \leq k$, be k pairwise disjoint r_i-regular graphs of order n_i, respectively. Then the adjacency spectrum of $G = H[G_1, G_2, ..., G_k]$ is given by*

$$Spec(G) = \left(\bigcup_{i=1}^{k} (Spec(G_i) \setminus \{r_i\})\right) \cup Spec(C_A(H)),$$

where

$$C_A(H) = [c_{ij}]_{k\times k} = \begin{cases} r_i, & \text{if } i = j \\ \sqrt{n_i n_j}, & \text{if } ij \in E(H) \\ 0, & \text{otherwise} \end{cases}$$

Since the complete graph K_n has eigenvalues $n-1$ and -1 with multiplicities 1 and $n-1$, respectively, while the complement graph $\overline{K}_n$ has 0 as its eigenvalue with multiplicity n, we apply this fact together with Theorem 3.1, Corollary 2.7, and Corollary 2.8. As a result, we obtain the adjacency spectrum and the energy of $E\Gamma(\mathbb{Z}_{p^{2m+1}})$ and $E\Gamma(\mathbb{Z}_{p^{2m}})$, respectively. Moreover, by Theorem 3.1 in [10], $E\Gamma(\mathbb{Z}_{p^2})$ is a complete graph; hence, to avoid this trivial case we assume $m \geq 2$ in the even power case $E\Gamma(\mathbb{Z}_{p^{2m}})$, while in the odd power case $E\Gamma(\mathbb{Z}_{p^{2m+1}})$ we allow $m \geq 1$.

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Theorem 3.2. *Let p be a prime and $m \geq 1$ be an integer. Then the adjacency spectrum of the exact zero-divisor graph of $\mathbb{Z}_{p^{2m+1}}$ is*

$$Spec(E\Gamma(\mathbb{Z}_{p^{2m+1}})) = \begin{Bmatrix} -\sqrt{p^{2m-1}}(p-1) & 0 & \sqrt{p^{2m-1}}(p-1) \\ m & p^{2m}-2m-1 & m \end{Bmatrix}.$$

Proof. Applying Theorem 3.1 to the generalized join representation given in Corollary 2.7, we first observe that 0 is an eigenvalue of $E\Gamma(\mathbb{Z}_{p^{2m+1}})$ with multiplicity

$$\sum_{i=1}^{2m}\left(\varphi(p^{2m+1-i})-1\right) = \sum_{i=1}^{2m} p^{2m+1-i} - \sum_{i=1}^{2m} p^{2m-i} - \sum_{i=1}^{2m} 1 = p^{2m}-2m-1.$$

Thus,

$$Spec(E\Gamma(\mathbb{Z}_{p^{2m+1}})) = \begin{Bmatrix} 0 \\ p^{2m}-2m-1 \end{Bmatrix} \cup Spec(C_A(\Phi_{p^{2m+1}})),$$

where $C_A(\Phi_{p^{2m+1}})$ is the matrix given by

$$\begin{array}{c} \\ p \\ p^2 \\ \vdots \\ p^{m-1} \\ p^m \\ p^{m+1} \\ p^{m+2} \\ \vdots \\ p^{2m-1} \\ p^{2m} \end{array}
\begin{array}{c} \begin{array}{ccccccccc} p & p^2 & \cdots & p^{m-1} & p^m & p^{m+1} & \cdots & p^{2m-1} & p^{2m} \end{array} \\
\left[\begin{array}{ccccccccc}
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \beta_{1,2m} \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \beta_{2,2m-1} & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & \beta_{m-1,m+2} & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & \beta_{m,m+1} & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & \beta_{m+1,m} & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & \beta_{m+2,m-1} & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \beta_{2m-1,2} & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
\beta_{2m,1} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0
\end{array}\right] \end{array}$$

Here, for $1 \leq i,j \leq 2m$, $\beta_{i,j} = \beta_{j,i} = \sqrt{n_i n_j}$. By Equation (2.4), this simplifies to

$$\beta_{i,j} = \sqrt{\varphi(p^{2m+1-i}) \cdot \varphi(p^{2m+1-j})} = \sqrt{p^{2m-1}}(p-1).$$

Therefore, the matrix $C_A(\Phi_{p^{2m+1}})$ has exactly two distinct eigenvalues,

$$\sqrt{p^{2m-1}}(p-1) \quad \text{and} \quad -\sqrt{p^{2m-1}}(p-1),$$

each with multiplicity m.

Combining these with the zero eigenvalues gives

$$Spec(E\Gamma(\mathbb{Z}_{p^{2m+1}})) = \begin{Bmatrix} -\sqrt{p^{2m-1}}(p-1) & 0 & \sqrt{p^{2m-1}}(p-1) \\ m & p^{2m}-2m-1 & m \end{Bmatrix}.$$

□

Corollary 3.3. *Let p be a prime and $m \geq 1$ be an integer. Then the energy of $E\Gamma(\mathbb{Z}_{p^{2m+1}})$ is*

$$E\big(E\Gamma(\mathbb{Z}_{p^{2m+1}})\big) = 2m\sqrt{p^{2m-1}}(p-1).$$

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Proof. From Theorem 3.2, the nonzero eigenvalues of $E\Gamma(\mathbb{Z}_{p^{2m+1}})$ are $\pm\sqrt{p^{2m-1}}(p-1)$, each occurring with multiplicity m. Hence,

$$E\big(E\Gamma(\mathbb{Z}_{p^{2m+1}})\big) = m\left|-\sqrt{p^{2m-1}}(p-1)\right| + m\left|\sqrt{p^{2m-1}}(p-1)\right| = 2m\sqrt{p^{2m-1}}(p-1).$$

□

Theorem 3.4. *Let p be a prime and $m \geq 2$ be an integer. Then the spectrum of the exact zero-divisor graph $E\Gamma(\mathbb{Z}_{p^{2m}})$ is given by*

$$\begin{Bmatrix} -p^{m-1}(p-1) & -1 & 0 & p^{m-1}(p-1)-1 & p^{m-1}(p-1) \\ m-1 & p^m - p^{m-1} - 1 & p^{2m-1} - p^m + p^{m-1} - 2m + 1 & 1 & m-1 \end{Bmatrix}.$$

Proof. Applying Theorem 3.1 to the join graph described in Corollary 2.8, we observe that 0 is an eigenvalue of $E\Gamma(\mathbb{Z}_{p^{2m}})$ with multiplicity

$$\sum_{i=1}^{m-1}\big(\varphi(p^{2m-i})-1\big) + \sum_{i=m+1}^{2m-1}\big(\varphi(p^{2m-i})-1\big) = p^{2m-1} - p^m + p^{m-1} - 2m + 1,$$

and -1 is an eigenvalue with multiplicity $p^m - p^{m-1} - 1$.

The remaining eigenvalues of $E\Gamma(\mathbb{Z}_{p^{2m}})$ correspond to those of the following matrix $C_A(\Phi_{p^{2m}})$:

$$\begin{array}{c} \\ p \\ p^2 \\ \\ p^{m-1} \\ p^m \\ p^{m+1} \\ \\ p^{2m-2} \\ p^{2m-1} \end{array}
\begin{array}{c} \begin{array}{ccccccccc} p & p^2 & & p^{m-1} & p^m & p^{m+1} & & p^{2m-2} & p^{2m-1} \end{array} \\
\begin{bmatrix}
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \beta_{1,2m-1} \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \beta_{2,2m-2} & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & \beta_{m-1,m+1} & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & r_m & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & \beta_{m+1,m-1} & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \beta_{2m-2,2} & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
\beta_{2m-1,1} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0
\end{bmatrix} \end{array}$$

where $r_m = p^{m-1}(p-1) - 1$ (by (2.4)), and for $1 \leq i, j \leq 2m-1$ with $i \neq j \neq m$, we have

$$\beta_{i,j} = \beta_{j,i} = \sqrt{n_i n_j} = \sqrt{\varphi(p^{2m-i})\varphi(p^{2m-j})} = p^{m-1}(p-1).$$

Thus, $C_A(\Phi_{p^{2m}})$ has eigenvalues $p^{m-1}(p-1)-1$, $p^{m-1}(p-1)$ and $-p^{m-1}(p-1)$, with multiplicities 1, $m-1$, and $m-1$, respectively.

Hence, $Spec(E\Gamma(\mathbb{Z}_{p^{2m}})) =$

$$\begin{Bmatrix} -p^{m-1}(p-1) & -1 & 0 & p^{m-1}(p-1)-1 & p^{m-1}(p-1) \\ m-1 & p^m - p^{m-1} - 1 & p^{2m-1} - p^m + p^{m-1} - 2m + 1 & 1 & m-1 \end{Bmatrix}.$$

□

Corollary 3.5. *Let p be a prime and $m \geq 2$ be an integer. Then the energy of $E\Gamma(\mathbb{Z}_{p^{2m}})$ is*

$$E\big(E\Gamma(\mathbb{Z}_{p^{2m}})\big) = 2mp^{m-1}(p-1) - 2.$$

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Proof. From Theorem 3.4, the nonzero eigenvalues of $E\Gamma(\mathbb{Z}_{p^{2m}})$ are $-p^{m-1}(p-1)$ with multiplicity $m-1$, -1 with multiplicity $p^m - p^{m-1} - 1$, $p^{m-1}(p-1)-1$ with multiplicity 1, and $p^{m-1}(p-1)$ with multiplicity $m-1$. Therefore,

$$\begin{aligned} E\big(E\Gamma(\mathbb{Z}_{p^{2m}})\big) &= (m-1)\,\left|-p^{m-1}(p-1)\right| + (p^m - p^{m-1} - 1)\,|-1| \\ &\quad + \left|p^{m-1}(p-1)-1\right| + (m-1)\,\left|p^{m-1}(p-1)\right| \\ &= 2(m-1)p^{m-1}(p-1) + 2\left(p^{m-1}(p-1)-1\right) \\ &= 2mp^{m-1}(p-1) - 2. \end{aligned}$$

□

Here we briefly recall the standard asymptotic notation. For two functions $f(m)$ and $g(m)$, we write $f(m) = \Theta(g(m))$ if there exist positive constants c_1, c_2 and M such that

$$c_1 g(m) \le f(m) \le c_2 g(m) \qquad \text{for all } m \ge M.$$

Remark 3.6. *Asymptotically, as $m \to \infty$, the energy of the extended zero-divisor graphs satisfies*

$$E\big(E\Gamma(\mathbb{Z}_{p^{2m+1}})\big) = \Theta\big(m\,p^{\,m-\frac{1}{2}}\big) \quad \text{and} \quad E\big(E\Gamma(\mathbb{Z}_{p^{2m}})\big) = \Theta\big(m\,p^{\,m-1}\big).$$

Thus, for a fixed prime p, both energies increase without bound as m grows, while for fixed m, both energies increase with p.
Moreover, by Corollaries 3.3 and 3.5, we have the explicit formulas

$$E\big(E\Gamma(\mathbb{Z}_{p^{2m+1}})\big) = 2m\,p^{\,m-\frac{1}{2}}(p-1), \qquad E\big(E\Gamma(\mathbb{Z}_{p^{2m}})\big) = 2m\,p^{\,m-1}(p-1) - 2.$$

Factoring $2m\,p^{\,m-1}(p-1)$ gives

$$\frac{E\big(E\Gamma(\mathbb{Z}_{p^{2m+1}})\big)}{2m\,p^{\,m-1}(p-1)} = \sqrt{p}, \qquad \frac{E\big(E\Gamma(\mathbb{Z}_{p^{2m}})\big)}{2m\,p^{\,m-1}(p-1)} = 1 - \frac{1}{m\,p^{\,m-1}(p-1)}.$$

Hence, for all primes $p \ge 2$ and integers $m \ge 2$, it follows that

$$E\big(E\Gamma(\mathbb{Z}_{p^{2m+1}})\big) \;>\; E\big(E\Gamma(\mathbb{Z}_{p^{2m}})\big)\,.$$

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