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THETA METHOD TO APPROXIMATE FIXED POINTS OF NONEXPANSIVE MULTIMAPS IN BANACH SPACES

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Abstract. In this paper, we introduce a novel iterative scheme that integrates the Mann iteration process with the implicit θ-method to approximate fixed points of nonexpansive multivalued mappings in Banach spaces. Under suitable assumptions, we establish both weak and strong convergence results for the proposed algorithm. Furthermore, we demonstrate the applicability of our method to variational inclusion problems and convex optimization problems. A numerical example is presented to illustrate the efficiency and effectiveness of the approach.

1. **Introduction**

The approximation of fixed points of non-linear mappings has long played a crucial role in nonlinear analysis, with broad applications in differential equations, variational inequalities, optimization, control theory, signal processing, and inverse problems. While fixed point theory for single-valued operators is well-developed, increasing attention has been devoted to multivalued mappings, which offer a flexible and powerful framework for modeling systems with uncertainty, non-uniqueness, or discontinuous dynamics.

Numerical methods for solving such problems are well-established, with the Euler methods (see [4–6, 22, 29]) forming a foundational class. A particularly effective technique is the θ-method, which interpolates between explicit and implicit Euler schemes:

$$\frac{x_{k+1}-x_k}{h} = \theta f(t,x_k) + (1-\theta) f(t,x_{k+1}) \quad \text{for } k = 1, \dots \tag{1.1}$$

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where $h > 0$ is a stepsize and $\theta \in [0, 1]$. The method is explicit for $\theta = 0$, implicit for $\theta = 1$, and recovers the trapezoidal rule when $\theta = 1/2$. Due to their accuracy and stability, the θ methods are widely used in the numerical treatment of stiff ODEs, partial differential equations, and stochastic differential equations (see [15, 28, 39, 41, 48]). In the realm of fixed point theory, the Mann iteration (see [23]) has been a cornerstone method for approximating fixed points of nonexpansive single-valued operators. To enhance convergence behavior, Xu et al. in [46] proposed an elegant hybrid scheme combining the Mann iteration with the method:

$$x_{n+1} = (1-\tau)x_n + \tau\left[\theta T x_n + (1-\theta)T x_{n+1}\right] \tag{1.2}$$

where $T : H \longrightarrow H$ is a nonexpansive mapping in a Hilbert space H. They proved the weak convergence of this scheme under suitable assumptions. Motivated by this approach and recent developments in multivalued analysis, in this paper, we focus on generalized fixed-point problems involving non-expansive multimaps, defined on Banach spaces. These problems are central in modern optimization theory and have important applications in various areas such as control theory, convex programming, economic equilibrium modeling, image reconstruction, and differential inclusions (see [2, 20, 21, 27, 30, 40, 44]).

The extension of iterative methods from single-valued operators to multi-valued operators presents non-trivial challenges. Key results by Nadler [26], Kirk [3], Panyanak [32], and others have provided theoretical foundations for such generalizations, using tools such as the Hausdorff metric and proximal mappings. A rich body of literature has proposed and analyzed various algorithms for approximating fixed points of nonexpansive set-valued mappings (see, e.g., [24, 34, 36, 38]).

In this paper, we propose a generalized θ-iteration process to approximate fixed points of nonexpansive multivalued mappings in Banach spaces. The proposed algorithm combines the classical Mann iteration with the implicit θ-method and is adapted to the multivalued framework using the Hausdorff metric. Under appropriate conditions, we establish both weak and strong convergence of the sequence generated by the method to a fixed point of the multimap. We further demonstrate the applicability of our results to variational inclusion problems and convex optimization problems, particularly those involving forward–backward splitting operators. To illustrate the practical effectiveness of the algorithm, a numerical example is provided, which confirms the stability and efficiency of the proposed approach.

The remainder of this paper is structured as follows. In Section 2, we recall essential definitions and preliminary results concerning Banach spaces and multivalued mappings. We also introduce the proposed generalized iterative algorithm. In Section 3, we establish several auxiliary lemmas and prove the main convergence theorems, including both weak and strong convergence results. Section 4 is devoted to the applications of our theoretical findings to variational inclusion problems and convex optimization models. Finally, we present a numerical example to illustrate the effectiveness and robustness of the proposed method.

2. Preliminaries

2.1. **Basic mathematical tools.** Let X be a real Banach space with the dual space X^* and $\langle \cdot, \cdot \rangle$ denote the duality pairing between X and X^*. Let $\varphi : [0, \infty) \longrightarrow [0, \infty)$ be a continuous strictly increasing function such that $\varphi(0) = 0$ and $\varphi(t) \to \infty$ as $t \to \infty$. Such a function is called a gauge. We associate with a gauge φ the duality map $J_\varphi : X \rightrightarrows X^*$ defined by Browder ([8]) as

$$J_\varphi(x) := \{x^* \in X^* : \langle x, x^* \rangle = \|x\| \, \|\varphi(x)\| \, , \|x^*\| = \varphi(\|x\|)\} \, , \forall x \in X.$$

The generalized duality map $J_p : X \rightrightarrows X^*$ of order p associated to the gauge function $\varphi(t) = t^{p-1}$ where $t \geq 0$ and $1 < p < \infty$, is given by

$$J_p(x) := \left\{ x^* \in X^* : \langle x, x^* \rangle = \|x\| \, \|x\| \, , \|x^*\| = \|x\|^{p-1} \right\} , \forall x \in X.$$

For more details on properties of p-duality, see ([1, 11, 42]).

In the case of $\varphi(t) = t$, we write J instead of J_2 which is called the normalized duality mapping defined by

$$J(x) := \left\{ x^* \in X^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2 \right\} , \forall x \in X.$$

For a Hilbert space H, its normalized duality map is identified with the identity map I. Recall that the modulus of convexity of X is defined as (see [11])

$$\delta_X(\varepsilon) = \inf \left\{ 1 - \frac{\|x + y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \varepsilon \right\} , \quad \varepsilon \in (0, 2] \, .$$

A Banach space X is said to be uniformly convex if $\delta_X(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$.

The following inequality is a characterization of uniform convexity.

Lemma 2.1 ([45]). *Let X be a uniformly convex Banach space. Then, for each fixed real number $r > 0$, there exists a continuous strictly increasing function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$, satisfying the inequality*

$$\|\lambda x + (1 - \lambda) y\|^2 \leq \lambda \|x\|^2 + (1 - \lambda) \|y\|^2 - \lambda(1 - \lambda)\varphi(\|x - y\|) \tag{2.1}$$

for all $x, y \in B_r$ such that $B_r = \{x \in X : \|x\| \leq r\}$ and $\lambda \in [0, 1]$.

The next Lemma is helpful.

Lemma 2.2 ([34]). *Let (α_n), (β_n) be two real sequences such that*

i) $0 \leq \alpha_n, \beta_n < 1$, *and* $\lim\limits_{n \to \infty} \beta_n = 0$;

ii) $\sum \alpha_n \beta_n = \infty$.

Let (γ_n) be a nonnegative real sequence such that $\sum \alpha_n \beta_n (1 - \beta_n) \gamma_n$ is bounded. Then (γ_n) has a subsequence which converges to zero.

In a Banach space X, we consider the Hausdorff metric induced by the metric d of X given as follows

$$h(A, B) := \max \{e(A, B); e(B, A)\}$$

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such that A and B are nonempty, closed and bounded subsets of X and $e(A,B) = \sup_{a\in A} d(a,B)$.

Let K be a subset of X, we said that K is proximinal if for each $x \in X$ there exists an element $k \in K$ such that

$$\|x-k\| = d(x,K).$$

It is well known that any weakly compact convex subset of a Banach space and any closed convex subset of a uniformly convex Banach space is proximinal. We shall denote by $\mathcal{P}(K)$ the family of nonempty proximinal subsets of K and we denote the set of nonempty compact subsets of K by $\mathcal{C}(K)$ and by $\mathcal{CB}(K)$ the class of all nonempty bounded and closed subsets of K.

Let $T : K \rightrightarrows K$ be a multimap, we denote by $\operatorname{Fix}(T)$ the set of fixed points of T such that $\operatorname{Fix}(T) := \{x \in K : x \in Tx\}$. Recall that the multimap T is said to be nonexpansive if for all $x, y \in K$, one has

$$h(T(x),T(y)) \leq \|x-y\|.$$

Also, the multimap $I-T$ is called demi-closed at 0 if for all $x_n \rightharpoonup x$, $y_n \longrightarrow 0$ with $y_n \in (I-T)(x_n)$, then $0 \in (I-T)(x)$ where $\rightharpoonup$ denote the weak convergence and $\longrightarrow$ denotes the strong convergence.

Let $T : K \to \mathcal{P}(K)$ be a multimap. The proximinal multifunction of T is the multimap $\mathcal{P}_T : K \rightrightarrows K$ defined by

$$\mathcal{P}_T(x) := \{y \in Tx : \|x-y\| = d(x,Tx)\} \qquad \forall x \in K.$$

Moreover, we have

$$p \in T(p) \Longleftrightarrow p \in \mathcal{P}_T(p) \Longleftrightarrow \mathcal{P}_T(p) = \{p\}. \tag{2.2}$$

So that, $\operatorname{Fix}(\mathcal{P}_T) = \operatorname{Fix}(T)$ and $d(x,T(x)) = d(x,\mathcal{P}_T(x))$ $\forall x \in K$. Noted that the nonexpansive property is not inherited between T and $\mathcal{P}_T$ as the following example shows.

Example 2.3. *Consider the multimap* $T : \left[0,\frac{2}{3}\right] \rightrightarrows \left[0,\frac{2}{3}\right]$*, such that*

$$T(x) := \begin{cases} \left\{\frac{1}{3}\right\} & \text{if } x \neq \frac{1}{3} \\ \left[\frac{1}{6},\frac{1}{2}\right] & \text{if } x = \frac{1}{3} \end{cases}.$$

Then, $\mathcal{P}_T(x) = \operatorname{Fix}(T) = \left\{\frac{1}{3}\right\}$ *which is a nonexpansive map however* T *is not nonexpansive. In fact, if* $x = \frac{5}{12}$*,* $y = \frac{1}{3}$ *we obtain*

$$h\left(T\left(\frac{5}{12}\right),T\left(\frac{1}{3}\right)\right) = \max\left\{|\frac{1}{3}-\frac{1}{6}|,|\frac{1}{3}-\frac{1}{2}|\right\} = \frac{1}{6} > \frac{1}{12} = d\left(\frac{5}{12},\frac{1}{3}\right).$$

Eslamian et al. in [14] proved the necessary condition of demi-closed property of the multimap $\mathcal{P}_T$ as follows.

Lemma 2.4. *Let* $T : K \to \mathcal{C}(K)$ *be a multimapping, such that* $\mathcal{P}_T$ *is nonexpansive. Then,* $I - \mathcal{P}_T$ *is demi-closed at* 0*, i.e., if* $x_n \rightharpoonup x$ *and* $z_n \in (I-\mathcal{P}_T)(x_n)$ *such that* z_n *converges strongly to* 0*, then* $x \in \mathcal{P}_T(x)$*.*

The Opial's condition is a very interesting tool in the sequel.

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Definition 2.5 ([31]). *A Banach space X is said to satisfy Opial's condition, if for any sequence (x_n) in X, $x_n \rightharpoonup x$ implies that*

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$$

for all $y \in X$ with $y \neq x$.

All Hilbert spaces and all l^p spaces $(1 < p < \infty)$ satisfy the Opial's condition whenever the $L^p([0, 2\pi])$ spaces with $1 < p \neq 2$ fail to satisfy Opial's condition.

2.2. **Problem formulation.** Let X be a uniformly convex Banach space and let K be a nonempty closed and convex subset of X. Consider the generalized fixed point problem:

$$\text{find } x^* \in K \text{ such that } x^* \in Tx^* \tag{2.3}$$

where $T : K \longrightarrow \mathcal{P}(K)$ is a multimap.

To solve the problem (2.3), we propose and analyze the convergence behavior of the following algorithm: given initial guess $x_0 \in K$ and compute

$$\begin{cases} x_{n+1} = (1-\tau_n)x_n + \tau_n v_n \\ v_n = \theta y_n + (1-\theta)z_n \end{cases} \tag{2.4}$$

where the sequences $y_n \in \mathcal{P}_T(x_n)$, $z_n \in \mathcal{P}_T(x_{n+1})$, $\theta \in (0,1)$ and $(\tau_n) \subset (0,1)$.

Let μ be an arbitrary point in K and set $F_\mu : K \rightrightarrows K$ such that

$$F_\mu(x) := (1-\tau_n)\mu + \tau_n\left(\theta\mathcal{P}_T(\mu) + (1-\theta)\mathcal{P}_T(x)\right).$$

Obviously, each F_μ is a contraction multimap with constant $(1-\theta)\tau_n < 1$. Indeed, for all x and y in K we have

$$h\left(F_\mu(x), F_\mu(y)\right) = \tau_n(1-\theta)h\left(\mathcal{P}_T(x), \mathcal{P}_{T(y)}\right) \leq \tau_n(1-\theta)d(x,y).$$

Since $\mathcal{P}_T$ is nonexpansive multimap. Hence, there exists a point $x \in K$ such that,

$$x \in F_\mu(x).$$

which expands to:

$$x \in (1-\tau_n)\mu + \tau_n\left(\theta\mathcal{P}_T(\mu) + (1-\theta)\mathcal{P}_T(x)\right).$$

Therefore, we check the existence of the iterative sequence $(x_n)_{n\in\mathbb{N}}$ generated by (2.4). It remains to prove the convergence results, which is the main aim of the next section.

3. Convergence analysis

3.1. **Technical Lemmas.** Before giving the convergence results, we present some technical lemmas.

Lemma 3.1. *Let X be a real Banach space and K a nonempty closed and convex subset of X. Let $T : K \to \mathcal{P}(K)$ be a multimap such that $\mathrm{Fix}(T) \neq \varnothing$ and $\mathcal{P}_T$ is nonexpansive. Then, for all $p \in \mathrm{Fix}(T)$ $\lim_{n\to\infty} \|x_n - p\|$ and $\lim_{n\to\infty} d(x_n, \mathcal{P}_T(x_n))$ exist where (x_n) is the sequence generated by scheme (2.4).*

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Proof. Let $p \in \mathrm{Fix}\,(T) = \mathrm{Fix}\,(\mathcal{P}_T)$. We have

$$\|x_n - y_n\| = d(x_n, Tx_n), \quad \|x_{n+1} - z_n\| = d(x_{n+1}, Tx_{n+1}). \tag{3.1}$$

Therefore,

$$\|x_{n+1} - p\| \le (1-\tau_n)\|x_n - p\| + \theta\tau_n \|y_n - p\| + (1-\theta)\tau_n \|z_n - p\|.$$

Since $\mathcal{P}_T$ is nonexpansive, it holds that

$$\|y_n - p\| \le h((\mathcal{P}_T(x_n), \mathcal{P}_T(p)) \le \|x_n - p\|, \tag{3.2}$$

and

$$\|z_n - p\| \le h((\mathcal{P}_T(x_{n+1}), \mathcal{P}_T(p)) \le \|x_{n+1} - p\|. \tag{3.3}$$

Thus,

$$\|x_{n+1} - p\| \le (1-\tau_n)\|x_n - p\| + \theta\tau_n \|x_n - p\| + (1-\theta)\tau_n \|x_{n+1} - p\| \tag{3.4}$$

that is

$$(1 - \tau_n(1-\theta))\|x_{n+1} - p\| \le (1 - \tau_n(1-\theta))\|x_n - p\|.$$

From the assumptions on the parameters θ and τ_n, it follows that $1 - \tau_n(1-\theta) > 0$ for all $n \ge 0$ and we get

$$\|x_{n+1} - p\| \le \|x_n - p\|.$$

Then, the sequence $(\|x_n - p\|)_{n\in\mathbb{N}}$ is nonincreasing and bounded below and hence $\lim\limits_{n\to\infty} \|x_n - p\|$ exists for any $p \in \mathrm{Fix}\,(T)$.

Similarly, we prove the monotonicity of the sequence $(d(x_n, \mathcal{P}_T(x_n)))_{n\in\mathbb{N}}$. We have

$$\begin{aligned} d(x_{n+1}, \mathcal{P}_T(x_{n+1})) &= d((1-\tau_n)x_n + \tau_n(\theta y_n + (1-\theta)z_n), \mathcal{P}_T(x_{n+1})) \\ &\le (1-\tau_n)d(x_n, \mathcal{P}_T(x_{n+1})) + \theta\tau_n d(y_n, \mathcal{P}_T(x_{n+1})) \end{aligned}$$

since $d(z_n, \mathcal{P}_T(x_{n+1})) = 0$. Thus

$$\begin{aligned} d(x_{n+1}, \mathcal{P}_T(x_{n+1})) \le{}& (1-\tau_n)d(x_n, \mathcal{P}_T(x_n)) + (1-\tau_n)h(\mathcal{P}_T(x_n), \mathcal{P}_T(x_{n+1})) \\ &+ \tau_n\theta h(\mathcal{P}_T(x_n), \mathcal{P}_T(x_{n+1})). \end{aligned}$$

Since $\mathcal{P}_T$ is nonexpansive we get

$$d(x_{n+1}, \mathcal{P}_T(x_{n+1})) \le (1-\tau_n)d(x_n, \mathcal{P}_T(x_n)) + (1 - \tau_n(1-\theta))\|x_{n+1} - x_n\|.$$

On the other side, we have

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|(1-\tau_n)x_n + \tau_n[\theta y_n + (1-\theta)z_n] - x_n\| \\ &\le \tau_n\theta\|y_n - x_n\| + \tau_n(1-\theta)\|z_n - x_n\| \\ &\le \tau_n\theta\|y_n - x_n\| + \tau_n(1-\theta)\|z_n - x_{n+1}\| + \tau_n(1-\theta)\|x_{n+1} - x_n\|. \end{aligned}$$

Equalities (3.1) implies that

$$(1 - \tau_n(1-\theta))\|x_{n+1} - x_n\| \le \tau_n\theta d(x_n, \mathcal{P}_T(x_n)) + \tau_n(1-\theta)d(x_{n+1}, \mathcal{P}_T(x_{n+1})),$$

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which leads to

$$d(x_{n+1}, \mathcal{P}_T(x_{n+1})) \leq (1-\tau_n)d(x_n, \mathcal{P}_T(x_n)) + \tau_n\theta d(x_n, \mathcal{P}_T(x_n)) + \tau_n(1-\theta)d(x_{n+1}, \mathcal{P}_T(x_{n+1})).$$

Therefore,

$$(1-\tau_n(1-\theta))d(x_{n+1}, \mathcal{P}_T(x_{n+1})) \leq (1-\tau_n(1-\theta))d(x_n, \mathcal{P}_T(x_n)).$$

As $1-\tau_n(1-\theta) > 0$, it yields that for all $n \geq 0$,

$$d(x_{n+1}, \mathcal{P}_T(x_{n+1})) \leq d(x_n, \mathcal{P}_T(x_n)).$$

Since $d(x_n, \mathcal{P}_T(x_n)) = d(x_n, T(x_n))$, then the sequence multimap is decreasing and bounded from below so that $\lim_{n\to\infty} d(x_n, T(x_n))$ exists. □

The following Lemma is useful. Consider $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ the continuous strictly increasing function with $\varphi(0) = 0$ given by Lemma 2.1 and let the sequences (x_n), (y_n), (z_n), (v_n) defined by (2.4).

Lemma 3.2. *Let X be a uniformly convex Banach space and let K be a nonempty closed convex subset of X. Consider the multimap $T : K \to \mathcal{P}(K)$ such that $\operatorname{Fix}(T) \neq \varnothing$ and $\mathcal{P}_T$ is nonexpansive. Hence,*

(i) $\sum_{n=0}^{\infty} \tau_n(1-\tau_n)\varphi(\|v_n - x_n\|) < \infty$;

(ii) $\sum_{n=0}^{\infty} \tau_n\varphi(\|y_n - z_n\|) < \infty$.

Proof. Since $\lim_{n\to+\infty} \|x_n - p\|$ exists for any $p \in \operatorname{Fix}(T)$, it follows that $(x_n - p)$, $(y_n - p)$ and $(z_n - p)$ are all bounded. We may assume that these sequences belong to B_r where $r > 0$. By (2.1), we get for any $p \in \operatorname{Fix}(T)$ and for all $n \geq 0$,

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|(1-\tau_n)x_n + \tau_n v_n - p\|^2 \\ &\leq (1-\tau_n)\|x_n - p\|^2 + \tau_n\|v_n - p\|^2 - \tau_n(1-\tau_n)\varphi(\|x_n - v_n\|)\end{aligned}$$

and

$$\|v_n - p\|^2 \leq \theta\|y_n - p\|^2 + (1-\theta)\|z_n - p\|^2 - \theta(1-\theta)\varphi(\|y_n - z_n\|).$$

Using properties (3.2) and (3.3), we get

$$\begin{aligned}\|x_{n+1} - p\|^2 \leq{}& (1-\tau_n)\|x_n - p\|^2 + \tau_n\theta\|x_n - p\|^2 + \tau_n(1-\theta)\|x_{n+1} - p\|^2 \\ &- \tau_n\theta(1-\theta)\varphi(\|y_n - z_n\|) - \tau_n(1-\tau_n)\varphi(\|x_n - v_n\|).\end{aligned}$$

From (3.4), it yields

$$\tau_n(1-\tau_n)\varphi(\|x_n - v_n\|) \leq (1-\tau_n(1-\theta))\left(\|x_n - p\|^2 - \|x_{n+1} - p\|^2\right),$$

and

$$\theta(1-\theta)\tau_n\varphi(\|y_n - z_n\|) \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2.$$

Since $(1-\theta)\tau_n \in (0,1)$, it turns out that

$$\tau_n(1-\tau_n)\varphi(\|x_n - v_n\|) \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2,$$

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then, we deduce (i). Also, since $0<\theta<1$, $\theta(1-\theta)>0$,

$$\tau_n\varphi\left(\|y_n-z_n\|\right)\leq\frac{1}{\theta(1-\theta)}\left(\|x_n-p\|^2-\|x_{n+1}-p\|^2\right)$$

and hence (ii) follows. □

Under the same divergence property, we get the next Lemma.

Lemma 3.3. *Assume that conditions of Lemma 3.1 are satisfied,* $\tau_n\to 0$ *and*

$$\sum_{n=0}^{\infty}\tau_n(1-\tau_n)=\infty. \tag{3.5}$$

Then the sequence (x_n) *generated by the iterative scheme (2.4) is an approximating fixed point of* T *that is,* $(d(x_n,T(x_n)))_{n\in\mathbb{N}}$ *converges to zero.*

Proof. From (2.4), we obtain

$$d(x_n,T(x_n))\leq\|x_n-v_n\|+(1+\theta)\|y_n-z_n\|.$$

From (i) and (ii) of Lemma (3.2), we deduce that

$$\sum_{n=0}^{\infty}\tau_n(1-\tau_n)\left(\varphi\left(\|v_n-x_n\|\right)+\varphi\left(\|y_n-z_n\|\right)\right)<\infty.$$

Applying Lemma 2.2 (with $\alpha_n:=1-\tau_n$ and $\beta_n:=\tau_n$), there exists a subsequence such that

$$\varphi\left(\|x_{n_k}-v_{n_k}\|\right)+\varphi\left(\|y_{n_k}-z_{n_k}\|\right)\longrightarrow 0 \text{ as } k\to\infty.$$

As φ is a continuous and strictly increasing function with $\varphi(0)=0$, it yields that

$$\lim_{k\to\infty}\|x_{n_k}-v_{n_k}\|=\lim_{k\to\infty}\|y_{n_k}-z_{n_k}\|=0$$

and consequently, $\lim\limits_{k\to\infty}d(x_{n_k},T(x_{n_k}))=\lim\limits_{k\to\infty}\|x_{n_k}-y_{n_k}\|=0$. We conclude from Lemma 3.1 that $\lim\limits_{n\to\infty}d(x_n,T(x_n))=0$. □

3.2. Convergence results.

Now, we present the main convergence result.

Theorem 3.4. *Let* X *be a uniformly convex Banach space satisfying Opial's condition and* K *be a nonempty closed convex subset of* X*. Let* $T:K\to\mathcal{P}(K)$ *be a multimap such that* $\operatorname{Fix}(T)\neq\varnothing$, $\mathcal{P}_T$ *is nonexpansive and* $I-\mathcal{P}_T$ *is demiclosed at* 0*. Assume all assumptions of Lemma 3.3 are satisfied. Then the sequence* (x_n) *generated by algorithm (2.4) converges weakly to a fixed point of* T*.*

Proof. To show that the sequence (x_n) converges weakly to a point in $\operatorname{Fix}(T)$, it suffices to show that (x_n) has a unique weak subsequential limit in $\operatorname{Fix}T$. According to Lemma 3.1, for each $p\in\operatorname{Fix}(T)$, $\lim\limits_{n\to\infty}\|x_n-p\|$ exists. Thus, the sequence (x_n) generated by algorithm (2.4) is bounded. So, there exists a subsequence (x_{n_k}) of (x_n) such that (x_{n_k}) converges weakly to some point $x^*\in K$. By Lemma 3.3, there exists $y_{n_k}\in T(x_{n_k})$ such that

$$\lim_{k\to\infty}\|x_{n_k}-y_{n_k}\|=\lim_{k\to\infty}d(x_{n_k},T(x_{n_k}))=\lim_{k\to\infty}d(x_{n_k},\mathcal{P}_T(x_{n_k}))=0.$$

Since, $I-\mathcal{P}_T$ is demiclosed at 0, we deduce that $x^* \in \mathcal{P}_T(x^*)$. Now, using Opial's condition, we prove the uniqueness of the fixed point x^*. Indeed, suppose there exists a subsequence (x_{n_k}) which converges weakly to $x_1^* \in \operatorname{Fix}(T)$ and there exists a subsequence $\left(x_{n_k'}\right)$ converges weakly to $x_2^* \in \operatorname{Fix}(T)$ such that $x_1^* \neq x_2^*$. Therefore, we have

$$\begin{aligned}\lim_{n\to\infty}\|x_n - x_1^*\| &= \lim_{k\to\infty}\|x_{n_k} - x_1^*\| \\ &< \lim_{k\to\infty}\|x_{n_k} - x_2^*\| = \lim_{n\to\infty}\|x_n - x_2^*\| = \lim_{k\to\infty}\left\|x_{n_k'} - x_2^*\right\| \\ &< \lim_{k\to\infty}\left\|x_{n_k'} - x_1^*\right\| = \lim_{n\to\infty}\|x_n - x_1^*\|\end{aligned}$$

which is a contradiction. Hence (x_n) converges weakly to a point in $\operatorname{Fix}(T)$. □

As a consequence, we obtain the following corollary in a real Hilbert space H.

Corollary 3.5. *Let $T : H \to \mathcal{P}(H)$ be a multimap such that $\operatorname{Fix}(T) \neq \varnothing$, $\mathcal{P}_T$ is nonexpansive and $I-\mathcal{P}_T$ is demiclosed at 0. Assume all assumptions of Lemma 3.3 are satisfied. Then the sequence (x_n) generated by algorithm (2.4) converges weakly to a fixed point of T.*

Remark 3.6. *If $T : H \to H$ is a singleton map satisfying conditions of Corollary 3.5. Then, we deduce the result given by Xu in* [46, Theorem 2.8].

Under compactness conditions, we obtain immediately the next result.

Theorem 3.7. *Let X be a real Banach space and K be a nonempty convex compact subset of X. Let $T : K \rightrightarrows K$ be a multimap with proximinal values such that $\operatorname{Fix}(T) \neq \varnothing$, $\mathcal{P}_T$ is nonexpansive and $I-\mathcal{P}_T$ is demiclosed at 0. Then the sequence (x_n) generated by algorithm (2.4) converges strongly to a fixed point of T.*

Proof. Since K is compact, then there exists a subsequence (x_{n_k}) of (x_n) which converges strongly to $\omega \in K$. By virtue to Lemma 3.3, we have $\lim\limits_{k\to\infty} d(x_{n_k}, \mathcal{P}_T(x_{n_k})) = 0$. Therefore,

$$\begin{aligned}d(\omega, \mathcal{P}_T(\omega)) &\leq \|\omega - x_{n_k}\| + d(x_{n_k}, \mathcal{P}_T(x_{n_k})) + h(\mathcal{P}_T(x_{n_k}), \mathcal{P}_T(\omega)) \\ &\leq d(x_{n_k}, \mathcal{P}_T(x_{n_k})) + 2\|x_{n_k} - \omega\| \longrightarrow 0 \text{ as } k \to \infty,\end{aligned}$$

thus $\omega \in \mathcal{P}_T(\omega)$. Therefore, we have $\lim\limits_{n\to\infty}\|x_n - \omega\|$ exists for all $\omega \in \operatorname{Fix}(T)$, hence ω is a strong limit of the sequence (x_n). □

Since the compactness condition is a strong assumption, we'll assume a weaker one in what follows. This assumption, known as the $(\mathcal{I})$ condition, is given in [35] as:

Definition 3.8. *Let K be a nonempty subset of a Banach space X. A nonexpansive mapping $T : K \longrightarrow \mathcal{CB}(K)$ is said to satisfy condition $(\mathcal{I})$ if there is a non-decreasing function $f : [0,\infty) \longrightarrow [0,\infty)$ with $f(0) = 0$, $f(t) > 0$ for $t \in (0,\infty)$ such that*

$$d(x, Tx) \geq f\left(d\left(x, \operatorname{Fix}(T)\right)\right)$$

for each $x \in K$.

Hence,

Theorem 3.9. *Let K be a nonempty convex closed subset of a uniformly convex Banach space X. Let $T : K \longrightarrow \mathcal{P}(K)$ be a multifunction such that $\mathrm{Fix}\,(T) \neq \varnothing$ and $\mathcal{P}_T$ is nonexpansive. Assume all assumptions of Lemma 3.3 are satisfied. If T satisfies condition $(\mathcal{I})$, then the sequence (x_n) generated by algorithm (2.4) converges strongly to a fixed point of T.*

Proof. Fix $p \in \mathrm{Fix}\,(T)$. From Lemma 3.1 we have $\lim\limits_{n\to\infty} \|x_n - p\| = l$ where l is a finite positive real number. If $l = 0$, the result is trivial. Now, suppose that $l > 0$ and for all $n \geq 0$ by taking infimum over the set $\mathrm{Fix}\,(T)$ we obtain

$$\inf_{p\in\mathrm{Fix}(T)} \|x_{n+1} - p\| \leq \inf_{p\in\mathrm{Fix}(T)} \|x_n - p\|$$

that is

$$d(x_{n+1}, \mathrm{Fix}\,(T)) \leq d(x_n, \mathrm{Fix}\,(T)).$$

Consequently, the sequence $(d(x_n, \mathrm{Fix}\,(T)))_{n\in\mathbb{N}}$ is nonincreasing and bounded below, then $\lim\limits_{n\to\infty} d(x_n, \mathrm{Fix}\,(T))$ exists. Since T satisfies condition $(\mathcal{I})$ and according to Lemma 3.3, we find

$$f\left(\lim_{n\to\infty} d(x_n, \mathrm{Fix}\,(T))\right) = \lim_{n\to\infty} f\left(d(x_n, \mathrm{Fix}\,(T))\right) \leq \lim_{n\to\infty} d(x_n, T\,(x_n))$$

Thereby from Definition 3.8, it yields that $\lim\limits_{n\to\infty} d(x_n, \mathrm{Fix}\,(T)) = 0$. Thus, for all $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have

$$d(x_n, \mathrm{Fix}\,(T)) < \frac{\varepsilon}{2}.$$

Specifically, we get $d(x_{n_0}, \mathrm{Fix}\,(T)) < \frac{\varepsilon}{2}$. In consequence, there exists some $p_\varepsilon \in \mathrm{Fix}\,(T)$ such that

$$\|x_n - p_\varepsilon\| < \frac{\varepsilon}{2}.$$

Since the sequence $(\|x_n - p\|)_{n\in\mathbb{N}}$ is decreasing for all $p \in \mathrm{Fix}\,(T)$, for all $m, n \geq n_0$ we have

$$\|x_{n+m} - x_n\| \leq \|x_{n+m} - p_\varepsilon\| + \|x_n - p_\varepsilon\| \leq 2\,\|x_{n_0} - p_\varepsilon\| < \varepsilon.$$

Therefore, (x_n) is a Cauchy sequence in closed subset K of a Banach space X which implies that (x_n) converges to some point q in K, that is $\lim\limits_{n\to\infty} x_n = q$. Moreover, we have

$$\begin{aligned} d(q, \mathcal{P}_T(q)) &\leq \|x_n - q\| + d(x_n, \mathcal{P}_T(x_n)) + h(\mathcal{P}_T(x_n), \mathcal{P}_T(q)) \\ &\leq d(x_n, \mathcal{P}_T(x_n)) + 2\,\|x_n - q\| \longrightarrow 0 \text{ as } n \to \infty. \end{aligned}$$

Hence (x_n) converges strongly to a fixed point of T. □

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4. **Applications**

4.1. **Variational inclusion problems.** Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$ and let C is a nonempty, closed, and convex subset of H. Consider the variational inclusion problem

$$\text{Find } x^* \in C \text{ such that } 0 \in Ax^* + Bx^* \tag{4.1}$$

where $A : C \longrightarrow H$ is a single-valued mapping and $B : \operatorname{dom}(B) \rightrightarrows H$ is a multimap.

It is well known (see [10]) that if B is a maximal monotone mapping with $\operatorname{dom}(B) = C$. Then for any given $\lambda > 0$, the solution of the inclusion problem (4.1) coincides with the fixed point of the operator $J_{B,\lambda}(I - \lambda A)$. Indeed, we have

$$0 \in Ax^* + Bx^* \Longleftrightarrow x^* \in J_{B,\lambda}(x^* - \lambda Ax^*) \Longleftrightarrow x^* \in \operatorname{Fix}(J_{B,\lambda}(I - \lambda A))$$

where $J_{B,\lambda} : H \longrightarrow \overline{\operatorname{dom}(B)}$ is the resolvent operator defined by

$$J_{B,\lambda}(x) = (I + \lambda B)^{-1}(x) \quad \forall x \in H.$$

Zeng ([49]) proved that if A is a strongly monotone, continuous mapping and B is a maximal monotone mapping. Then for each $z \in H$, the equation $z \in Ax + Bx$ has a unique solution x^*. Furthermore, the operator

$$T(x) = J_{B,\lambda}(x - \lambda Ax) \qquad (\lambda > 0)$$

is single-valued and firmly nonexpansive. Therefore, $\operatorname{Fix}(T) \neq \varnothing$ and $\mathcal{P}_T(x) = T(x)$ is nonexpansive mapping and $I - \mathcal{P}_T(x)$ is demi-closed operator. Consequently, we can apply the θ-method (2.4) to solve the variational inclusion (4.1).

Theorem 4.1. *Let C be a nonempty, closed and convex subset of real Hilbert space H, and let $A : C \longrightarrow H$ be a strongly monotone, continuous single-valued mapping, and $B : \operatorname{dom}(B) \rightrightarrows H$ be a maximal monotone multimap with $\operatorname{dom}(B) = C$. For $x_0 \in C$ chosen arbitrarily, calculate*

$$x_{n+1} = (1 - \tau_n)x_n + \tau_n\left[\theta J_{B,\lambda}(x_n - \lambda Ax_n) + (1 - \theta)J_{B,\lambda}(x_{n+1} - \lambda Ax_{n+1})\right] \quad \forall n \in \mathbb{N} \tag{4.2}$$

Furthermore, assume that all assumptions of Lemma 3.3 are satisfied. Then $\operatorname{Fix}(T) \neq \varnothing$ and the sequence (x_n) generated by algorithm (4.2) converges weakly to the unique solution of problem (4.1).

If $B = 0$ and $A = \nabla f$ such that ∇f is L-Lipschitz and α-strongly monotone, then problem (4.1) reduces to the convex minimization problem

$$\text{Find } x^* \in C \text{ such that } f(x^*) = \min_{x \in C} f(x) \tag{4.3}$$

where $f : C \longrightarrow H$ is a convex and fréchet differentiable function. Hence, the solution of convex minimization problem (4.3) is determined with the fixed point of the metric projection operator $P_C(I - \lambda\nabla f)$ such that $\lambda > 0$ (see [47]), thus we have

$$f(x^*) = \min_{x \in C} f(x) \Longleftrightarrow x^* = P_C(x^* - \lambda\nabla f(x^*))$$

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Putting $T(x) = P_C\left(x - \lambda \nabla f(x)\right)$ is a nonexpansive and demi-closed operator. Further, if C is a closed and convex set then the metric projection operator is a singleton. Therefore, we have $\mathrm{Fix}(T) \neq \varnothing$ and $\mathcal{P}_T(x) = T(x)$ for all $x \in C$. Hence, we deduce the following convergence result.

Theorem 4.2. *Let C be a nonempty, closed and convex subset of real Hilbert space H, and let $f : C \longrightarrow H$ be a convex and differentiable function, such that ∇f is L-lipschitz and α-strongly monotone operator. Choose $x_0 \in C$ and compute*

$$x_{n+1} = (1-\tau_n)x_n + \tau_n\left[\theta P_C\left(x_n - \lambda \nabla f(x_n)\right) + (1-\theta)P_C\left(x_{n+1} - \lambda \nabla f(x_{n+1})\right)\right] \quad \forall n \in \mathbb{N} \tag{4.4}$$

Moreover, assume that all assumptions of divergence in Lemma 3.3 are satisfied. Then the sequence (x_n) generated by algorithm (4.4) converges weakly to the unique solution of the problem (4.3).

In addition, if $A = \nabla f$ and $B = \partial g$ such that $f : H \longrightarrow \mathbb{R}$ is a convex and differentiable function with L-Lipschitz continuous gradient and $g : H \to \mathbb{R} \cup \{+\infty\}$ is a convex and lower semi-continuous function where ∂g is the subdifferential of g defined by

$$\partial g(x) = \left\{v \in H : \langle y - x; v\rangle \leq g(y) - g(x), \forall y \in H\right\}.$$

The inclusion problem (4.1) reduces to the convex minimization problem

$$\min_{x \in H} f(x) + g(x) \Longleftrightarrow 0 \in \nabla f\left(x^*\right) + \partial g\left(x^*\right) \tag{4.5}$$

where $prox_g$ is the proximity operator defined by

$$prox_g(x) = \arg\min\left\{g(y) + \frac{1}{2}\left\|x - y\right\|_2^2\right\}$$

which might be empty, a singleton, or a set with multiple vectors (for more details see the references [7, 9, 12, 25]).

For $x \in H$ and $\lambda > 0$ we have (see [43] and references therein)

$$\begin{aligned} x^* \in \arg\min_{x \in H} f(x) + g(x) &\Longleftrightarrow x^* \in J_{\partial g, \lambda}\left(x^* - \lambda \nabla f\left(x^*\right)\right) \\ &\Longleftrightarrow x^* = prox_{\lambda g}\left(I - \lambda \nabla f\right)\left(x^*\right). \end{aligned}$$

Now, consider the multimapping $T : H \rightrightarrows H$ such that $T(x) = prox_{\lambda g}\left(I - \lambda \nabla f\right)(x)$. Therefore, the solutions of problem (4.5) coincide with the fixed points of the set-valued forward-backward operator T. Under the Lipschitz property of the operator ∇f and the coercive property of the function $f + g$, Combettes [12] proved that for all $\lambda > 0$ the operator $prox_{\lambda g}\left(I - \lambda \nabla f\right)$ is a singleton. Consequently, we can apply our convergence result to approximate the solution of the convex optimization problem (4.5).

Theorem 4.3. *Let $f, g : H \longrightarrow \mathbb{R} \cup \{+\infty\}$ be two proper convex functions such that f is a differentiable function with L-Lipschitz continuous gradient and g is a lower semi-continuous function where $f + g$ is coercive. Suppose $\lambda > 0$ and $x_0 \in H$ be fixed. By the generalized θ-method, for every $n \in \mathbb{N}$ set:*

$$x_{n+1} = (1-\tau_n)x_n + \tau_n\left[\theta prox_{\lambda g}(x_n - \lambda \nabla f(x_n)) + (1-\theta)prox_{\lambda g}(x_{n+1} - \lambda \nabla f(x_{n+1}))\right]. \tag{4.6}$$

Moreover, assume that all assumptions of Lemma 3.3 are fulfilled. Then the sequence (x_n) *generated by algorithm* (4.6) *converges weakly to a solution of* (4.5).

4.2. **Numerical example.**

Example 4.4. *Let* $X = \mathbb{R}$. *Define the multifunction* $T : X \rightrightarrows X$ *as follows*

$$Tx = \left[1, 1 + \frac{|x|}{2}\right]$$

Thus, $\operatorname{Fix}(T) = \{x \in X : x \in Tx\} = [1, 2]$ *and*

$$\begin{aligned} \mathcal{P}_T(x) &= \{y \in Tx : |x - y| = d(x, Tx)\} \\ &= \begin{cases} \{1 + \frac{x}{2}\} & \textit{if } x > 2 \\ \{x\} & \textit{if } x \in [1, 2] \\ \{1\} & \textit{if } x < 1 \end{cases} \end{aligned}$$

$\mathcal{P}_T$ *is nonexpansive and demiclosed at* 0. *The study is composed of three cases.*
1**st **case: *If* $x > 2$, *we have* $y_n = 1 + \frac{x_n}{2} \in \mathcal{P}_T(x_n)$ *and* $z_n = 1 + \frac{x_{n+1}}{2} \in \mathcal{P}_T(x_{n+1})$. *Then, we obtain the following processus, given the initial point* x_0 *and calculate:*

$$\begin{cases} x_{n+1} = (1 - \tau_n)x_n + \tau_n v_n \\ v_n = \theta\left(1 + \frac{x_n}{2}\right) + (1 - \theta)\left(1 + \frac{x_{n+1}}{2}\right) \end{cases} \tag{4.7}$$

Algorithm (4.7) *converges to* $x^* = 2 \in \operatorname{Fix}(T)$.

We execute algorithm (4.7) *for different choices of the initial guesses, the parameters* (τ_n, θ) *and we study the convergence behavior of the generalized* θ*-method in the 100 first iterations as shown in the above results.*

x_0	3	5	10
x^*	2.0004	2.0011	2.0028

TABLE 1. Influence of initial guesses with $\tau_n = \frac{n}{(n+3)^{3/2}}$, $\theta = 0$

θ	0	0.5	0.6	0.7	1
x^*	2.0118	2.0136	2.0140	2.0144	2.0155

TABLE 2. Influence of the parameter θ with $\tau_n = \frac{1}{(5n+3)^{1/2}}$, $x_0 = 3$

τ_n	$\frac{n}{(2n+5)^{3/2}}$	$\frac{1}{(n+2)^{4/5}}$	$\frac{1}{(5n+3)^{1/2}}$	$\frac{n}{(n+3)^{3/2}}$
x^*	2.0711	2.0284	2.0136	2.0005

TABLE 3. Influence of τ_n with $\theta = 0.5$, $x_0 = 3$

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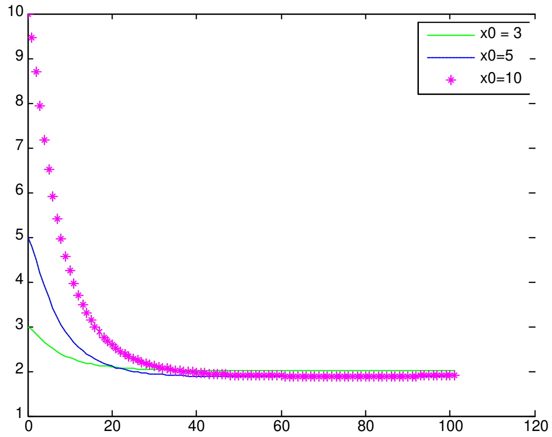

Figure 1. Convergence behavior with different values of initial guesses

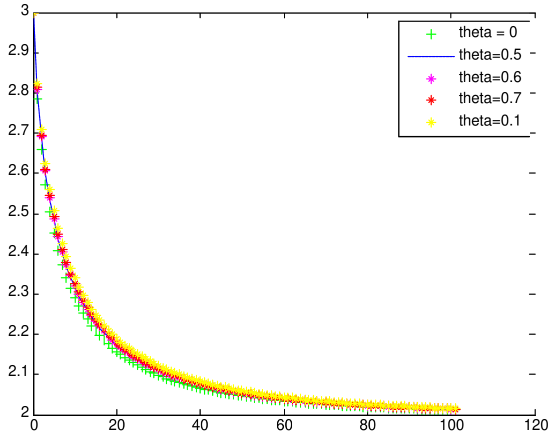

Figure 2. Convergence behavior for different values of θ

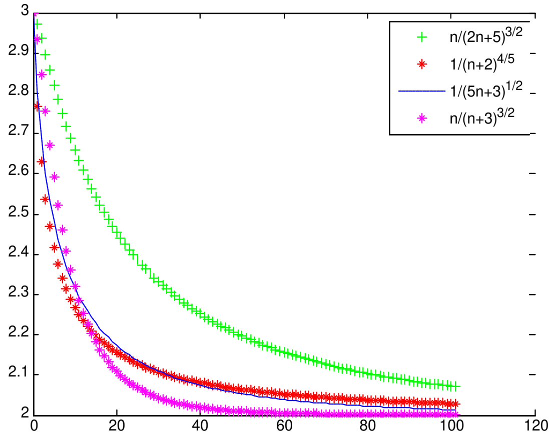

Figure 3. Convergence behavior for different values of τ_n

For initial guesses, the algorithm (4.7) *exhibits convergence to the upper bound of the fixed point interval, and the results in Table* 1 *and Figure* 1 *show that the distance to the fixed point slightly* *increases with larger initial values. This highlights the sensitivity of the method to the starting point, especially outside the fixed point set. Figure 2 and Table 2 demonstrate the influence of the inertial parameter θ, where an increase in θ leads to a slower convergence, suggesting a trade-off between stability and speed. Table 3 and Figure 3 further confirm that the choice of the relaxation sequence τ_n significantly affects convergence: faster-decaying sequences such as $\tau_n = \frac{n}{(n+3)^{3/2}}$ enhance the precision of the limit point.*

2^{nd} case: *If $x \in [1,2]$, we have $y_n = x_n \in \mathcal{P}_T(x_n)$ and $z_n = x_{n+1} \in \mathcal{P}_T(x_{n+1})$. Then, we obtain the following processus, given the initial point $x_0 \in [1,2]$ and calculate*

$$\left\{ \begin{array}{l} x_{n+1} = (1-\tau_n)x_n + \tau_n v_n \\ \\ v_n = \theta x_n + (1-\theta)x_{n+1} \end{array} \right. \tag{4.8}$$

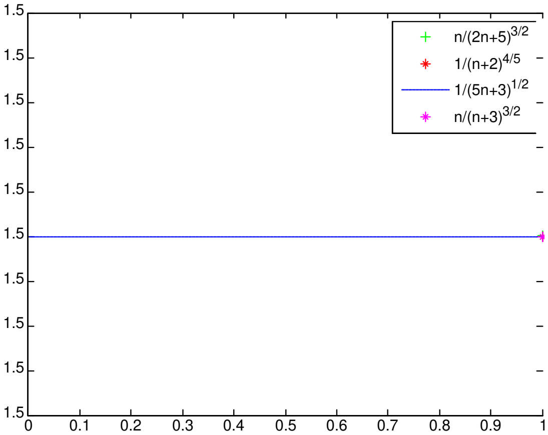

FIGURE 4. convergence behavior of Algorithm (4.8) with $\theta = 1$ and $x_0 = 1.5$

For initial guesses inside the fixed point set, algorithm (4.8) simplifies, and the sequence trivially converges to the initial point itself, as shown in Figure 4.

3^{rd} case: *If $x < 1$, we have $y_n = 1 = z_n \in \mathcal{P}_T(x_n) = \mathcal{P}_T(x_{n+1})$ where the iterative scheme becomes: given the initial point x_0 and,*

$$x_{n+1} = (1-\tau_n)x_n + \tau_n \tag{4.9}$$

For initial values, the projection leads to a constant value 1, and the algorithm (4.9) reduces to a simple relaxation scheme that ensures convergence to the left endpoint of Fix (T)*, as illustrated in Figure 5.*

Finally, the proposed algorithm demonstrates both robustness and convergence across a range of initializations and parameter settings, with empirical findings in agreement with the theoretical framework of the generalized θ-method.

5. Conclusion

In this work, we introduced a generalized iterative scheme that combines the Mann iteration with the implicit θ-method for approximating fixed points of nonexpansive multivalued mappings in Banach

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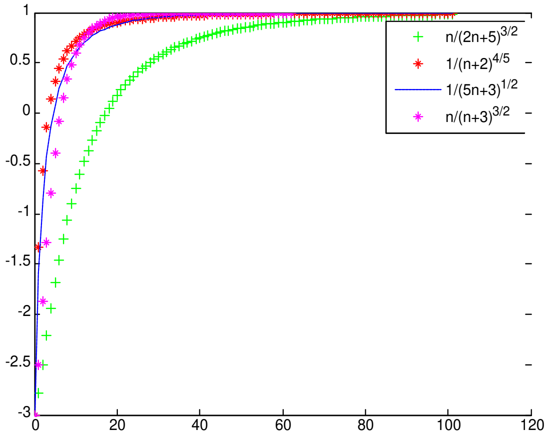

FIGURE 5. convergence behavior of Algorithm (4.9) with $\theta = 0.7$ and $x_0 = -3$

spaces. We established both weak and strong convergence under suitable conditions and demonstrated that our results extend existing methods in the literature. Applications to variational inclusion and convex optimization problems were presented, supported by numerical experiments. These experiments showed that while the initial guess has a limited effect, the method's convergence is sensitive to parameter choices, highlighting the importance of their proper selection.

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