**Journal of the Iranian Mathematical Society**
ISSN (on-line): 2717-1612
J. Iran. Math. Soc. 6 (2025), no. 2, 107-113

# SPACEABILITY OF THE SET OF SURJECTIVE BOUNDED OPERATORS ON BANACH FUNCTION SPACES

A. R. BAGHERI SALEC*, S. M. TABATABAIE AND A. M. JUWEED

*Dedicated to Prof. A. T.-M. Lau*

Abstract. In this work, we prove that the set of all surjective bounded linear operators on a Banach function space in the context of a class of general sets, is spaceable. The obtained results not only prepare some new applications, but also improve the previous versions given for standard Banach sequence spaces.

## 1. **Introduction**

A subset $S$ of a Banach space $\mathcal{X}$ is called *spaceable* in $\mathcal{X}$ if there exists a closed infinite-dimensional linear subspace $V$ of $\mathcal{X}$ such that $V \subseteq S \cup \{0\}$. Also, $S$ is called *lineable*, if $S \cup \{0\}$ contains an infinite-dimensional linear subspace of $\mathcal{X}$. These concepts were first introduced and studied by Aron, Gurariy, and Seoane-Sepúlveda in [1]. Since then, much research has been done on this topic in several branches of mathematics; see [3, 5] and their references to observe some of these results. Among other research, recently, Aron, Bernal-González, Jiménez-Rodríguez, Muñoz-Fernández, and Seoane-Sepúlveda in [2] proved an interesting result about spaceability of the set of surjective bounded linear operators on the sequence spaces $\ell^p$, where $1 \leq p \leq \infty$. Throughout this paper, the set of all bounded linear operators on a Banach space $\mathcal{X}$ will be denoted by $\mathcal{B}(\mathcal{X})$. Also, $\mathcal{SB}(\mathcal{X})$ and $\mathcal{IB}(\mathcal{X})$ denote the set of surjective and the set of injective elements of $\mathcal{B}(\mathcal{X})$, respectively.

---

Communicated by Hamid Reza Ebrahimi Vishki

MSC(2020): Primary: 15A03; Secondary: 46B87.

Keywords: Spaceable sets, Banach function spaces, Banach sequence spaces, surjective operators.

Received: 15 March 2025, Accepted: 14 July 2025.

*Corresponding author

DOI: https://dx.doi.org/10.30504/jims.2025.512613.1250

**Theorem 1.1** ([2, Theorem 4.1 and Corollary 3.4])**.** *Assume that* $p \in [1,\infty]$*. Then, the sets* $\mathcal{SB}(\ell^p)$ *and* $\mathcal{IB}(c_0)$ *are spaceable in* $\mathcal{B}(\ell^p)$*.*

Diniz, Fávaro, Pellegrino and Raposo in [4] extended the above results by replacing the sequence space $\ell^p$ with a $c_{00}$-dense standard Banach sequence space. We begin by recalling the fundamental definition of standard Banach sequence spaces, following the framework established in [4].

**Definition 1.2** ([4])**.** *Let* $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ *be a non-trivial Banach space. Then an infinite-dimensional Banach space* $\mathcal{X}$ *of* $\mathcal{Y}$*-valued sequences is called a* standard Banach sequence space *over* $\mathcal{Y}$*, if the following conditions hold:*

(1) *There is some* $C > 0$ *such that for every* $x = (x_i)_{i=1}^{\infty} \in \mathcal{X}$ *and* $i \in \mathbb{N}$*,*

$$\|x_i\|_{\mathcal{Y}} \leq C\|x\|_{\mathcal{X}} \;;$$

(2) *For every* $x := (x_i)_{i=1}^{\infty} \in \mathcal{X}$ *and any subsequence* $(x_{n_k})_{k=1}^{\infty}$ *of* $x$ *we have* $(x_{n_k})_{k=1}^{\infty} \in \mathcal{X}$ *and* $\|(x_{n_k})_{k=1}^{\infty}\|_{\mathcal{X}} \leq \|x\|_{\mathcal{X}}$ *;*

(3) *If* $(x_i)_{i=1}^{\infty} \in \mathcal{X}$*, and* $n_1 < n_2 < \dots$ *in* $\mathbb{N}$*, then the sequence* $(y_i)_{i=1}^{\infty}$ *belongs to* $\mathcal{X}$*, where* $y_i := x_k$ *if* $i = n_k$ *for some* $k$*, and* $y_i := 0$ *if* $i \notin \{n_1, n_2, \dots\}$*. Moreover,*

$$\|y_i\|_{\mathcal{Y}} \leq \|x_i\|_{\mathcal{Y}}.$$

The results of this article are inspired by the following theorem. By a $c_{00}$*-dense standard Banach sequence space*, we mean a standard Banach sequence space that has $c_{00}$ as a dense subspace.

**Theorem 1.3** ([4, Theorem 1.2 and Theorem 1.3])**.** *Let* $E$ *be a* $c_{00}$*-dense standard Banach sequence space. Then, the set* $\mathcal{SB}(E)$ *is spaceable in* $\mathcal{B}(E)$*. Furthermore, if* $V$ *is an infinite-dimensional Banach space, then the set*

$$\{T : V \to E : T \text{ is linear, bounded and injective}\}$$

*is spaceable in* $\mathcal{B}(V, E)$*.*

Inspired by the above-mentioned results, in this article we present an extension of Theorem 1.3 to some more general categories of Banach function spaces. In this regard, we give a version of the above facts for operators defined on a densely $(\Omega; \mathcal{Y})$-space; see Definition 2.1. Elements of these spaces are not necessarily sequences but also can be functions on a general set.

## 2. Main results

In this section, first we introduce a generalization of standard Banach sequence spaces.

**Definition 2.1.** *Let* $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ *be a non-trivial Banach space, and* $\Omega$ *be a non-empty set. Let* $M_0(\Omega; \mathcal{Y})$ *be the space of all* $\mathcal{Y}$*-valued functions on* $\Omega$*, equipped with pointwise addition and scalar multiplication, and* $\mathcal{X}$ *be a linear subspace of* $M_0(\Omega; \mathcal{Y})$ *equipped with a complete norm* $\|\cdot\|_{\mathcal{X}}$*. Assume that the following hold:*

(1) *There is some constant* $C > 0$*, such that for each* $f \in \mathcal{X}$ *and* $t \in \Omega$*,* $\|f(t)\|_{\mathcal{Y}} \leq C\|f\|_{\mathcal{X}}$*;*

DOI: https://dx.doi.org/10.30504/jims.2025.512613.1250

(2) *There is a collection* $\{\Omega_k\}_{k=1}^{\infty}$ *of disjoint non-empty subsets of* $\Omega$*, such that for each* $k \in \mathbb{N}$*, there is a bijective mapping* $\alpha_k : \Omega \to \Omega_k$ *with* $f \circ \alpha_k \in \mathcal{X}$ *and* $\|f \circ \alpha_k\|_{\mathcal{X}} \leq \|f\|_{\mathcal{X}}$ *for all* $f \in \mathcal{X}$*;*

(3) *For each* $k \in \mathbb{N}$ *and* $g \in \mathcal{X}$*, if we define* $g_{(k)} : \Omega \to \mathcal{Y}$ *as*

$$g_{(k)}(t) = \begin{cases} g \circ \alpha_k^{-1}(t) & \text{if } t \in \Omega_k \\ 0 & \text{if } t \notin \Omega_k, \end{cases}$$

*then* $g_{(k)} \in \mathcal{X}$ *and*

$$\|g_{(k)}(t)\|_{\mathcal{Y}} \leq \|g(t)\|_{\mathcal{Y}}, \qquad (t \in \Omega). \tag{2.1}$$

*Then,* $\mathcal{X}$ *is called an* $(\Omega;\mathcal{Y})$-space related to $\{(\Omega_k, \alpha_k)\}_{k=1}^{\infty}$.

*Moreover, if*

$$\Omega_0 := \Omega \setminus \bigcup_{k=1}^{\infty} \Omega_k,$$

*we say that* $\mathcal{X}$ *is a* densely $(\Omega;\mathcal{Y})$*-space related to* $\{(\Omega_k, \alpha_k)\}_{k=1}^{\infty}$ *whenever the linear span of the set of all elements* $f \in \mathcal{X}$ *such that for some* $k \in \mathbb{N}$ *we have* $\sigma(f) \subseteq \Omega_k \cup \Omega_0$*, is dense in* $\mathcal{X}$*, where*

$$\sigma(f) := \{t \in \Omega : \ f(t) \neq 0\}.$$

**Remark 2.2.** *Assume that* $\{\Omega_k\}_{n=1}^{\infty}$ *is a partition of* $\mathbb{N}$ *such that for any* $k$*,* $\Omega_k$ *is infinite, and let for each* $k$*,* $\alpha_k : \mathbb{N} \to \Omega_k$ *be a strictly increasing bijection. Then, every standard sequence space over* $\mathcal{Y}$ *is an* $(\mathbb{N}, \mathcal{Y})$*-space related to* $\{(\Omega_k, \alpha_k)\}_{k=1}^{\infty}$*.*

**Remark 2.3.** *Let* $\mathcal{X}$ *be an* $(\Omega;\mathcal{Y})$*-space related to* $\{(\Omega_k, \alpha_k)\}_{k=1}^{\infty}$*. Let* $f \in \mathcal{X}$ *and* $k \in \mathbb{N}$*. Then,*

$$f_{(k)} \circ \alpha_k = \chi_{\Omega_k} f,$$

*so, by conditions* (2)-(3) *in Definition* 2.1*, we have* $\chi_{\Omega_k} f \in \mathcal{X}$*, and* $\|\chi_{\Omega_k} f\|_{\mathcal{X}} \leq \|f_{(k)}\|_{\mathcal{X}}$*.*

**Remark 2.4.** *Note that the condition* (2) *in Definition* 1.2 *of a* $c_{00}$*-dense standard Banach sequence space which was introduced in* [4] *is stronger than the corresponding condition (2) in Definition* 2.1*, because the collection of strictly increasing sequences of natural numbers is uncountable.*

Now we are ready to state the main theorem of this paper. The above remark shows that the following theorem is a generalization of the first part of Theorem 1.3.

**Theorem 2.5.** *Let* $\mathcal{X}$ *be a densely* $(\Omega;\mathcal{Y})$*-space related to* $\{(\Omega_k, \alpha_k)\}_{k=1}^{\infty}$*. Then, the set* $\mathcal{SB}(\mathcal{X})$ *is spaceable in* $\mathcal{B}(\mathcal{X})$*.*

*Proof.* For each $k \in \mathbb{N}$, we define the mapping $S_k : \mathcal{X} \to \mathcal{X}$ with

$$S_k(f) := f \circ \alpha_k$$

for all $f \in \mathcal{X}$. Then, by the condition (2) in Definition 2.1, we have

$$\|S_k(f)\|_{\mathcal{X}} = \|f \circ \alpha_k\|_{\mathcal{X}} \leq \|f\|_{\mathcal{X}}$$

DOI: https://dx.doi.org/10.30504/jims.2025.512613.1250

for all $f \in \mathcal{X}$. This implies that $\{S_k : k \in \mathbb{N}\} \subseteq \mathcal{B}(\mathcal{X})$, and

$$\|S_k\| \leq 1, \text{ for all } k \in \mathbb{N}. \tag{2.2}$$

Note that $S_k$ is surjective for all $k$. Indeed, for every $g \in \mathcal{X}$, by the condition (3) in Definition 2.1 we have $f := g_{(k)} \in \mathcal{X}$ and

$$S_k(f)(t) = (f \circ \alpha_k)(t) = g_{(k)}(\alpha_k(t)) = (g \circ \alpha_k^{-1})(\alpha_k(t)) = g(t)$$

for all $t \in \Omega$, because $\alpha_k(t) \in \Omega_k$, thus $S_k(f) = g$. Fix some arbitrary $n \in \mathbb{N}$. We show that

$$\left\{\sum_{k=1}^{n} b_k S_k : (b_1, \ldots, b_n) \in \mathbb{C}^n \setminus \{0\}\right\} \subseteq \mathcal{SB}(\mathcal{X}).$$

For this, denote

$$S := \sum_{k=1}^{n} b_k S_k$$

with $(b_1, \ldots, b_n) \in \mathbb{C}^n \setminus \{0\}$. Clearly, $S$ is linear. We prove that the formula of $S_k$'s guarantees that $S$ is surjective. For this, assume that $g \in \mathcal{X}$, and let $k$ be an index with $b_k \neq 0$. Put $f := b_k^{-1} g_{(k)}$ (for definition of $g_{(k)}$ see the condition (3) in Definition 2.1). Then, $f \in \mathcal{X}$ and since the sets $\Omega_j$'s are disjoint and $\alpha_j(\Omega) \subseteq \Omega_j$, for each $t \in \Omega$ we have

$$\begin{aligned} S(f)(t) &= \sum_{j=1}^{n} b_j S_j(f)(t) = \sum_{j=1}^{n} b_j \ (f \circ \alpha_j)(t) \\ &= \sum_{j=1}^{n} b_k^{-1} b_j g_{(k)}(\alpha_j(t)) = b_k^{-1} b_k g_{(k)}(\alpha_k(t)) = g(t). \end{aligned}$$

This implies that $S(f) = g$, and so $S$ is surjective. For linearly independence of the set $\{S_k : k \in \mathbb{N}\}$, assume that $m \in \mathbb{N}$, $c_1, \ldots, c_m \in \mathbb{C}$ and $\sum_{j=1}^{m} c_j S_j = 0$. If one of the numbers $c_1, \ldots, c_m$ is non-zero, then $(c_1, \ldots, c_m) \in \mathbb{C}^m \setminus \{0\}$. Thus, by the above argument, $\sum_{j=1}^{m} c_j S_j$ is surjective and so non-zero, a contradiction. This means that the set $\{S_k : k \in \mathbb{N}\}$ is a linearly independent subset of $\mathcal{B}(\mathcal{X})$. Now, we define

$$V := \left\{\sum_{k=1}^{\infty} b_k S_k : (b_k)_k \in \ell^1\right\}. \tag{2.3}$$

In the sequel, we will show that $V \subseteq \mathcal{SB}(\mathcal{X}) \bigcup \{0\}$, and also $V$ is a closed infinite-dimensional linear subspace of $\mathcal{B}(\mathcal{X})$. Let $0 \neq (b_j)_j \in \ell^1$, and set $S := \sum_{j=1}^{\infty} b_j S_j$. First, note that

$$\sum_{j=1}^{\infty} \|b_j S_j\| = \sum_{j=1}^{\infty} |b_j| \, \|S_j\| \leq \sum_{j=1}^{\infty} |b_j| < \infty$$

because for each $j$ by (2.2) we have $\|S_j\| \leq 1$. Hence, $\sum_{j=1}^{\infty} b_j S_j$ is absolutely convergent in $\mathcal{B}(\mathcal{X})$, and since $\mathcal{B}(\mathcal{X})$ is a Banach space with the operator norm, the series $\sum_{j=1}^{\infty} b_j S_j$ is convergent in $\mathcal{B}(\mathcal{X})$. This means that $S \in \mathcal{B}(\mathcal{X})$. Now, let $g \in \mathcal{X}$. Since $(b_j)_j \neq 0$, there is an index $k$ such that $b_k \neq 0$. Setting $f := b_k^{-1} g_{(k)}$, similar to the above argument, one can see that $S(f) = g$. Therefore, $V \subseteq \mathcal{SB}(\mathcal{X}) \bigcup \{0\}$. Trivially, $V$ is infinite-dimensional because $\{S_k : k \in \mathbb{N}\}$ is an infinite linearly independent subset of

DOI: https://dx.doi.org/10.30504/jims.2025.512613.1250

$V$. Let $V_0$ be the closure of V in $\mathcal{B}(\mathcal{X})$. Clearly, $V_0$ is a closed infinite dimensional subspace of $\mathcal{B}(\mathcal{X})$. Assume that $S$ is an arbitrary element of $V_0 \setminus \{0\}$. It would be enough to prove that $S$ is surjective. Pick some $g \in \mathcal{X}$. So, there is $(T_n)_n$ in $V$ such that $\|T_n - S\| \to 0$ as $n \to \infty$. For each $n \in \mathbb{N}$, put

$$T_n := \sum_{k=1}^{\infty} b_k^{(n)} S_k,$$

where $(b_k^{(n)})_k \in \ell^1$. Then, for any $f \in \mathcal{X}$,

$$\|T_n(f) - S(f)\|_{\mathcal{X}} \leq \|T_n - S\| \, \|f\|_{\mathcal{X}}.$$

This implies that

$$S(f) = \lim_{n\to\infty} T_n(f) = \lim_{n\to\infty} \left( \sum_{k=1}^{\infty} b_k^{(n)} S_k \right)(f) = \lim_{n\to\infty} \sum_{k=1}^{\infty} b_k^{(n)} S_k(f)$$

for all $f \in \mathcal{X}$. By the condition (1) in Definition 2.1,

$$\left\| S(f)(t) - \sum_{k=1}^{\infty} b_k^{(n)} S_k(f)(t) \right\|_{\mathcal{Y}} \leq \left\| S(f) - \sum_{k=1}^{\infty} b_k^{(n)} S_k(f) \right\|_{\mathcal{X}}$$

for all $t \in \Omega$ and $n \in \mathbb{N}$. Hence,

$$S(f)(t) = \lim_{n\to\infty} \sum_{k=1}^{\infty} b_k^{(n)} S_k(f)(t)$$

for all $t \in \Omega$. So, by definition of the operators $S_k$,

$$S(f)(t) = \lim_{n\to\infty} \sum_{k=1}^{\infty} b_k^{(n)} \, f(\alpha_k(t)) \tag{2.4}$$

for all $t \in \Omega$ and $f \in \mathcal{X}$. Let $\Gamma$ be the set of all elements $f \in \mathcal{X}$ that for some $k \in \mathbb{N}$ we have $\sigma(f) \subseteq \Omega_k \cup \Omega_0$. Then, by the hypothesis, $\operatorname{span}(\Gamma)$ is dense in $\mathcal{X}$. Hence, since $S$ is continuous and linear, there is an element $h \in \Gamma$ such that $S(h) \neq 0$. There is an index $l \in \mathbb{N}$ such that $\sigma(h) \subseteq \Omega_l \cup \Omega_0$. In this case, for some $t \in \Omega$ we have

$$0 \neq S(h)(t) = \lim_{n\to\infty} \sum_{k=1}^{\infty} b_k^{(n)} S_k(h)(t) = \lim_{n\to\infty} \sum_{k=1}^{\infty} b_k^{(n)} h(\alpha_k(t)) = \lim_{n\to\infty} b_l^{(n)} h(\alpha_l(t)).$$

Hence, $\lim_{n\to\infty} b_l^{(n)} \neq 0$. Now, we define the function $f : \Omega \to \mathcal{Y}$ by

$$f := \left( \lim_{n\to\infty} b_l^{(n)} \right)^{-1} g_{(l)}.$$

Then $f \in \mathcal{X}$ and by (2.4) for each $t \in \Omega$ we have

$$\begin{aligned} S(f)(t) &= \lim_{n\to\infty} \sum_{k=1}^{\infty} b_k^{(n)} \left( \lim_{n\to\infty} b_l^{(n)} \right)^{-1} g_{(l)}(\alpha_k(t)) \\ &= \lim_{n\to\infty} b_l^{(n)} \left( \lim_{n\to\infty} b_l^{(n)} \right)^{-1} g_{(l)}(\alpha_l(t)) = g(t), \end{aligned}$$

and so $S(f) = g$. Hence, $V_0$ is a closed infinite dimensional subspace of $\mathcal{B}(\mathcal{X})$ and

$$V_0 \subseteq \{0\} \cup \mathcal{SB}(\mathcal{X}). \tag{2.5}$$

DOI: https://dx.doi.org/10.30504/jims.2025.512613.1250

Now, we show that $V_0 \neq \mathcal{B}(\mathcal{X})$. In contrast, assume that $V_0 = \mathcal{B}(\mathcal{X})$. Then, by (2.5), $\{0\} \cup \mathcal{SB}(\mathcal{X}) = \mathcal{B}(\mathcal{X})$. On the other hand, the above conclusion shows that $\mathcal{B}(\mathcal{X})$ (and so $\mathcal{X}$) is infinite dimensional. If $0 \neq u \in \mathcal{X}$ is fixed, then by Hahn-Banach Theorem, there is some $h \in \mathcal{X}^*$ such that $h(u) \neq 0$. Then, the operator $T : \mathcal{X} \to \mathcal{X}$ defined by $T(x) := h(x)u$ for all $x \in \mathcal{X}$, is a non-surjective non-zero element of $\mathcal{B}(\mathcal{X})$, a contradiction. This implies that $V_0 \neq \mathcal{B}(\mathcal{X})$. Therefore, by definition, the set $\mathcal{SB}(\mathcal{X})$ is spaceable in $\mathcal{B}(\mathcal{X})$. □

Note that in the above proof, we have not used the relation (2.1). So, we can conclude the next result.

**Corollary 2.6.** *Let $p \geq 1$, $G$ be a discrete group and $H$ be a subgroup of $G$ such that the quotient space $\frac{G}{H}$ is infinite. Then, $\mathcal{SB}(\ell^p(G))$ is spaceable in $\mathcal{B}(\ell^p(G))$.*

*Proof.* Assume that $H$ is a subgroup of a discrete group $G$ such that $\frac{G}{H}$ is infinite. Then, there are $a_1, a_2, \ldots \in G$ such that for every distinct $k, j \in \mathbb{N}$, $a_kH \cap a_jH = \varnothing$. Now, for each $k \in \mathbb{N}$ we define

$$\alpha_k : G \to a_kH, \qquad x \mapsto a_kx.$$

Then, easily by some calculation, one can verify that except for the relation (2.1), all conditions for $\ell^p(G)$ related to $\{(a_kH, \alpha_k)\}_{k=1}^{\infty}$ to be a $(G;\mathbb{C})$-space hold. Moreover, since the collection of distinct cosets of $H$ regarding the group $G$ is a partition of $G$, the linear span of the set of all elements of $f \in \ell^p(G)$ such that for some $k \in \mathbb{N}$ we have $\sigma(f) \subseteq a_kH \cup \Omega_0$, is dense in $\ell^p(G)$. Therefore, the statement would be a conclusion of Theorem 2.5. □

For the next example, recall that $\ell^p(\mathbb{R})$ consists all functions $f : \mathbb{R} \to \mathbb{C}$ with $\sum_{x\in\mathbb{R}} |f(x)|^p < \infty$. So, it can not be covered by the results of [4].

**Example 2.7.** *Consider the additive group $\mathbb{R}$ equipped with the discrete topology. Then, $\frac{\mathbb{R}}{\mathbb{Z}}$ is uncountable because for each distinct $\alpha, \beta \in (0,1)$ we have $\alpha + \mathbb{Z} \neq \beta + \mathbb{Z}$. Thus, thanks to the above corollary, the set $\mathcal{SB}(\ell^p(\mathbb{R}))$ is spaceable in $\mathcal{B}(\ell^p(\mathbb{R}))$.*

The following corollary is an improvement of Theorem 1.3. Just note that here the arbitrary subsequences in the conditions (2)-(3) of Definition 1.2 have been replaced by some subsequences with specified indexes.

**Corollary 2.8.** *Assume that $\mathcal{X}$ is a $c_{00}$-dense infinite-dimensional Banach space of $\mathcal{Y}$-valued sequences such that there is some $C > 0$ such that for every $x = (x_i)_{i=1}^{\infty} \in \mathcal{X}$ and $i \in \mathbb{N}$, $\|x_i\|_{\mathcal{Y}} \leq C\|x\|_{\mathcal{X}}$. Also, assume that there are distinct numbers $n_k^{(j)} \in \mathbb{N}$, where $j, k \in \mathbb{N}$, with*

$$\{n_k^{(j)} : j, k \in \mathbb{N}\} \subseteq \mathbb{N},$$

*and $n_k^{(j)} < n_{k+1}^{(j)}$ for all $j, k \in \mathbb{N}$ such that for all $x = (x_i)_{i=1}^{\infty} \in \mathcal{X}$ and each $j \in \mathbb{N}$, the followings hold:*

1. *$(x_{n_k^{(j)}})_{k=1}^{\infty} \in \mathcal{X}$ and $\|(x_{n_k^{(j)}})_{k=1}^{\infty}\|_{\mathcal{X}} \leq \|x\|_{\mathcal{X}}$,*
2. *if for every $i \in \mathbb{N}$ we set $y_i^{(j)} := x_k$ whenever for some $k$, $i = n_k^{(j)}$ and $y_i^{(j)} := 0$ otherwise, then $(y_i^{(j)})_{i=1}^{\infty} \in \mathcal{X}$ and $\|y_i^{(j)}\|_{\mathcal{Y}} \leq \|x_i\|_{\mathcal{Y}}$ for all $i \in \mathbb{N}$.*

*Assume that the span of all sequences* $(x_n)_n \in \mathcal{X}$ *such that for some* $j$*,*

$$\{n \in \mathbb{N} : x_n \neq 0\} \subseteq \{n_k^{(j)} : k \in \mathbb{N}\} \cup (\mathbb{N} \setminus \{n_k^{(s)} : k, s \in \mathbb{N}\}),$$

*is dense in* $\mathcal{X}$*. Then,* $\mathcal{SB}(\mathcal{X})$ *is spaceable in* $\mathcal{B}(\mathcal{X})$*.*

## Acknowledgements

The authors would like to thank the referee of this paper for his/her helpful remarks and suggestions which improved this article.

## REFERENCES

[1] R. M. Aron, V. I. Gurariy, and J. B. Seoane-Sepúlveda, Lineability and spaceability of sets of functions on ℝ, *Proc. Am. Math. Soc.* **133** (2005), no. 3, 795–803.

[2] R. M. Aron, L. Bernal-González, P. Jiménez-Rodríguez, G. Muñoz-Fernández, and J.B. Seoane-Sepúlveda, On the size of special families of linear operators, *Linear Algebra Appl.* **544** (2018) 186–205.

[3] R. M. Aron, L. Bernal-González, D. Pellegrino, and J. B. Seoane-Sepúlveda, Lineability: The Search for Linearity in Mathematics, *Monographs and Research Notes in Mathematics*, CRC, Boca Raton, 2016.

[4] D. Diniz, V. V. Fávaro, D. Pellegrino and A. Raposo, Spaceability of the sets of surjective and injective operators between sequence spaces, *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat RACSAM* **114** (2020), no. 4, 114–194.

[5] V. V. Fávaro, D. Pellegrino and D. Tomaz, Lineability and spaceability: a new approach, *Bull. Braz. Math. Soc. New Ser.* **51** (2020), no. 1, 27–46.

**AliReza Bagheri Salec**
Department of Mathematics, University of Qom, P.O. Box 37161, Qom, Iran.
Email: `r-bagheri@qom.ac.ir`

**Seyyed Mohammad Tabatabaie**
Department of Mathematics, University of Qom, P.O. Box 37161, Qom, Iran.
Email: `sm.tabatabaie@qom.ac.ir`

**Amer Mahdi Juweed**
Department of Mathematics, University of Qom, P.O. Box 37161, Qom, Iran.
Email: `ameralnasry16@gmail.com`

DOI: https://dx.doi.org/10.30504/jims.2025.512613.1250