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# A STUDY OF 3-DERIVATIONS AND THEIR ALGEBRAIC IMPLICATIONS

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Abstract. This paper explores the concept of 3-derivations in the context of algebras. Building on prior work that established the invariance of primitive ideals, prime ideals, and minimal prime ideals under derivations, we extend these results to the case of 3-derivations. In particular, we show that several properties of primitive and prime ideals, previously proven for derivations, also hold in the setting of 3-derivations. Furthermore, we examine 3-derivations on triangular Banach algebras and show that a linear map $\mathcal{D} : \mathcal{T} \to \mathcal{T}$ qualifies as a 3-derivation if it satisfies certain structural conditions.

## 1. **Introduction**

A derivation on an algebra $\mathcal{A}$ is a linear map $\mathcal{D} : \mathcal{A} \to \mathcal{A}$ satisfying $\mathcal{D}(ab) = a\mathcal{D}(b) + \mathcal{D}(a)b$, for $a, b \in \mathcal{A}$. In [6], a concept of $n$-derivation and its relationship with derivation for each $n \geq 2$ were investigated. This paper provides a brief exploration of the concept of $n$-derivation for each $n \geq 3$, for primitive ideals, prime ideals, and triangular of Banach algebras.

In [3], it was shown that primitive ideals remain invariant under a bounded derivation. Similarly, in [1,4], it was demonstrated that continuous derivations on Banach algebras preserve primitive ideals. Studies in [4,7] investigated the invariance of prime and minimal prime ideals under a derivation. We extend these results to 3-derivations.

In [2], derivations on a triangular Banach algebras $\mathcal{T} = \begin{pmatrix} \mathcal{A} & \mathcal{M} \\ 0 & \mathcal{B} \end{pmatrix}$ was explored for the case where $\mathcal{A}$ is a unital Banach algebra, $\mathcal{B}$ has a bounded approximate identity, and $\mathcal{M}$ is an essential $\mathcal{A}, \mathcal{B}$-module. Here, we characterize a 3-derivations $\mathcal{D} : \mathcal{T} \to \mathcal{T}$.

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## 2. Notations and preliminaries

A $n$-derivation on an algebra $\mathcal{A}$ is a linear map $\mathcal{D}:\mathcal{A}\to\mathcal{A}$, satisfying

$$\begin{aligned}\mathcal{D}(a_1a_2a_3\dots a_n) &= \mathcal{D}(a_1)a_2a_3\dots a_n + a_1\mathcal{D}(a_2)a_3\dots a_n+\dots+\\ &\quad + a_1a_2a_3\dots\mathcal{D}(a_n),\quad (a_1,a_2,a_3,\dots a_n\in\mathcal{A}).\end{aligned}$$

**Definition 2.1.** *Let $\mathcal{A}$ be an algebra. The quotient of a left ideal $\mathcal{I}$ in an algebra $\mathcal{A}$ is*

$$\mathcal{I}:\mathcal{A}=\{a\in\mathcal{A}\,:\,a\mathcal{A}\subset\mathcal{I}\}.$$

*A left ideal $\mathcal{I}$ in $\mathcal{A}$ is modular if there exists $u\in\mathcal{A}$ with $a-au\in\mathcal{I}$, $(a\in\mathcal{A})$. A maximal modular left ideal in $\mathcal{A}$ is a maximal element in the family of modular left ideals in $\mathcal{A}$. The quotient of a maximal modular left ideal is called a primitive ideal. $\mathcal{A}$ is called primitive if $0$ is a primitive ideal. Intersection of the primitive ideals of $\mathcal{A}$ is called radical (jacobson) $\mathcal{A}$, and denote by $\Re(\mathcal{A})$, $\mathcal{A}$ is called semisimple if $\Re(\mathcal{A})=0$.*

**Definition 2.2.** *An ideal $\mathcal{P}$ of a algebra $\mathcal{A}$ is called prime if for ideals $\mathcal{I}$ and $\mathcal{J}$ in $\mathcal{A}$ that $\mathcal{I}\mathcal{J}\subset\mathcal{P}$, then either $\mathcal{I}\subset\mathcal{P}$ or $\mathcal{J}\subset\mathcal{P}$. An ideal of algebra $\mathcal{A}$ is called minimal prime ideal if it is minimal in the family of prime ideals in $\mathcal{A}$. Intersection of the minimal prime ideals of $\mathcal{A}$ is denoted by $\mathfrak{P}(\mathcal{A})$. An algebra $\mathcal{A}$ is called prime algebra if $0$ is a prime ideal in $\mathcal{A}$.*

**Proposition 2.3** ([4],1.3.42)**.** *Let $\mathcal{P}$ be an ideal in an algebra $\mathcal{A}$. Then the following conditions on $\mathcal{P}$ are equivalent*

1. *$\mathcal{P}$ is a prime ideal;*
2. *whenever $a\mathcal{A}b\subset\mathcal{P}$, then either $a\in\mathcal{P}$ or $b\in\mathcal{P}$;*
3. *whenever $\mathcal{I}$ and $\mathcal{J}$ are both left ideals in $\mathcal{A}$ that $\mathcal{I}\mathcal{J}\subset\mathcal{P}$, then either $\mathcal{I}\subset\mathcal{P}$ or $\mathcal{J}\subset\mathcal{P}$.*

**Definition 2.4.** *Let $\mathcal{A}$ be an algebra. An element $a\in\mathcal{A}$ is called idempotent if $a^2=a$, and is called nilpotent if $a^n=0$ for a $n\in\mathbb{N}$. The set of nilpotent elements of $\mathcal{A}$ is denoted by $\mathfrak{N}(\mathcal{A})$. In the commutative algebra $\mathcal{A}$, the nilradical is an ideal consisting of nilpotent elements.*

**Definition 2.5.** *Let $\mathcal{A}$ be an unital Banach algebra.*
*If $a\in\mathcal{A}$, the spectrum of $a$ is*

$$\sigma(a)=\{\lambda\in\mathbb{C}\,:\,\lambda e-a\ \textit{is not invertible}\}.$$

*Note that in the unital Banach algebra $\mathcal{A}$, $\sigma(a)$ is a nonempty and compact subset of $\mathbb{C}$, for each $a\in\mathcal{A}$.*
*If $a\in\mathcal{A}$, the spectral radius of $a$ is*

$$\rho(a)=\sup\{|\lambda|\,:\,\lambda\in\sigma(a)\}.$$

*If $\sigma(a)=\emptyset$, we take $\rho(a)=0$.*
*If $\rho(a)=0$, then $a\in\mathcal{A}$ is quasi nilpotent. The set of quasi nilpotent elements of $\mathcal{A}$ is denote by $\mathcal{D}_0(\mathcal{A})$.*

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**Definition 2.6.** *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras, $\mathcal{M}$ be a essential Banach $\mathcal{A},\mathcal{B}$-module, we define triangular Banach algebra $\mathcal{T}$ as*

$$\mathcal{T}=\begin{pmatrix}\mathcal{A} & \mathcal{M}\\ 0 & \mathcal{B}\end{pmatrix}=\left\{\begin{pmatrix}a & m\\ 0 & b\end{pmatrix}\ :\ a\in\mathcal{A},\, b\in\mathcal{B},\, m\in\mathcal{M}\right\}.$$

*The linear space $\mathcal{T}$ with the usual $2\times 2$ matrix addition and formal multiplication and with the norm $\|\begin{pmatrix}a & m\\ 0 & b\end{pmatrix}\|=\|a\|_{\mathcal{A}}+\|b\|_{\mathcal{B}}+\|m\|_{\mathcal{M}}$ is a Banach algebra.*

## 3. Sinclair result for 3-derivations

In this section, we show that any primitive ideal and prime ideal are invariant under any 3-derivation. We first extend some results about derivations to 3-derivations.

**Lemma 3.1.** *Let $\mathcal{D}$ be a $3-$derivation on an algebra $\mathcal{A}$ with $\mathcal{A}^2=\mathcal{A}$, where $\mathcal{A}^2=lin\{a_1a_2\,:\,a_1,a_2\in\mathcal{A}\}$, and $n\in\mathbb{N}$.*

1. *Let $\mathcal{I}$ be an ideal in $\mathcal{A}$. Then $\mathcal{D}(\mathcal{I})/\mathcal{I}$ is an ideal in $\mathcal{A}/\mathcal{I}$;*
2. *Let $\mathcal{I}$ be an ideal in $\mathcal{A}$, and that $a_1,\dots,a_n\in\mathcal{I}$. Then $\mathcal{D}^n(a_1\dots a_n)-n!(\mathcal{D}a_1)\dots(\mathcal{D}a_n)\in\mathcal{I}$.*

*Proof.* (1):Take $a=a_1a_2\in\mathcal{A}$ and $b\in\mathcal{I}$. Then $\mathcal{D}(ba_1a_2)=\mathcal{D}(b)a_1a_2+b\mathcal{D}(a_1)a_2+ba_1\mathcal{D}(a_2)$, so $\mathcal{D}(b)a_1a_2=\mathcal{D}(ba_1a_2)-b\mathcal{D}(a_1)a_2-ba_1\mathcal{D}(a_2)\in\mathcal{D}(\mathcal{I})+\mathcal{I}$, hence $\mathcal{D}(\mathcal{I})/\mathcal{I}$ is a right ideal of $\mathcal{A}/\mathcal{I}$. Similarly it is a left ideal.
(2): We claim that, for each $k\geq 3$, we have $\mathcal{D}^j(b_1\dots b_k)\in\mathcal{I}$ whenever $1\leq j\leq k-1$ and $b_1\dots b_k\in\mathcal{I}$. The claim is true for $k=3$, if $j=1$, $\mathcal{D}(b_1b_2b_3)=\mathcal{D}(b_1)b_2b_3+b_1\mathcal{D}(b_2)b_3+b_1b_2\mathcal{D}(b_3)\in\mathcal{I}$ and if $j=2$, $\mathcal{D}^2(b_1b_2b_3)=\mathcal{D}(\mathcal{D}(b_1b_2b_3))=\mathcal{D}^2(b_1)b_2b_3+\mathcal{D}(b_1)\mathcal{D}(b_2)b_3+\mathcal{D}(b_1)b_2\mathcal{D}(b_3)+\mathcal{D}(b_1)\mathcal{D}(b_2)b_3+b_1\mathcal{D}^2(b_2)b_3+b_1\mathcal{D}(b_2)\mathcal{D}(b_3)+\mathcal{D}(b_1)b_2\mathcal{D}(b_3)+b_1\mathcal{D}(b_2)\mathcal{D}(b_3)+b_1b_2\mathcal{D}^2(b_3)\in\mathcal{I}$. By induction suppose that the claim holds for $k$ and that $b_1,\dots,b_{k+1}\in\mathcal{I}$. Then, by Proposition 2.3 of [6], for $j\leq k$,

$$\mathcal{D}^j(b_1\dots b_{k-1}b_kb_{k+1})=\sum_{k_1+k_2+k_3=j}\binom{j}{k_1,k_2,k_3}\mathcal{D}^{k_1}(b_1\dots b_{k-1})\mathcal{D}^{k_2}(b_k)\mathcal{D}^{k_3}(b_{k+1}),$$

where $\binom{j}{k_1,k_2,k_3}=\frac{j!}{k_1!k_2!k_3!}$ and $\mathcal{D}^0=I$ it is the identity operator. If $k_1=0$, the right side belong to $\mathcal{I}$. If $k_1=k-1$ we have $(k-1)+k_2+k_3\leq k$, so $k_2$ or $k_3=0$ then $\mathcal{D}^{k_2}(b_k)$ or $\mathcal{D}^{k_3}(b_{k+1})\in\mathcal{I}$. So $\mathcal{D}^j(b_1\dots b_{k+1})\in\mathcal{I}$.
We now prove the result by induction on $n$. It holds for $n=3$,
$\mathcal{D}^3(a_1a_2a_3)=\sum_{k_1+k_2+k_3=3}\frac{3!}{k_1!k_2!k_3}\mathcal{D}^{k_1}(a_1)\mathcal{D}^{k_2}(a_2)\mathcal{D}^{k_3}(a_3)$, if $k_1$, $k_2$ or $k_3=0$, then
$\mathcal{D}^{k_1}(a_1)\mathcal{D}^{k_2}(a_2)\mathcal{D}^{k_3}(a_3)\in\mathcal{I}$, otherwise $k_1=k_2=k_3=1$, and so $\mathcal{D}^3(a_1a_2a_3)-3!\mathcal{D}(a_1)\mathcal{D}(a_2)\mathcal{D}(a_3)\in\mathcal{I}$.
Assume that the result is true for $n$, and take $a_1,\dots,a_{n+1}\in\mathcal{I}$.

$$\mathcal{D}^{n+1}(a_1\dots a_{n-1}a_na_{n+1})=\sum_{\substack{k_1+k_2+k_3=\\ n+1}}\binom{n+1}{k_1,k_2,k_3}\mathcal{D}^{k_1}(a_1\dots a_{n-1})\mathcal{D}^{k_2}(a_n)\mathcal{D}^{k_3}(a_{n+1}),$$

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all terms on the right hand side of this equality belong to $\mathcal{I}$, except perhaps the case $k_1 = n-1$, $k_2 = k_3 = 1$, therefore $\mathcal{D}^{n+1}(a_1 \dots a_{n+1}) - \binom{n+1}{n-1,1,1}\mathcal{D}^{n-1}(a_1 \dots a_{n-1})\mathcal{D}(a_n)\mathcal{D}(a_{n+1}) \in \mathcal{I}$, and so

$$\begin{aligned}\mathcal{D}^{n+1}(a_1 \dots a_{n+1}) - (n+1)!(\mathcal{D}a_1)\dots(\mathcal{D}a_{n+1}) = \mathcal{D}^{n+1}&(a_1 \dots a_{n+1})\\ &- n(n+1)\mathcal{D}^{n-1}(a_1 \dots a_{n-1})(\mathcal{D}a_n)(\mathcal{D}a_{n+1})\\ &+ n(n+1)(\mathcal{D}^{n-1}(a_1 \dots a_{n-1})\\ &- (n-1)!(\mathcal{D}a_1)\dots(\mathcal{D}a_{n-1}))(\mathcal{D}a_n)(\mathcal{D}a_{n+1}),\end{aligned}$$

belongs to $\mathcal{I}$, by $\mathcal{D}^{n-1}(a_1 \dots a_{n-1}) - (n-1)!(\mathcal{D}a_1)\dots(\mathcal{D}a_{n-1}) \in \mathcal{I}$, and assumption of induction. Therefore, for each $n \geq 3$, $\mathcal{D}^n(a_1 \dots a_n) - n!(\mathcal{D}a_1)\dots(\mathcal{D}a_n) \in \mathcal{I}$. □

**Theorem 3.2.** *Let $\mathcal{D}$ be a continuous $3-$derivation on a Banach algebra $\mathcal{A}$, and $\mathcal{P}$ be a primitive ideal of $\mathcal{A}$. Then $\mathcal{D}(\mathcal{P}) \subset \mathcal{P}$.*

*Proof.* By Lemma 2.2.28($iii$) of [4], $\mathcal{P}$ is a closed ideal. As Lemma 3.1(1), $\mathcal{D}(\mathcal{P})/\mathcal{P}$ is an ideal in $\mathcal{A}/\mathcal{P}$. Let $a \in \mathcal{P}$, and set $x = \mathcal{D}(a) + \mathcal{P} \in \mathcal{A}/\mathcal{P}$. Take $n \in \mathbb{N}$, by Lemma 3.1(2), we have $n!x^n = n!(\mathcal{D}(a))^n + \mathcal{P} = \mathcal{D}^n(a^n) + \mathcal{P}$, and so

$$\begin{aligned}\|x^n\| &\leq \frac{1}{n!}\|\mathcal{D}^n(a^n) + \mathcal{P}\|\\ &\leq \frac{1}{n!}\|\mathcal{D}^n(a^n)\|\\ &\leq \frac{1}{n!}\|\mathcal{D}^n\|\|a^n\|\\ &\leq \frac{1}{n!}\|\mathcal{D}\|^n\|a\|^n,\end{aligned}$$

thus $\|x^n\|^{1/n} \leq \frac{1}{\sqrt[n]{n!}}\|\mathcal{D}\|\|a\| \to 0 \,\text{as}\, n \to \infty$, and hence $(\mathcal{D}(a) + \mathcal{P}) \subset \mathcal{D}_0(\mathcal{A}/\mathcal{P})$. By Proposition 1.5.32($iii$) of [4], since $\mathcal{A}/\mathcal{P}$ is semisimple, that is, $x \in rad(\mathcal{A}/\mathcal{P}) = 0$, so $\mathcal{D}(a) \in \mathcal{P}$. Therefore $\mathcal{D}(\mathcal{P}) \subset \mathcal{P}$. □

In [5], there is a result about the largest ideal contained in a given ideal that is invariant under a derivation $\mathcal{D}$. Here , a modified result is given for a 3-derivations.

**Lemma 3.3.** *Let $\mathcal{D}$ be a 3-derivation on an algebra $\mathcal{A}$, $\mathcal{I}$ be an ideal of $\mathcal{A}$ and $\mathcal{J} = \{x \in \mathcal{A}, n \geq 0 : \underbrace{\mathcal{D}\dots\mathcal{D}}_{n\text{-times}}x \in \mathcal{I}\}$.*

1. *$\mathcal{J}$ is the largest ideal of $\mathcal{A}$ contained in $\mathcal{I}$ which is stable under $\mathcal{D}$.*
2. *If $\mathcal{I}$ is prime, then $\mathcal{J}$ is prime.*

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*Proof.* (1):Let $a, b \in \mathcal{A}$, $x \in \mathcal{J}$. We show that for each $n \in \mathbb{N}$, $\underbrace{\mathcal{D}\ldots\mathcal{D}}_{\text{n-times}}(xab)$ and $\underbrace{\mathcal{D}\ldots\mathcal{D}}_{\text{n-times}}(abx)$ belongs to $\mathcal{I}$, by induction on $n$. Let it hold for $n-1$, we show that it holds for $n$

$$\begin{aligned}\underbrace{\mathcal{D}\ldots\mathcal{D}}_{\text{n-times}}(xab) &= \underbrace{\mathcal{D}\ldots\mathcal{D}}_{\text{n-1-times}}(\mathcal{D}(xab)) \\ &= \underbrace{\mathcal{D}\ldots\mathcal{D}}_{\text{n-1-times}}((\mathcal{D}x)ab + x(\mathcal{D}a)b + xa(\mathcal{D}b)),\end{aligned}$$

since $((\mathcal{D}x)ab + x(\mathcal{D}a)b + xa(\mathcal{D}b))$ belongs to $\mathcal{J}$. And according to the assumption of induction, the right side belongs to $\mathcal{I}$. So $\underbrace{\mathcal{D}\ldots\mathcal{D}}_{\text{n-times}}(xab) \in \mathcal{I}$. It is obvious for $n = 0$, since $x \in \mathcal{J} \subset \mathcal{I}$, so $xab \in \mathcal{I}$. Therefore, $\underbrace{\mathcal{D}\ldots\mathcal{D}}_{\text{n-times}}(xab) \in \mathcal{I}$. Similarly, show that $\underbrace{\mathcal{D}\ldots\mathcal{D}}_{\text{n-times}}(abx) \in \mathcal{I}$. Since $\mathcal{A}^2 = \mathcal{A}$, this show that $\mathcal{J}$ is ideal.

(2): Let $\mathcal{I}$ be prime. To see that $\mathcal{J}$ is prime, suppose $a, b \in \mathcal{A}$ such that $a\mathcal{A}b \subset \mathcal{J}$. Let $b \notin \mathcal{J}$, then there exist $m > 0$ such that $\mathcal{D}^m b \notin \mathcal{I}$, we show for each $n$, $\mathcal{D}^n a \in \mathcal{I}$, by induction on $n$. Let it holds for $n-1$. Let $s$ be the smallest $m$ such that $\mathcal{D}^m b \notin \mathcal{I}$.

For each $x \in \mathcal{A}$, $\mathcal{D}^{n+s}(axb) = \sum_{i+j+l=n+s} \binom{n+s}{i,j,l}(\mathcal{D}^i a)(\mathcal{D}^j x)(\mathcal{D}^l b)$. If $i < n$ or $l < s$ we have $\mathcal{D}^i(a) \in \mathcal{I}$ or $\mathcal{D}^l(b) \in \mathcal{I}$ and so $\sum_{i+j+l=n+s} \binom{n+s}{i,j,l}(\mathcal{D}^i a)(\mathcal{D}^j x)(\mathcal{D}^l b) \in \mathcal{I}$. If $i \geq n$ and $l \geq s$ then the possible value for $j$ is only 0, therefore in the sum $\sum_{i+j+l=n+s} \binom{n+s}{i,j,l}(\mathcal{D}^i a)(\mathcal{D}^j x)(\mathcal{D}^l b)$ any term belong to $\mathcal{I}$, except perhaps for the term $(\mathcal{D}^i a)x(\mathcal{D}^l b)$, on the other hand $\mathcal{D}^{n+s}(axb) \in \mathcal{I}$ therefore the term $(\mathcal{D}^n a)x(\mathcal{D}^s b)$ have to belong to $\mathcal{I}$ since $x$ is arbitrary, $(\mathcal{D}^n a)\mathcal{A}(\mathcal{D}^s b) \subset \mathcal{I}$, according to the assumption $\mathcal{I}$ is prime, then $(\mathcal{D}^n a)$ or $(\mathcal{D}^s b)$, belongs to $\mathcal{I}$. Since $b$ is not belongs to $\mathcal{J}$, so $(\mathcal{D}^s b)$ is not belongs to $\mathcal{I}$. Therefore $(\mathcal{D}^n a)$ belongs to $\mathcal{I}$. Similar to the process above, can be show that $a \in \mathcal{I}$, so $a \in J$. □

**Theorem 3.4.** *Let $\mathcal{D}$ be a 3-derivation on an algebra $\mathcal{A}$. Then $\mathcal{D}(\mathfrak{P}(\mathcal{A})) \subset \mathfrak{P}(\mathcal{A})$.*

*Proof.* Let $\mathcal{P}$ minimal prime ideal in $\mathcal{A}$, $\mathcal{Q} = \{a \in \mathcal{A}, n \geq 0 \ : \underbrace{\mathcal{D}\ldots\mathcal{D}}_{\text{n-times}}(a) \in \mathcal{P}\}$.

As above Lemma (3.3), $\mathcal{Q}$ is the largest ideal in $\mathcal{A}$, $\mathcal{D}(\mathcal{Q}) \subset \mathcal{Q} \subset \mathcal{P}$, for any each $\mathcal{P}$, $\mathfrak{P}(\mathcal{A}) \subset \mathcal{P}$ and so $\mathcal{D}(\mathfrak{P}(\mathcal{A})) \subset \mathcal{D}(\mathcal{P}) \subset \mathcal{P}$. Therefore $\mathcal{D}(\mathfrak{P}(\mathcal{A})) \subset \mathfrak{P}(\mathcal{A})$. □

**Corollary 3.5.** *Let $\mathcal{D}$ be a 3-derivation on a commutative algebra $\mathcal{A}$. Then $\mathfrak{N}(\mathcal{A}) = \mathfrak{P}(\mathcal{A})$ and $\mathcal{D}(\mathfrak{N}(\mathcal{A})) \subset \mathfrak{N}(\mathcal{A})$.*

**Corollary 3.6.** *Let $\mathcal{D}$ be a 3-derivation on a commutative algebra $\mathcal{A}$. Then $\mathfrak{R}(\mathcal{A}) = \mathfrak{P}(\mathcal{A})$ and $\mathcal{D}(\mathfrak{R}(\mathcal{A})) \subset \mathfrak{R}(\mathcal{A})$.*

## 4. 3-derivations on triangular banach algebras

In [2], the nature of derivation on triangular of Banach algebras was investigated, here we give some related result for 3-derivations.

**Theorem 4.1.** *Let $\mathcal{A}$ be a unital Banach algebra, $\mathcal{B}$ be a Banach algebra with bounded approximate identity $(e_\beta)_\beta$, and $\mathcal{M}$ be a essential Banach $\mathcal{A},\mathcal{B}$-module, and $\mathcal{D} : \mathcal{T} \to \mathcal{T}$ be a 3-derivation. Then,*

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*there exist* $m_0 \in \mathcal{M}$*,* 3*-derivations* $\mathcal{D}_1 : \mathcal{A} \to \mathcal{A}$*,* $\mathcal{D}_2 : \mathcal{B} \to \mathcal{B}$*, and continuous linear map* $\tau : \mathcal{M} \to \mathcal{M}$ *and a net* $(b_\beta)_\beta \subset \mathcal{B}$*, such that for each* $a \in \mathcal{A}$*,* $b \in \mathcal{B}$*,* $m \in \mathcal{M}$ *we have*

(1) $\lim_\beta(b_\beta b + b b_\beta) = 0$;

(2) $\mathcal{D}(\begin{pmatrix} 1_\mathcal{A} & 0 \\ 0 & 0 \end{pmatrix}) = \begin{pmatrix} 0 & m_0 \\ 0 & 0 \end{pmatrix}$;

(3) $\mathcal{D}(\begin{pmatrix} 0 & 0 \\ 0 & e_\beta \end{pmatrix}) = \begin{pmatrix} 0 & -m_0 e_\beta \\ 0 & b_\beta \end{pmatrix}$;

(4) $\mathcal{D}(\begin{pmatrix} a & m \\ 0 & b \end{pmatrix}) = \begin{pmatrix} \mathcal{D}_1(a) & am_0 - m_0 b + \tau(m) \\ 0 & \mathcal{D}_2(b) \end{pmatrix}$;

(5) $\tau(a_1 a_2 m) = \mathcal{D}_1(a_1) a_2 m + a_1 \mathcal{D}_1(a_2) m + a_1 a_2 \tau(m)$;

(6) $\tau(m b_1 b_2) = \tau(m) b_1 b_2 + m \mathcal{D}_2(b_1) b_2 + m b_1 \mathcal{D}_2(b_2)$;

(7) $\tau(amb) = \mathcal{D}_1(a) mb + a\tau(m) b + am\mathcal{D}_2(b)$.

*Conversely, if maps* $\mathcal{D}_1 : \mathcal{A} \to \mathcal{A}$*,* $\mathcal{D}_2 : \mathcal{B} \to \mathcal{B}$ *be* 3*-derivations, and map* $\tau : \mathcal{M} \to \mathcal{M}$ *satisfying in* (5)*,* (6) *and* (7)*, then the map* $\mathcal{D} : \mathcal{T} \to \mathcal{T}$ *with definition* $\mathcal{D}(\begin{pmatrix} a & m \\ 0 & b \end{pmatrix}) = \begin{pmatrix} \mathcal{D}_1(a) & am_0 - m_0 b + \tau(m) \\ 0 & \mathcal{D}_2(b) \end{pmatrix}$ *is a* 3*-derivation.*

*Proof.* First, let $\mathcal{D} : \mathcal{T} \to \mathcal{T}$ be a derivation. We define the derivations $\mathcal{D}_1 : \mathcal{A} \to \mathcal{A}$, $\mathcal{D}_2 : \mathcal{B} \to \mathcal{B}$ and map $\tau : \mathcal{M} \to \mathcal{M}$ as follows

$$\mathcal{D}_1(a) = P_1(\mathcal{D}(\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix})), \tag{4.1.1}$$

$$\mathcal{D}_2(b) = P_2(\mathcal{D}(\begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix})), \tag{4.1.2}$$

$$\tau(m) = P_3(\mathcal{D}(\begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix})), \tag{4.1.3}$$

where $P_1(\begin{pmatrix} a & m \\ 0 & b \end{pmatrix}) = a$, $P_2(\begin{pmatrix} a & m \\ 0 & b \end{pmatrix}) = b$, $P_3(\begin{pmatrix} a & m \\ 0 & b \end{pmatrix}) = m$.

We show that $\mathcal{D}_1 : \mathcal{A} \to \mathcal{A}$ is a 3-derivation. For each $a_1, a_2, a_3 \in \mathcal{A}$, say $\mathcal{D}(\begin{pmatrix} a_1 & 0 \\ 0 & 0 \end{pmatrix}) = \begin{pmatrix} a_1{}' & m_1{}' \\ 0 & b_1{}' \end{pmatrix}$, $\mathcal{D}(\begin{pmatrix} a_2 & 0 \\ 0 & 0 \end{pmatrix}) = \begin{pmatrix} a_2{}' & m_2{}' \\ 0 & b_2{}' \end{pmatrix}$, $\mathcal{D}(\begin{pmatrix} a_3 & 0 \\ 0 & 0 \end{pmatrix}) = \begin{pmatrix} a_3{}' & m_3{}' \\ 0 & b_3{}' \end{pmatrix}$,

$$\begin{aligned} \mathcal{D}_1(a_1 a_2 a_3) &= P_1(\mathcal{D}(\begin{pmatrix} a_1 a_2 a_3 & 0 \\ 0 & 0 \end{pmatrix})) \\ &= P_1(\mathcal{D}(\begin{pmatrix} a_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a_2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a_3 & 0 \\ 0 & 0 \end{pmatrix})) \\ &= P_1(\mathcal{D}(\begin{pmatrix} a_1 & 0 \\ 0 & 0 \end{pmatrix}) \begin{pmatrix} a_2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a_3 & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

$$
\begin{aligned}
&+\begin{pmatrix} a_1 & 0 \\ 0 & 0 \end{pmatrix} \mathcal{D}(\begin{pmatrix} a_2 & 0 \\ 0 & 0 \end{pmatrix}) \begin{pmatrix} a_3 & 0 \\ 0 & 0 \end{pmatrix} \\
&+\begin{pmatrix} a_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a_2 & 0 \\ 0 & 0 \end{pmatrix} \mathcal{D}(\begin{pmatrix} a_3 & 0 \\ 0 & 0 \end{pmatrix})) \\
&= P_1(\begin{pmatrix} {a_1}' & {m_1}' \\ 0 & {b_1}' \end{pmatrix} \begin{pmatrix} a_2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a_3 & 0 \\ 0 & 0 \end{pmatrix} \\
&+\begin{pmatrix} a_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} {a_2}' & {m_2}' \\ 0 & {b_2}' \end{pmatrix} \begin{pmatrix} a_3 & 0 \\ 0 & 0 \end{pmatrix} \\
&+\begin{pmatrix} a_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a_2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} {a_3}' & {m_3}' \\ 0 & {b_3}' \end{pmatrix}) \\
&= P_1(\begin{pmatrix} {a_1}'a_2a_3 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} a_1{a_2}'a_3 & 0 \\ 0 & 0 \end{pmatrix} \\
&+\begin{pmatrix} a_1a_2{a_3}' & a_1a_2{m_3}' \\ 0 & 0 \end{pmatrix}) \\
&= P_1(\begin{pmatrix} {a_1}'a_2a_3 + a_1{a_2}'a_3 + a_1a_2{a_3}' & a_1a_2{m_3}' \\ 0 & 0 \end{pmatrix}) \\
&= {a_1}'a_2a_3 + a_1{a_2}'a_3 + a_1a_2{a_3}',
\end{aligned}
$$

on the other hand

$$
\begin{aligned}
\mathcal{D}_1(a_1)a_2a_3 + a_1\mathcal{D}_1(a_2)a_3 + a_1a_2\mathcal{D}_1(a_3) &= P_1(\mathcal{D}(\begin{pmatrix} a_1 & 0 \\ 0 & 0 \end{pmatrix}))a_2a_3 \\
&+ a_1P_1(\mathcal{D}(\begin{pmatrix} a_2 & 0 \\ 0 & 0 \end{pmatrix}))a_3 \\
&+ a_1a_2P_1(\mathcal{D}(\begin{pmatrix} a_3 & 0 \\ 0 & 0 \end{pmatrix})) \\
&= P_1(\begin{pmatrix} {a_1}' & {m_1}' \\ 0 & {b_1}' \end{pmatrix})a_2a_3 \\
&+ a_1P_1(\begin{pmatrix} {a_2}' & {m_2}' \\ 0 & {b_2}' \end{pmatrix})a_3 \\
&+ a_1a_2P_1(\begin{pmatrix} {a_3}' & {m_3}' \\ 0 & {b_3}' \end{pmatrix}) \\
&= {a_1}'a_2a_3 + a_1{a_2}'a_3 + a_1a_2{a_3}',
\end{aligned}
$$

therefore $\mathcal{D}_1(a_1a_2a_3) = \mathcal{D}_1(a_1)a_2a_3 + a_1\mathcal{D}_1(a_2)a_3 + a_1a_2\mathcal{D}(a_3)$, i.e. $\mathcal{D}_1$ is a 3-derivation. In a similar way, it can be show that $\mathcal{D}_2$ is 3-derivation by the definition of (4.1.2).

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(2):By the above calculations with $a_1 = a_2 = a_3 = 1_{\mathcal{A}}$ we have $\mathcal{D}(\begin{pmatrix} 1_{\mathcal{A}} & 0 \\ 0 & 0 \end{pmatrix}) = \begin{pmatrix} 0 & m_0 \\ 0 & 0 \end{pmatrix}$ where $m_0 = {m_3}'$.

(3):Say $\mathcal{D}(\begin{pmatrix} 0 & 0 \\ 0 & e_\beta \end{pmatrix}) = \begin{pmatrix} a_\beta & m_\beta \\ 0 & b_\beta \end{pmatrix}$, $\mathcal{D}(\begin{pmatrix} 0 & 0 \\ 0 & e_\alpha \end{pmatrix}) = \begin{pmatrix} a_\alpha & m_\alpha \\ 0 & b_\alpha \end{pmatrix}$, we have

$$\begin{aligned}
0 = \mathcal{D}(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}) &= \mathcal{D}(\begin{pmatrix} 1_{\mathcal{A}} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & e_\beta \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & e_\alpha \end{pmatrix}) \\
&= \mathcal{D}(\begin{pmatrix} 1_{\mathcal{A}} & 0 \\ 0 & 0 \end{pmatrix})\begin{pmatrix} 0 & 0 \\ 0 & e_\beta \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & e_\alpha \end{pmatrix} \\
&+ \begin{pmatrix} 1_{\mathcal{A}} & 0 \\ 0 & 0 \end{pmatrix}\mathcal{D}(\begin{pmatrix} 0 & 0 \\ 0 & e_\beta \end{pmatrix})\begin{pmatrix} 0 & 0 \\ 0 & e_\alpha \end{pmatrix} \\
&+ \begin{pmatrix} 1_{\mathcal{A}} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & e_\beta \end{pmatrix}\mathcal{D}(\begin{pmatrix} 0 & 0 \\ 0 & e_\alpha \end{pmatrix}) \\
&= \begin{pmatrix} 0 & m_0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & e_\beta \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & e_\alpha \end{pmatrix} \\
&+ \begin{pmatrix} 1_{\mathcal{A}} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a_\beta & m_\beta \\ 0 & b_\beta \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & e_\alpha \end{pmatrix} \\
&+ \begin{pmatrix} 1_{\mathcal{A}} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & e_\beta \end{pmatrix}\begin{pmatrix} a_\alpha & m_\alpha \\ 0 & b_\alpha \end{pmatrix} \\
&= \begin{pmatrix} 0 & m_0 e_\beta e_\alpha \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & m_\beta e_\alpha \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & m_0 e_\beta e_\alpha + m_\beta e_\alpha \\ 0 & 0 \end{pmatrix},
\end{aligned}$$

so $m_0 e_\beta e_\alpha + m_\beta e_\alpha = 0$, then $\lim_\alpha m_0 e_\beta e_\alpha + m_\beta e_\alpha = 0$ thus $m_\beta = -m_0 e_\beta$, so

$$\begin{aligned}
\mathcal{D}(\begin{pmatrix} 0 & 0 \\ 0 & e_\beta e_\alpha e_\alpha \end{pmatrix}) &= \mathcal{D}(\begin{pmatrix} 0 & 0 \\ 0 & e_\beta \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & e_\alpha \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & e_\alpha \end{pmatrix}) \\
&= \mathcal{D}(\begin{pmatrix} 0 & 0 \\ 0 & e_\beta \end{pmatrix})\begin{pmatrix} 0 & 0 \\ 0 & e_\alpha \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & e_\alpha \end{pmatrix} \\
&+ \begin{pmatrix} 0 & 0 \\ 0 & e_\beta \end{pmatrix}\mathcal{D}(\begin{pmatrix} 0 & 0 \\ 0 & e_\alpha \end{pmatrix})\begin{pmatrix} 0 & 0 \\ 0 & e_\alpha \end{pmatrix} \\
&+ \begin{pmatrix} 0 & 0 \\ 0 & e_\beta \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & e_\alpha \end{pmatrix}\mathcal{D}(\begin{pmatrix} 0 & 0 \\ 0 & e_\alpha \end{pmatrix})
\end{aligned}$$

$$\begin{aligned}
&= \begin{pmatrix} a_\beta & m_\beta \\ 0 & b_\beta \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & e_\alpha \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & e_\alpha \end{pmatrix} \\
&+ \begin{pmatrix} 0 & 0 \\ 0 & e_\beta \end{pmatrix} \begin{pmatrix} a_\alpha & m_\alpha \\ 0 & b_\alpha \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & e_\alpha \end{pmatrix} \\
&+ \begin{pmatrix} 0 & 0 \\ 0 & e_\beta \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & e_\alpha \end{pmatrix} \begin{pmatrix} a_\alpha & m_\alpha \\ 0 & b_\alpha \end{pmatrix} \\
&= \begin{pmatrix} 0 & m_\beta e_\alpha e_\alpha \\ 0 & b_\beta e_\alpha e_\alpha \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & e_\beta b_\alpha e_\alpha \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & e_\beta e_\alpha b_\alpha \end{pmatrix} \\
&= \begin{pmatrix} 0 & m_\beta e_\alpha e_\alpha \\ 0 & b_\beta e_\alpha e_\alpha + e_\beta b_\alpha e_\alpha + e_\beta e_\alpha b_\alpha \end{pmatrix},
\end{aligned}$$

since $b_\beta = \lim_\alpha b_\beta e_\alpha e_\alpha + e_\beta b_\alpha e_\alpha + e_\beta e_\alpha b_\alpha = b_\beta + \lim_\alpha (e_\beta b_\alpha e_\alpha + e_\beta e_\alpha b_\alpha)$ and $\lim_\alpha (\mathcal{D}(\begin{pmatrix} 0 & 0 \\ 0 & e_\beta e_\alpha e_\alpha \end{pmatrix})) = \mathcal{D}(\begin{pmatrix} 0 & 0 \\ 0 & e_\beta \end{pmatrix}) = \begin{pmatrix} a_\beta & m_\beta \\ 0 & b_\beta \end{pmatrix}$, so $a_\beta = 0$, $b_\beta = b_\beta$. Therefore $\mathcal{D}(\begin{pmatrix} 0 & 0 \\ 0 & e_\beta \end{pmatrix}) = \begin{pmatrix} 0 & -m_0 e_\beta \\ 0 & b_\beta \end{pmatrix}$.

(4):If $b = m = 0$, for each $a \in \mathcal{A}$ say $\mathcal{D}(\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}) = \begin{pmatrix} a' & m' \\ 0 & b' \end{pmatrix}$,so

$$\begin{aligned}
\mathcal{D}(\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}) &= \mathcal{D}(\begin{pmatrix} 1_\mathcal{A} a 1_\mathcal{A} & 0 \\ 0 & 0 \end{pmatrix}) \\
&= \mathcal{D}(\begin{pmatrix} 1_\mathcal{A} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1_\mathcal{A} & 0 \\ 0 & 0 \end{pmatrix}) \\
&= \mathcal{D}(\begin{pmatrix} 1_\mathcal{A} & 0 \\ 0 & 0 \end{pmatrix}) \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1_\mathcal{A} & 0 \\ 0 & 0 \end{pmatrix} \\
&+ \begin{pmatrix} 1_\mathcal{A} & 0 \\ 0 & 0 \end{pmatrix} \mathcal{D}(\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}) \begin{pmatrix} 1_\mathcal{A} & 0 \\ 0 & 0 \end{pmatrix} \\
&+ \begin{pmatrix} 1_\mathcal{A} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \mathcal{D}(\begin{pmatrix} 1_\mathcal{A} & 0 \\ 0 & 0 \end{pmatrix}) \\
&= \begin{pmatrix} 0 & m_0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1_\mathcal{A} & 0 \\ 0 & 0 \end{pmatrix} \\
&+ \begin{pmatrix} 1_\mathcal{A} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a' & m' \\ 0 & b' \end{pmatrix} \begin{pmatrix} 1_\mathcal{A} & 0 \\ 0 & 0 \end{pmatrix} \\
&+ \begin{pmatrix} 1_\mathcal{A} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & m_0 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} a' & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & am_0 \\ 0 & 0 \end{pmatrix}
\end{aligned}$$

$$= \begin{pmatrix} a' & am_0 \\ 0 & 0 \end{pmatrix},$$

so $a' = a'$, $m' = am_0$ and $b' = 0$. Therefore $\mathcal{D}(\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}) = \begin{pmatrix} \mathcal{D}_1(a) & am_0 \\ 0 & 0 \end{pmatrix}$, where $a' = \mathcal{D}_1(a)$.

If $a = m = 0$, for each $b \in \mathcal{B}$ put $\mathcal{D}(\begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}) = \begin{pmatrix} a' & m' \\ 0 & b' \end{pmatrix}$, then

$$\begin{aligned}
\mathcal{D}(\begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}) &= \mathcal{D}(\begin{pmatrix} 0 & 0 \\ 0 & \lim_\beta e_\beta b e_\beta \end{pmatrix}) \\
&= \mathcal{D}(\lim_\beta (\begin{pmatrix} 0 & 0 \\ 0 & e_\beta \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & e_\beta \end{pmatrix})) \\
&= \lim_\beta (\mathcal{D}(\begin{pmatrix} 0 & 0 \\ 0 & e_\beta \end{pmatrix}) \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & e_\beta \end{pmatrix} \\
&+ \begin{pmatrix} 0 & 0 \\ 0 & e_\beta \end{pmatrix} \mathcal{D}(\begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}) \begin{pmatrix} 0 & 0 \\ 0 & e_\beta \end{pmatrix} \\
&+ \begin{pmatrix} 0 & 0 \\ 0 & e_\beta \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} \mathcal{D}(\begin{pmatrix} 0 & 0 \\ 0 & e_\beta \end{pmatrix})) \\
&= \lim_\beta (\begin{pmatrix} 0 & -m_0 e_\beta \\ 0 & b_\beta \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & e_\beta \end{pmatrix} \\
&+ \begin{pmatrix} 0 & 0 \\ 0 & e_\beta \end{pmatrix} \begin{pmatrix} a' & m' \\ 0 & b' \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & e_\beta \end{pmatrix} \\
&+ \begin{pmatrix} 0 & 0 \\ 0 & e_\beta \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} 0 & -m_0 e_\beta \\ 0 & b_\beta \end{pmatrix}) \\
&= \lim_\beta (\begin{pmatrix} 0 & -m_0 e_\beta b e_\beta \\ 0 & b_\beta b e_\beta \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & e_\beta b' e_\beta \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & e_\beta b b_\beta \end{pmatrix}) \\
&= \begin{pmatrix} 0 & \lim_\beta -m_0 e_\beta b e_\beta \\ 0 & \lim_\beta (b_\beta b e_\beta + e_\beta b' e_\beta + e_\beta b b_\beta) \end{pmatrix},
\end{aligned}$$

so $a' = 0$, $m' = -\lim_\beta m_0 e_\beta b e_\beta$ thus $m' = -m_0 b$, and $b' = \lim_\beta (b_\beta b e_\beta + e_\beta b' e_\beta + e_\beta b b_\beta)$ thus $\lim_\beta (b_\beta b + b b_\beta) = 0$. This proves (1), and $b' = b'$. Therefore $\mathcal{D}(\begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}) = \begin{pmatrix} 0 & -m_0 b \\ 0 & \mathcal{D}_2(b) \end{pmatrix}$, where $b' = \mathcal{D}_2(b)$.

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If $a = b = 0$, for each $m \in \mathcal{M}$, say $\mathcal{D}(\begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}) = \begin{pmatrix} a' & m' \\ 0 & b' \end{pmatrix}$,

$$\begin{aligned}
\mathcal{D}(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}) &= \mathcal{D}(\begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1_{\mathcal{A}} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1_{\mathcal{A}} & 0 \\ 0 & 0 \end{pmatrix}) \\
&= \mathcal{D}(\begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}) \begin{pmatrix} 1_{\mathcal{A}} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1_{\mathcal{A}} & 0 \\ 0 & 0 \end{pmatrix} \\
&+ \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \mathcal{D}(\begin{pmatrix} 1_{\mathcal{A}} & 0 \\ 0 & 0 \end{pmatrix}) \begin{pmatrix} 1_{\mathcal{A}} & 0 \\ 0 & 0 \end{pmatrix} \\
&+ \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1_{\mathcal{A}} & 0 \\ 0 & 0 \end{pmatrix} \mathcal{D}(\begin{pmatrix} 1_{\mathcal{A}} & 0 \\ 0 & 0 \end{pmatrix}) \\
&= \begin{pmatrix} a' & m' \\ 0 & b' \end{pmatrix} \begin{pmatrix} 1_{\mathcal{A}} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1_{\mathcal{A}} & 0 \\ 0 & 0 \end{pmatrix} \\
&+ \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & m_0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1_{\mathcal{A}} & 0 \\ 0 & 0 \end{pmatrix} \\
&+ \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1_{\mathcal{A}} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & m_0 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} a' & 0 \\ 0 & 0 \end{pmatrix},
\end{aligned}$$

so $a' = 0$. Let, for each $0 \neq b_1, b_2 \in \mathcal{B}$ which have an approximate identity $(e_\beta)_\beta$, in this case, say $\mathcal{D}(\begin{pmatrix} 0 & 0 \\ 0 & b_1 \end{pmatrix}) = \begin{pmatrix} 0 & -m_0 b_1 \\ 0 & \mathcal{D}_2(b_1) \end{pmatrix}$, $\mathcal{D}(\begin{pmatrix} 0 & 0 \\ 0 & b_2 \end{pmatrix}) = \begin{pmatrix} 0 & -m_0 b_2 \\ 0 & \mathcal{D}_2(b_2) \end{pmatrix}$,

$$\begin{aligned}
\mathcal{D}(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}) &= \mathcal{D}(\begin{pmatrix} 0 & 0 \\ 0 & b_1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & b_2 \end{pmatrix} \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}) \\
&= \mathcal{D}(\begin{pmatrix} 0 & 0 \\ 0 & b_1 \end{pmatrix}) \begin{pmatrix} 0 & 0 \\ 0 & b_2 \end{pmatrix} \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \\
&+ \begin{pmatrix} 0 & 0 \\ 0 & b_1 \end{pmatrix} \mathcal{D}(\begin{pmatrix} 0 & 0 \\ 0 & b_2 \end{pmatrix}) \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \\
&+ \begin{pmatrix} 0 & 0 \\ 0 & b_1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & b_2 \end{pmatrix} \mathcal{D}(\begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}) \\
&= \begin{pmatrix} 0 & -m_0 b_1 \\ 0 & \mathcal{D}_2(b_1) \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & b_2 \end{pmatrix} \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \\
&+ \begin{pmatrix} 0 & 0 \\ 0 & b_1 \end{pmatrix} \begin{pmatrix} 0 & -m_0 b_2 \\ 0 & \mathcal{D}_2(b_2) \end{pmatrix} \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}
\end{aligned}$$

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$$+\begin{pmatrix}0 & 0\\ 0 & b_1\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & b_2\end{pmatrix}\begin{pmatrix}a' & m'\\ 0 & b'\end{pmatrix}$$
$$=\begin{pmatrix}0 & 0\\ 0 & b_1b_2b'\end{pmatrix},$$

so $b_1b_2b'=0$, by applying the bounded approximate identity $e_\beta$, we get $b'=0$. Therefore $\mathcal{D}(\begin{pmatrix}0 & m\\ 0 & 0\end{pmatrix}) = \begin{pmatrix}0 & \tau(m)\\ 0 & 0\end{pmatrix}$, where $\tau(m)=m'$.

(6):

We have,

$$\begin{aligned}\begin{pmatrix}0 & \tau(mb_1b_2)\\ 0 & 0\end{pmatrix} = \mathcal{D}(\begin{pmatrix}0 & mb_1b_2\\ 0 & 0\end{pmatrix}) &= \mathcal{D}(\begin{pmatrix}0 & m\\ 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & b_1\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & b_2\end{pmatrix})\\
&= \mathcal{D}(\begin{pmatrix}0 & m\\ 0 & 0\end{pmatrix})\begin{pmatrix}0 & 0\\ 0 & b_1\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & b_2\end{pmatrix}\\
&+ \begin{pmatrix}0 & m\\ 0 & 0\end{pmatrix}\mathcal{D}(\begin{pmatrix}0 & 0\\ 0 & b_1\end{pmatrix})\begin{pmatrix}0 & 0\\ 0 & b_2\end{pmatrix}\\
&+ \begin{pmatrix}0 & m\\ 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & b_1\end{pmatrix}\mathcal{D}(\begin{pmatrix}0 & 0\\ 0 & b_2\end{pmatrix})\\
&= \begin{pmatrix}0 & \tau(m)\\ 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & b_1\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & b_2\end{pmatrix}\\
&+ \begin{pmatrix}0 & m\\ 0 & 0\end{pmatrix}\begin{pmatrix}0 & -m_0b_1\\ 0 & \mathcal{D}_2(b_1)\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & b_2\end{pmatrix}\\
&+ \begin{pmatrix}0 & m\\ 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & b_1\end{pmatrix}\begin{pmatrix}0 & -m_0b_2\\ 0 & \mathcal{D}_2(b_2)\end{pmatrix}\\
&= \begin{pmatrix}0 & \tau(m)b_1b_2 + m\mathcal{D}_2(b_1)b_2 + mb_1\mathcal{D}_2(b_2)\\ 0 & 0\end{pmatrix},\end{aligned}$$

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so $\tau(mb_1b_2) = \tau(m)b_1b_2 + m\mathcal{D}_2(b_1)b_2 + mb_1\mathcal{D}_2(b_2)$. The proof of (5) and (7) is similar to (6). Conversely, for each $\begin{pmatrix} a_1 & m_1 \\ 0 & b_1 \end{pmatrix}, \begin{pmatrix} a_2 & m_2 \\ 0 & b_2 \end{pmatrix}, \begin{pmatrix} a_3 & m_3 \\ 0 & b_3 \end{pmatrix} \in \mathcal{T}$, we have

$$\mathcal{D}(\begin{pmatrix} a_1 & m_1 \\ 0 & b_1 \end{pmatrix} \begin{pmatrix} a_2 & m_2 \\ 0 & b_2 \end{pmatrix} \begin{pmatrix} a_3 & m_3 \\ 0 & b_3 \end{pmatrix})$$

$$= \mathcal{D}(\begin{pmatrix} a_1a_2a_3 & a_1a_2m_3 + a_1m_2b_3 + m_1b_1b_3 \\ 0 & b_1b_2b_3 \end{pmatrix})$$

$$= \begin{pmatrix} \mathcal{D}_1(a_1a_2a_3) & \tau(a_1a_2m_3 + a_1m_2b_3 + m_1b_2b_3) \\ 0 & \mathcal{D}_2(b_1b_2b_3) \end{pmatrix}$$

$$= \begin{pmatrix} \mathcal{D}_1(a_1)a_2a_3 + a_1\mathcal{D}_1(a_2)a_3 + a_1a_2\mathcal{D}_1(a_3) & \begin{array}{l} (\mathcal{D}_1(a_1)a_2m_3 + a_1\mathcal{D}_1(a_2)m_3 + a_1a_2\tau(m_3) \\ +\mathcal{D}_1(a_1)m_2b_3 + a_1\tau(m_2)b_3 + a_1m_2\mathcal{D}_2(b_3) \\ +\tau(m_1)b_2b_3 + m_1\mathcal{D}_2(b_2)b_3 + m_1b_2\mathcal{D}_2(b_3)) \end{array} \\ 0 & \mathcal{D}_2(b_1)b_2b_3 + b_1\mathcal{D}_2(b_2)b_3 + b_1b_2\mathcal{D}_2(b_3) \end{pmatrix}$$

$$= \mathcal{D}(\begin{pmatrix} a_1 & m_1 \\ 0 & b_1 \end{pmatrix}) \begin{pmatrix} a_2 & m_2 \\ 0 & b_2 \end{pmatrix} \begin{pmatrix} a_3 & m_3 \\ 0 & b_3 \end{pmatrix}$$
$$+ \begin{pmatrix} a_1 & m_1 \\ 0 & b_1 \end{pmatrix} \mathcal{D}(\begin{pmatrix} a_2 & m_2 \\ 0 & b_2 \end{pmatrix}) \begin{pmatrix} a_3 & m_3 \\ 0 & b_3 \end{pmatrix}$$
$$+ \begin{pmatrix} a_1 & m_1 \\ 0 & b_1 \end{pmatrix} \begin{pmatrix} a_2 & m_2 \\ 0 & b_2 \end{pmatrix} \mathcal{D}(\begin{pmatrix} a_3 & m_3 \\ 0 & b_3 \end{pmatrix}),$$

so $\mathcal{D} : \mathcal{T} \to \mathcal{T}$ is a 3-derivation. □

The following example indicate that the existence of bounded approximate identity for $\mathcal{B}$ is essential and can not be omitted.

**Example 4.2.** *Let* $\mathcal{A} = \mathcal{M} = \mathbb{M}_{3\times 3}(\mathbb{C})$ *and* $\mathcal{B} = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} : x, y, z \in \mathbb{C} \right\}$, $\mathcal{D}_1 = ad_{a_0}$ *where* $a_0 \in \mathcal{A}$ *and* $\mathcal{D}_2 = id_{\mathcal{B}}$. *Then* $\mathcal{D}(\begin{pmatrix} a & m \\ 0 & b \end{pmatrix}) = \begin{pmatrix} \mathcal{D}_1(a) & 0 \\ 0 & \mathcal{D}_2(b) \end{pmatrix}$, *is not a* 3*-derivation. This demonstrates that the existence of a bounded approximate identity for* $\mathcal{B}$ *is an necessary condition for the preceding theorem.*

## References

[1] A. M. Sinclair, Continuous Derivations on Banach Algebras, *Proc. Am. Math. Soc.* 20 (1969) 166–170.

[2] B. E. Forrest and L. W. Marcoux, Derivations of Triangular Banach Algebras, *Indiana Univ. Math. J.* 45 (1996), no. 2, 441–462.

[3] G. J. Murphy, Aspects of the Theory of Derivations, *Banach Center Publ.* 30, (1992) 267–275.

DOI: https://dx.doi.org/10.30504/jims.2025.520461.1257

[4] H. G. Dales, Banach Algebras and Automatic Continuity, London Math. Soc. Monogr. Ser., Oxford University Press, New York, 2000.
[5] J. Dixmier, Enveloping Algebras, North-Holland Publishing Co., Amsterdam-New York-Oxford, 1977.
[6] M. H. Sattari, On n-Derivations, *Sahand Commun. Math. Anal.* **3** (2016), no. 2, 107–115.
[7] T. Creedon, Products of Derivations, *Proc. Edinb. Math. Soc.* **41** (1998) 407-410.

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