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SOME FAMILIAR GRAPHS ON THE RINGS OF MEASURABLE FUNCTIONS

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Dedicated to Prof. O. A. S. Karamzadeh

Abstract. On the ring of real-valued measurable functions $\mathcal{M}(X,\mathcal{A})$, we redefine the co-maximal graph $\Gamma_2'(\mathcal{M}(X,\mathcal{A}))$, the annihilator graph $AG(\mathcal{M}(X,\mathcal{A}))$ and the weakly zero-divisor graph $W\Gamma(\mathcal{M}(X,\mathcal{A}))$ with the help of a measure μ defined on the measurable space $(X,\mathcal{A})$. First we observe that the vertex set of $\Gamma_2'(\mathcal{M}(X,\mathcal{A}))$ is equal to the vertex set of the zero-divisor graph $\Gamma(\mathcal{M}(X,\mathcal{A}))$ of $\mathcal{M}(X,\mathcal{A})$. We show that, $\Gamma_2'(\mathcal{M}(X,\mathcal{A}))$ and $\Gamma(\mathcal{M}(X,\mathcal{A}))$ are not isomorphic as graphs in general, nevertheless we deduce a sufficient condition for them to be isomorphic as graphs. We establish a condition for which $\Gamma_2'(\mathcal{M}(X,\mathcal{A}))$, $\Gamma(\mathcal{M}(X,\mathcal{A}))$ and $AG(\mathcal{M}(X,\mathcal{A}))$ are equal. We further characterise the non-atomicity of the measure space $(X,\mathcal{A},\mu)$ by some graph-theoretic properties of $\Gamma_2'(\mathcal{M}(X,\mathcal{A}))$ and $AG(\mathcal{M}(X,\mathcal{A}))$. Moreover, we realize that $W\Gamma(\mathcal{M}(X,\mathcal{A}))$ is a complete partite graph.

1. Introduction

Suppose that $(X,\mathcal{A})$ is a measurable space. A real-valued function f on X is said to be measurable, if for any $r\in\mathbb{R}$, $f^{-1}(r,\infty)$ is a measurable set. The collection $\mathcal{M}(X,\mathcal{A})$ of all real-valued measurable functions on X forms a commutative lattice-ordered ring with unity with respect to the usual pointwise operations on X [see [11,12]]. For each $A\subset X$, the characteristic function of A is denoted by 1_A and is defined by

$$1_A(x)=\begin{cases}1 & \text{if } x\in A\\ 0 & \text{otherwise}\end{cases}$$

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If A is a singleton set $\{x\}$, then we simply write 1_A as 1_x. Clearly, $1_A \in \mathcal{M}(X, \mathcal{A})$ if and only if $A \in \mathcal{A}$. Let μ be a measure on $(X, \mathcal{A})$. We always consider $\mu(X) > 0$, otherwise it renders everything to triviality. $A \in \mathcal{A}$ is called an atom, if $\mu(A) > 0$ and A can not be written as the union of two disjoint measurable sets with positive measure [see [9]]. The measure space $(X, \mathcal{A}, \mu)$ or the measure μ is said to be atomic, if every measurable sets with positive measure contains an atom. On the contrary, $(X, \mathcal{A}, \mu)$ is called non-atomic, if it contains no atom. Beck, in [5], was the first one who introduced the zero-divisor graph defined over a ring. In particular, the zero-divisor graph over the ring $C(X)$ of all real-valued continuous functions over a topological space X was introduced in [3]. Later, the co-maximal graph [in [8]], the annihilator graph [in [1]], the weakly zero-divisor graph [in [7]], etc. over a ring are introduced to discover the possible interaction between the properties of the ring with those of the associated graphs. Here, we redefine those above-mentioned graphs on the ring $\mathcal{M}(X, \mathcal{A})$ in the "almost everywhere sense" concerning the measure μ.

In section 2, certain parameters, viz. the distance between two vertices, the diameter, the eccentricity of a vertex, the girth of the graph, etc., of the co-maximal graph of $\mathcal{M}(X, \mathcal{A})$ are calculated. A necessary and sufficient condition for the co-maximal graph of $\mathcal{M}(X, \mathcal{A})$ to be complete bipartite is proved [Theorem 2.11]. We also prove that the triangulatedness of the co-maximal graph characterises the non-atomocity of the measure μ [Corollary 2.15]. The zero-divisor graph of $\mathcal{M}(X, \mathcal{A})$ is recently introduced in [4]. We also find out that the vertex set of the co-maximal graph of $\mathcal{M}(X, \mathcal{A})$ is equal to the vertex set of the zero-divisor graph [Remark 2.3]. This motivates us to find out the interrelation between these two graphs, if there is any.

In section 3, we introduce a much smaller subgraph of the co-maximal graph of $\mathcal{M}(X, \mathcal{A})$ which exhibits a similar behaviour as the parent graph. A similar subgraph for the zero-divisor graph of $\mathcal{M}(X, \mathcal{A})$ is also introduced. Through these subgraphs, we find out a sufficient condition for the zero-divisor graph and co-maximal graph of $\mathcal{M}(X, \mathcal{A})$ to be equal up to isomorphism [Theorem 3.10]. We further show that these two graphs of $\mathcal{M}(X, \mathcal{A})$ over the Lebesgue measure space (over $\mathbb{R}$) are isomorphic. However, the zero-divisor graph and the co-maximal graph of $\mathcal{M}(X, \mathcal{A})$ over the counting measure space are not isomorphic in general [Theorem 3.19].

In sections 4, we examine the annihilator graph of $\mathcal{M}(X, \mathcal{A})$. Eventually, we find out that the zero-divisor graph and the co-maximal graph are subgraphs of the annihilator graph of $\mathcal{M}(X, \mathcal{A})$ [Theorem 4.5]. We also show when these two graphs are equal to the annihilator graph of $\mathcal{M}(X, \mathcal{A})$ [Theorem 4.7]. We establish that the non-atomicity of the measure space follows from the hypertriangulatedness of the annihilator graph [Theorem 4.12]. The condition when and only when the annihilator graph is complemented is also found out [Theorem 4.14]. We also determine the dominating number of the annihilator graph [Theorem 4.16]. We further explore the weakly zero-divisor graph over a redefined vertex set of $\mathcal{M}(X, \mathcal{A})$ in section 5. We show that a weakly zero-divisor graph is a complete partite graph [Theorem 5.3].

2. **Prerequisites**

Let G be a simple graph with vertex set V. The distance $d(u,v)$ between two distinct vertices u,v is the length of the shortest path joining them in G. The eccentricity $ecc(v)$ of a vertex v is the maximum distance of v from other vertices of G. The diameter $diam(G)$ is the maximum distance between a pair of vertices, and the girth $gr(G)$ is the length of the smallest cycle in G. G is triangulated or hypertriangulated according as every vertex is a vertex of a triangle or every edge is an edge of a triangle in G. For two distinct vertices $u,v \in V$, $c(u,v)$ denotes the length of the smallest cycle in G containing u,v. Two vertices u,v are said to be orthogonal in G, denoted by $u \perp v$, if u,v are adjacent and $c(u,v) > 3$. G is called a complemented graph, if for every $u \in V$, there exists $v \in V$ such that $u \perp v$. A complemented graph is called uniquely complemented, if whenever $u \perp v$ and $u \perp w$ in G for $u,v,w \in V$, then v,w are adjacent to the same set of vertices in G. The clique number $cl(G)$ of G is the maximum cardinality of complete subgraphs of G. A subset V_1 of V is a dominating set in G, if for every $u \in V \setminus V_1$, u,v are adjacent in G for some $v \in V_1$ and the dominating number $dt(G) = min\{|V_1| : V_1 \text{ is a dominating set in } G\}$. A subset V_1 of V is a total dominating set in G, if for every $u \in V$, u,v are adjacent in G for some $v \in V_1$ and the total dominating number $dt_t(G) = min\{|V_1| : V_1 \text{ is a total dominating set in } G\}$. For a cardinal number α, let $|A_\alpha| = \alpha$. Then G is said to be α-colorable, if there is a map $\psi : V \to A_\alpha$ such that $\psi(u) \neq \psi(v)$ whenever u,v are adjacent in G. The chromatic number $\chi(G)$ of G is $min\{\alpha : G \text{ is } \alpha\text{-colorable}\}$. For any subset V' of V, the induced subgraph G' of G induced by V' is the graph whose vertex set is V' and two vertices in G' are adjacent, if they are adjacent in G. Let G_1, G_2 be two graphs having V_1, V_2 as sets of vertices, respectively. A bijection map $\psi : V_1 \to V_2$ is called a graph isomorphism between two graphs G_1, G_2, if it preserves the adjacency relations. For more graph-related terms, we refer to [10].

Let R be a commutative ring. The zero-divisor graph $\Gamma(R)$ of R has its vertex set as the set of non-zero zero-divisors with two distinct vertices x,y being adjacent if and only if $x.y = 0$ [see [2,5]]. The co-maximal graph $\Gamma'_2(R)$ of R, defined in [8], has its vertices as elements of R, where two distinct vertices x and y are adjacent if and only if the sum of the principal ideals generated by x,y is R. Later in [6], the vertex set of the co-maximal graph of R is redefined to be $R \setminus [U(R) \cup J(R)]$, here $U(R)$ and $J(R)$ be the set of all units in R and the Jacobson radical of R respectively. The annihilator graph $AG(R)$ of R, introduced in [1], is a supergraph of $\Gamma(R)$ having the same set of vertices and two distinct vertices x,y are adjacent if and only if $ann(x) \cup ann(y) \subsetneqq ann(x \cdot y)$, where $ann(a) = \{r \in R : a \cdot r = 0\}$ is the annihilator ideal of a in R. The weakly zero-divisor graph $W\Gamma(R)$ of R, introduced in [7], is also a supergraph of $\Gamma(R)$ with the same set of vertices where two distinct vertices x,y are adjacent if and only if there exist $a \in ann(x) \setminus \{0\}$ and $b \in ann(y) \setminus \{0\}$ such that $a \cdot b = 0$.

3. **The co-maximal graph $\Gamma'_2(\mathcal{M}(X,\mathcal{A}))$ of $\mathcal{M}(X,\mathcal{A})$**

For each $f \in \mathcal{M}(X,\mathcal{A})$, $Z(f) = \{x \in X : f(x) = 0\}$ denotes the zero-set of f in X. An $f \in \mathcal{M}(X,\mathcal{A})$ which is non-zero almost everywhere on X is said to be a zero-divisor in $\mathcal{M}(X,\mathcal{A})$, if there exists $g \in \mathcal{M}(X,\mathcal{A})$, non-zero almost everywhere on X, such that $f \cdot g \equiv 0$ almost everywhere on X. Let

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$\mathscr{D}(\mathcal{M}(X,\mathcal{A}))$ (in short, $\mathscr{D}$) be the set of all zero-divisors of $\mathcal{M}(X,\mathcal{A})$. Then

$$\mathscr{D}(\mathcal{M}(X,\mathcal{A})) = \{f \in \mathcal{M}(X,\mathcal{A}) : \mu(Z(f)) > 0 \text{ and } \mu(X \setminus Z(f)) > 0\}$$

The following result can be proved using routine arguments.

Theorem 3.1. *The following statements are equivalent:*

1. *Every measurable set with positive measure is an atom;*
2. *X itself is an atom;*
3. *$\mathscr{D}(\mathcal{M}(X,\mathcal{A}))$ is an empty set.*

The Dirac measure space is an example of such measure space where X is itself an atom. In view of Theorem 2.1, throughout this article, we make the convention that $\mathscr{D}(\mathcal{M}(X,\mathcal{A})) \neq \emptyset$, i.e., X is not an atom.

Definition 3.2.

1. *A function $f \in \mathcal{M}(X,\mathcal{A})$ is called a μ-unit in $\mathcal{M}(X,\mathcal{A})$, if $\mu(Z(f)) = 0$. We denote the set of all μ-units in $\mathcal{M}(X,\mathcal{A})$ by $U(\mathcal{M}(X,\mathcal{A}))$.*
2. *An ideal I in $\mathcal{M}(X,\mathcal{A})$ is said to be a μ-principal ideal in $\mathcal{M}(X,\mathcal{A})$, if I is a principal ideal generated by a μ-unit in $\mathcal{M}(X,\mathcal{A})$.*

The Jacobson radical, $J(\mathcal{M}(X,\mathcal{A}))$, of $\mathcal{M}(X,\mathcal{A})$ which is the intersection of all maximal ideals of $\mathcal{M}(X,\mathcal{A})$ is $\{0\}$. In the almost everywhere sense, let $J(\mathcal{M}(X,\mathcal{A})) = \{f \in \mathcal{M}(X,\mathcal{A}) : \mu(X \setminus Z(f)) = 0\}$. The co-maximal graph $\Gamma'_2(\mathcal{M}(X,\mathcal{A}))$ (in short, Γ'_2) of $\mathcal{M}(X,\mathcal{A})$ is defined on the vertex set $\mathcal{M}(X,\mathcal{A}) \setminus [U(\mathcal{M}(X,\mathcal{A})) \cup J(\mathcal{M}(X,\mathcal{A}))]$. Two distinct vertex f, g in $\Gamma'_2(\mathcal{M}(X,\mathcal{A}))$ are adjacent if and only if the sum of the principal ideals generated by f, g is a μ-principal ideal in $\mathcal{M}(X,\mathcal{A})$.

Remark 3.3. *The vertex set of the co-maximal graph $\Gamma'_2(\mathcal{M}(X,\mathcal{A}))$ is $\mathscr{D}(\mathcal{M}(X,\mathcal{A}))$. Hence the vertex sets of the zero-divisor graph and the co-maximal graph of $\mathcal{M}(X,\mathcal{A})$ are equal.*

The adjacency relation in $\Gamma'_2(\mathcal{M}(X,\mathcal{A}))$ in the measure-theoretic scenario is given by the following theorem.

Theorem 3.4. *Let $f, g \in \mathscr{D}(\mathcal{M}(X,\mathcal{A}))$. Then f, g are adjacent in $\Gamma'_2(\mathcal{M}(X,\mathcal{A}))$ if and only if $\mu(Z(f) \cap Z(g)) = 0$.*

Proof. If f, g are adjacent in Γ'_2, then $\mu(Z(f) \cap Z(g)) = 0$ follows from $\langle f^2 + g^2 \rangle \subseteq \langle f \rangle + \langle g \rangle$, where $\langle h \rangle$ denotes the principal ideal in $\mathcal{M}(X,\mathcal{A})$ generated by h. Conversely let $\mu(Z(f) \cap Z(g)) = 0$. It suffices to show that $\langle f \rangle + \langle g \rangle = \langle f^2 + g^2 \rangle$. Clearly, $f \in \langle f^2 + g^2 \rangle$, because $f = (f^2 + g^2).f_1$, where

$$f_1(x) = \begin{cases} \frac{f(x)}{f^2(x)+g^2(x)} & \text{if } x \notin Z(f) \cap Z(g) \\ 0 & \text{if } x \in Z(f) \cap Z(g) \end{cases}$$

and similarly, $g \in \langle f^2 + g^2 \rangle$. □

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The following theorem provides condition under which a pair of vertices of $\Gamma_2'(\mathcal{M}(X,\mathcal{A}))$ admits of a third vertex adjacent to both of them.

Theorem 3.5. *Let $f, g \in \mathscr{D}(\mathcal{M}(X,\mathcal{A}))$. Then there is a vertex in $\mathscr{D}(\mathcal{M}(X,\mathcal{A}))$ adjacent to both f, g in $\Gamma_2'(\mathcal{M}(X,\mathcal{A}))$ if and only if $\mu(X \setminus Z(f) \cap X \setminus Z(g)) > 0$.*

Proof. If $\mu(X \setminus Z(f) \cap X \setminus Z(g)) > 0$, then $1_{Z(f)\cup Z(g)} \in \mathscr{D}$ and is adjacent to both f, g in Γ_2'. Conversely let $h \in \mathscr{D}$ be adjacent to both f, g in Γ_2'. Then $\mu(Z(h) \cap (Z(f) \cup Z(g))) = 0$ and so, $\mu(Z(h) \cap X \setminus (Z(f) \cup Z(g))) = \mu(Z(h)) > 0$, as $h \in \mathscr{D}$. Consequently, $\mu(X \setminus Z(f) \cap X \setminus Z(g)) > 0$. $\square$

Corollary 3.6. *Every edge $f - g$ in $\Gamma_2'(\mathcal{M}(X,\mathcal{A}))$ is either an edge of a triangle or an edge of a square according as $\mu(X \setminus Z(f) \cap X \setminus Z(g)) > 0$ or $\mu(X \setminus Z(f) \cap X \setminus Z(g)) = 0$.*

Theorem 3.7. *For any two distinct non-adjacent vertices f, g in $\Gamma_2'(\mathcal{M}(X,\mathcal{A}))$,*

$$d(f,g) = \begin{cases} 2 & \text{if } \mu(Z(f) \cap Z(g)) > 0 \text{ and } \mu(X \setminus Z(f) \cap X \setminus Z(g)) > 0 \\ 3 & \text{if } \mu(Z(f) \cap Z(g)) > 0 \text{ and } \mu(X \setminus Z(f) \cap X \setminus Z(g)) = 0 \end{cases}$$

We omit the proof since this can be done by using Theorem 2.4 and Theorem 2.5.

Corollary 3.8. *$\Gamma_2'(\mathcal{M}(X,\mathcal{A}))$ is a connected graph.*

The following two lemmas will be employed to determine when the graph $\Gamma_2'(\mathcal{M}(X,\mathcal{A}))$ becomes complete bipartite and their proofs can be made using standard measure theoretic arguments.

Lemma 3.9. *Let $f, g \in \mathscr{D}(\mathcal{M}(X,\mathcal{A}))$.*

1. *If $\mu(Z(f) \triangle Z(g)) = 0$, then f, g are not adjacent in $\Gamma_2'(\mathcal{M}(X,\mathcal{A}))$.*
2. *If $\mu(Z(f) \setminus A) = 0$ and $\mu(Z(g) \cap A) = 0$ for some $A \in \mathcal{A}$, then f, g are adjacent in $\Gamma_2'(\mathcal{M}(X,\mathcal{A}))$.*
3. *If f, g are adjacent in $\Gamma_2'(\mathcal{M}(X,\mathcal{A}))$, then for any $f_1 \in \mathscr{D}(\mathcal{M}(X,\mathcal{A}))$ with $\mu(Z(f_1) \triangle Z(f)) = 0$ and for any $g_1 \in \mathscr{D}(\mathcal{M}(X,\mathcal{A}))$ with $\mu(Z(g_1) \triangle Z(g)) = 0$, f_1, g_1 are adjacent in $\Gamma_2'(\mathcal{M}(X,\mathcal{A}))$.*

Lemma 3.10. *If X can be partitioned into two atoms A, B, then $\mathscr{D}(\mathcal{M}(X,\mathcal{A}))$ can be partitioned into $\{f \in \mathscr{D}(\mathcal{M}(X,\mathcal{A})) : \mu(Z(f) \triangle A) = 0\}$ and $\{f \in \mathscr{D}(\mathcal{M}(X,\mathcal{A})) : \mu(Z(f) \triangle B) = 0\}$.*

Theorem 3.11. *$\Gamma_2'(\mathcal{M}(X,\mathcal{A}))$ is a complete bipartite graph if and only if X can be partitioned into two atoms.*

Proof. If Γ_2' is a complete bipartite graph, then it has a bipartion, say V_1, V_2. Let $f \in V_1$ and $g \in V_2$. Then by Theorem 2.5, $\mu(X \setminus Z(f) \cap X \setminus Z(g)) = 0 = \mu(Z(f) \cap Z(g))$. We claim that $Z(f)$ is an atom. For this let $A \in \mathcal{A}$ with $\mu(A) > 0$. If $\mu(X \setminus A) = 0$, then $\mu(X \setminus A \cap Z(f)) = 0$. Now let $\mu(X \setminus A) > 0$. Then $1_A, 1_{X\setminus A} \in \mathscr{D}$ and $1_A, 1_{X\setminus A}$ are adjacent in Γ_2'. If $1_A \in V_1$, then $1_{X\setminus A} \in V_2 \implies 1_{X\setminus A}, f$ are adjacent, i.e., $\mu(A \cap Z(f)) = 0$. If $1_A \in V_2$, then $1_A, f$ are adjacent and so, $\mu(X \setminus A \cap Z(f)) = 0$. In any case, either $\mu(A \cap Z(f)) = 0$ or $\mu(X \setminus A \cap Z(f)) = 0$. Consequently, $Z(f)$ is an atom. Similarly, $Z(g)$ is also an atom. Let $A = Z(f) \setminus Z(g) = Z(f) \setminus [Z(f) \cap Z(g)]$ and $B = Z(g) \cup X \setminus Z(f) = Z(g) \cup [X \setminus Z(f) \cap X \setminus Z(g)]$. Then $X = A \sqcup B$ and A, B are atoms. Conversely let A, B be two atoms in X such that $X = A \sqcup B$.

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Let $V_1 = \{f \in \mathscr{D} : \mu(Z(f)\triangle A) = 0\}$ and $V_2 = \{f \in \mathscr{D} : \mu(Z(f)\triangle B) = 0\}$. By Lemma 2.10, $\mathscr{D} = V_1 \sqcup V_2$. Again by Lemma 2.9(1), V_1, V_2 are stable sets in Γ_2'. Let $f \in V_1$ and $g \in V_2$. Then $\mu(Z(f)\triangle A) = 0 = \mu(Z(g)\triangle B) \implies \mu(Z(f) \setminus A) = 0 = \mu(Z(g) \setminus B) = \mu(Z(g) \cap A)$, as $B = X \setminus A$. By Lemma 2.9(2), f, g are adjacent in Γ_2'. Consequently, Γ_2' is a complete bipartite graph. $\square$

Remark 3.12. *X can be partitioned into two atoms if and only if for each $f \in \mathscr{D}(\mathcal{M}(X, \mathcal{A}))$, both $Z(f)$ and $X \setminus Z(f)$ are atoms.*

The zero divisor graph $\Gamma(\mathcal{M}(X, \mathcal{A}))$ (in short Γ) of $\mathcal{M}(X, \mathcal{A})$ is a simple graph defined on the vertex set $\mathscr{D}(\mathcal{M}(X, \mathcal{A}))$ in which two vertices f, g are adjacent if and only if $f \cdot g \equiv 0$ almost everywhere on X, i.e., $\mu(X \setminus Z(f) \cap X \setminus Z(g)) = 0$. If we replace $Z(f)$ by $X \setminus Z(f)$ and make adjustment as in the proof of Theorem 2.11, then we can show that $\Gamma(\mathcal{M}(X, \mathcal{A}))$ is a complete bipartite graph if and only if X can be partitioned into two atoms. Hence, the following result is immediate.

Corollary 3.13. *The following statements are equivalent:*

(1) *$\Gamma(\mathcal{M}(X, \mathcal{A}))$ is a complete bipartite graph;*
(2) *$\Gamma_2'(\mathcal{M}(X, \mathcal{A}))$ is a complete bipartite graph;*
(3) *X can be partitioned into two atoms.*

The next theorem gives rise to a condition for $\Gamma_2'(\mathcal{M}(X, \mathcal{A}))$ to be triangulated.

Theorem 3.14. *An $f \in \mathscr{D}(\mathcal{M}(X, \mathcal{A}))$ is a vertex of a triangle in $\Gamma_2'(\mathcal{M}(X, \mathcal{A}))$ if and only if $X \setminus Z(f)$ is not an atom.*

Proof. First let $X \setminus Z(f)$ be not an atom. Then there exist $A, B \in \mathcal{A}$ with $\mu(A), \mu(B) > 0$ such that $X \setminus Z(f) = A \sqcup B$. Then $1_{X\setminus A}, 1_{X\setminus B} \in \mathscr{D}$ and $f - 1_{X\setminus A} - 1_{X\setminus B} - f$ is a triangle in Γ_2'. Conversely let f be a vertex of a triangle in Γ_2', i.e., there exist $g, h \in \mathscr{D}$ such that $f - g - h - f$ is a triangle in Γ_2'. Then $\mu(Z(f) \cap Z(g)) = 0$ and so, $\mu(X \setminus Z(f) \cap Z(g)) > 0$, otherwise $\mu(Z(g)) = 0$. Similarly, $\mu(X \setminus Z(f) \cap Z(h)) > 0$. Let $A = X \setminus Z(f) \cap Z(g)$ and $B = X \setminus Z(f) \cap X \setminus Z(g)$. Then $\mu(A) > 0$ and $X \setminus Z(f) = A \sqcup B$. Since g, h are adjacent, $\mu(Z(g) \cap Z(h)) = 0 \implies \mu(A \cap Z(h)) = 0$. Now, $X \setminus Z(f) \cap Z(h) = (A \cap Z(h)) \sqcup (B \cap Z(h)) \implies \mu(B \cap Z(h)) = \mu(X \setminus Z(f) \cap Z(h)) > 0 \implies \mu(B) > 0$. Hence, $X \setminus Z(f)$ is not an atom. $\square$

Corollary 3.15. *$\Gamma_2'(\mathcal{M}(X, \mathcal{A}))$ is triangulated if and only if μ is non-atomic.*

For each $f \in \mathscr{D}(\mathcal{M}(X, \mathcal{A}))$, $1_{Z(f)} \in \mathscr{D}(\mathcal{M}(X, \mathcal{A}))$ and $f, 1_{Z(f)}$ are adjacent in $\Gamma_2'(\mathcal{M}(X, \mathcal{A}))$. So, by Theorem 2.5, we get the following result.

Theorem 3.16. *For every $f \in \mathscr{D}(\mathcal{M}(X, \mathcal{A}))$, there exists an edge containing f which is not an edge of any triangle in $\Gamma_2'(\mathcal{M}(X, \mathcal{A}))$. In particular, $\Gamma_2'(\mathcal{M}(X, \mathcal{A}))$ is not hypertriangulated.*

We get the formulae of several parameters of $\Gamma_2'(\mathcal{M}(X, \mathcal{A}))$ using the above arguments.

Theorem 3.17.

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(1) *The eccentricity of* f *in* $\Gamma_2'(\mathcal{M}(X,\mathcal{A}))$ *is* $\begin{cases} 2 & \textit{if } Z(f) \textit{ is an atom} \\ 3 & \textit{otherwise} \end{cases}$;

(2) *The diameter of* $\Gamma_2'(\mathcal{M}(X,\mathcal{A}))$ *is* $\begin{cases} 2 & \textit{if } X \textit{ can be partitioned into two atoms} \\ 3 & \textit{otherwise} \end{cases}$;

(3) *The girth of* $\Gamma_2'(\mathcal{M}(X,\mathcal{A}))$ *is* $\begin{cases} 4 & \textit{if } X \textit{ can be partitioned into two atoms} \\ 3 & \textit{otherwise} \end{cases}$;

(4) *For distinct vertices* f,g *in* $\Gamma_2'(\mathcal{M}(X,\mathcal{A}))$,

$$c(f,g) = \begin{cases} 3 & \textit{if } \mu(Z(f)\cap Z(g)) = 0 \textit{ and } \mu(X\setminus Z(f)\cap X\setminus Z(g)) > 0 \\ 4 & \textit{if } \mu(Z(f)\cap Z(g)) = 0 \textit{ and } \mu(X\setminus Z(f)\cap X\setminus Z(g)) = 0 \\ & \textit{or if } \mu(Z(f)\cap Z(g)) > 0 \textit{ and } \mu(X\setminus Z(f)\cap X\setminus Z(g)) > 0 \\ 6 & \textit{if } \mu(Z(f)\cap Z(g)) > 0 \textit{ and } \mu(X\setminus Z(f)\cap X\setminus Z(g)) = 0 \end{cases}.$$

By Theorem 2.5, we get the condition for a pair of vertices to be orthogonal in $\Gamma_2'(\mathcal{M}(X,\mathcal{A}))$.

Theorem 3.18. *Let* $f,g \in \mathscr{D}(\mathcal{M}(X,\mathcal{A}))$. *Then* $f \perp g$ *in* $\Gamma_2'(\mathcal{M}(X,\mathcal{A}))$ *if and only if* $\mu(Z(f)\cap Z(g)) = 0 = \mu(X\setminus Z(f)\cap X\setminus Z(g))$.

For each $f \in \mathscr{D}$, as $f \perp 1_{Z(f)}$, $\Gamma_2'(\mathcal{M}(X,\mathcal{A}))$ is a complemented graph. To check that whether $\Gamma_2'(\mathcal{M}(X,\mathcal{A}))$ is uniquely complemented or not, we need the following lemma whose proof can be done using Theorem 2.5.

Lemma 3.19. *Let* $f,g \in \mathscr{D}(\mathcal{M}(X,\mathcal{A}))$. *Then* f,g *are adjacent to the same set of vertices in* $\Gamma_2'(\mathcal{M}(X,\mathcal{A}))$ *if and only if* $\mu(Z(f)\triangle Z(g)) = 0$.

Theorem 3.20. $\Gamma_2'(\mathcal{M}(X,\mathcal{A}))$ *is a uniquely complemented graph.*

Proof. Let $f,g,h \in \mathscr{D}$ be such that $f \perp g$ and $f \perp h$ in Γ_2'. In view of Lemma 2.19, it suffices to show that $\mu(Z(g)\triangle Z(h)) = 0$. If possible let $\mu(Z(g)\cap X\setminus Z(h)) > 0$, then $k = 1_{X\setminus Z(g)\cup Z(h)} \in \mathscr{D}$. Now, $Z(k) = Z(g)\cap X\setminus Z(h)$ and so, $Z(k)\cap Z(h) = \emptyset \implies \mu(Z(k)\cap Z(h)) = 0$ and also, $Z(k)\cap Z(f) \subset Z(g)\cap Z(f) \implies \mu(Z(k)\cap Z(f)) = 0$, as $f \perp g$. Therefore, k is adjacent to both f,h, which contradicts that $f \perp h$. Therefore, $\mu(Z(g)\cap X\setminus Z(h)) = 0$ and similarly, $\mu(X\setminus Z(g)\cap Z(h)) = 0$. Hence, $\mu(Z(g)\triangle Z(h)) = 0$. □

4. Special subgraph

We know that, for any pair of subsets A,B of X, $A\triangle B = (X\setminus A)\triangle(X\setminus B)$. We consider a relation "$\sim$" on $\mathscr{D}(\mathcal{M}(X,\mathcal{A}))$ defined by: "$f \sim g$ if and only if $\mu(Z(f)\triangle Z(g)) = 0$". It can easily be shown that $\sim$ is an equivalence relation on $\mathscr{D}(\mathcal{M}(X,\mathcal{A}))$ and hence it makes a partition on $\mathscr{D}(\mathcal{M}(X,\mathcal{A}))$. For each $f \in \mathscr{D}(\mathcal{M}(X,\mathcal{A}))$, let $[f]$ be the equivalence class of the equivalence relation $\sim$ on $\mathscr{D}(\mathcal{M}(X,\mathcal{A}))$. It is also clear that for each $f \in \mathscr{D}(\mathcal{M}(X,\mathcal{A}))$, $f \sim 1_{X\setminus Z(f)}$. Let V be a subset of $\mathscr{D}(\mathcal{M}(X,\mathcal{A}))$ such that:

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1. every element of V is of the form 1_A where $\mu(A), \mu(X \setminus A) > 0$.
2. $1_A, 1_B$ are distinct elements in V if and only if $\mu(A \triangle B) > 0$.

Therefore, V is the collection of distinct class representatives corresponding to the equivalence relation $\sim$ on $\mathscr{D}(\mathcal{M}(X, \mathcal{A}))$. Let G_2 be the induced subgraph of $\Gamma_2'(\mathcal{M}(X, \mathcal{A}))$ whose set of vertices is V. We get the following results on using Lemma 2.9.

Theorem 4.1.

1. *For each $1_A \in V$, $[1_A]$ is a stable set in $\Gamma_2'(\mathcal{M}(X, \mathcal{A}))$.*
2. *For any two distinct stable sets $[1_A], [1_B]$ in $\Gamma_2'(\mathcal{M}(X, \mathcal{A}))$, either $[1_A] \sqcup [1_B]$ is a stable set or $[1_A] \sqcup [1_B]$ forms a complete bipartite subgraph of $\Gamma_2'(\mathcal{M}(X, \mathcal{A}))$.*

Thus $\Gamma_2'(\mathcal{M}(X, \mathcal{A}))$ is a $|V|$-partite graph such that for any two stable sets V_1, V_2, either $V_1 \sqcup V_2$ is a stable set or $V_1 \sqcup V_2$ is a complete bipartite subgraph of $\Gamma_2'(\mathcal{M}(X, \mathcal{A}))$. Therefore, G_2 is a subgraph of $\Gamma_2'(\mathcal{M}(X, \mathcal{A}))$ where we choose exactly one vertex from each stable sets and two vertices in G_2 are adjacent if and only if the corresponding two stable sets make a complete bipartite subgraph. This observation leads to the following set of facts for G_2:

1. Let $1_A, 1_B \in V$. Then there is a vertex in V adjacent to both $1_A, 1_B$ in G_2 if and only if $\mu(A \cap B) > 0$;
2. The distance between two distinct vertices $1_A, 1_B$ in G_2 is given by

$$d(1_A, 1_B) = \begin{cases} 1 & \text{if } \mu(X \setminus A \cap X \setminus B) = 0 \\ 2 & \text{if } \mu(X \setminus A \cap X \setminus B) > 0 \text{ and } \mu(A \cap B) > 0 \\ 3 & \text{if } \mu(X \setminus A \cap X \setminus B) > 0 \text{ and } \mu(A \cap B) = 0 \end{cases};$$

3. G_2 is a connected graph;
4. $|V| = 2$, i.e., $G_2 = K_2$ if and only if X can be partitioned into two atoms;
5. If $|V| > 2$, then the diameter of G_2 is 3;
6. If $|V| > 2$, then the eccentricity of $1_A \in V$ in G_2 is $\begin{cases} 2 & \text{if } X \setminus A \text{ is an atom} \\ 3 & \text{otherwise} \end{cases}$;
7. A vertex $1_A \in V$ is a vertex of a triangle in G_2, if and only if A is not an atom;
8. If $|V| > 2$, then the girth of G_2 is 3;
9. G_2 is triangulated if and only if μ is a non-atomic measure;
10. G_2 is not hypertriangulated;
11. $1_A, 1_B \in V$ are orthogonal in G_2 if and only if $\mu(A \cap B) = 0 = \mu(X \setminus A \cap X \setminus B)$;
12. For every $1_A \in V$ there exists unique $1_B \in V$ such that $1_A \perp 1_B$ in G_2 [uniqueness follows from Lemma 2.19]. Therefore, G_2 is uniquely complemented.

We now compare the clique number and the chromatic number of the graphs $\Gamma_2'(\mathcal{M}(X, \mathcal{A}))$ and G_2. Since G_2 is a subgraph of Γ_2', $cl(G_2) \leq cl(\Gamma_2')$ and $\chi(G_2) \leq \chi(\Gamma_2')$.

Theorem 4.2. *The clique number of G_2 and $\Gamma_2'(\mathcal{M}(X, \mathcal{A}))$ are the same.*

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Proof. Let M be a complete subgraph of Γ'_2. Let M' be the subgraph of G_2 whose vertex set is $\{1_A \in V : 1_A \sim f \text{ for some vertex } f \text{ in } M\}$. Clearly, M' is a complete subgraph of G_2 and $|M| = |M'|$. Consequently, $cl(G_2) = cl(\Gamma'_2)$. □

Theorem 4.3. *The chromatic number of G_2 and $\Gamma'_2(\mathcal{M}(X, \mathcal{A}))$ are equal.*

Proof. Let G_2 be α-colorable. For each $f \in \mathscr{D}$, there exists $1_A \in V$ such that $f \sim 1_A$. Since G_2 is α-colorable, 1_A has already been colored by some color. We color $f \in \mathscr{D}$ by the color of $1_A \in V$. In this way, we can color all the vertices in $\mathscr{D}$ by using α-many colors. It only remains to show that this coloring on Γ'_2 is consistent. Let $f, g \in \mathscr{D}$ be colored by the same color, say the color of 1_A. Then by our method of coloring of Γ'_2, $f \sim 1_A$ and $g \sim 1_A$. By Theorem 3.1, $[1_A]$ is a stable set in Γ'_2 and hence, f, g are non-adjacent in Γ'_2. Consequently, Γ'_2 is α-colorable and so, $\chi(\Gamma'_2) \leq \chi(G_2)$. □

Using the above two theorems, we can compute the clique number and the chromatic number of $\Gamma'_2(\mathcal{M}(X, \mathcal{A}))$, as seen in the following example.

Example 4.4. *In the counting measure space $(X, \mathcal{P}(X), c)$, $f \sim g$ if and only if $Z(f) = Z(g)$. Hence, $V = \{1_A : A \neq \emptyset, X\}$. Clearly, $\{1_{X\setminus\{x\}} : x \in X\}$ is a complete subgraph of G_2. Therefore, $cl(G_2) \geq |X|$. We color each $1_{X\setminus\{x\}} \in V$ by distinct colors. Let $1_A \in V \setminus \{1_{X\setminus\{x\}} : x \in X\}$. Then $|Z(1_A)| \geq 2$. We then color 1_A by one of the color of $1_{X\setminus\{x\}}$, where $x \in Z(1_A)$. Suppose $1_A, 1_B \in V$ are colored by the same color, say by the color of $1_{X\setminus\{x\}}$. Then by our definition, $x \in Z(1_A) \cap Z(1_B)$. Consequently, $1_A, 1_B$ are not adjacent in G_2. Hence we get a consistent coloring of G_2 by $|X|$-many color. Therefore $\chi(G_2) \leq |X|$. Hence, $cl(G_2) = |X| = \chi(G_2)$. A graph is said to be weakly perfect, if the clique number and the chromatic number of the graph are equal. Therefore, $\Gamma'_2(\mathcal{M}(X, \mathcal{P}(X)))$ is a weakly perfect graph.*

We draw a comparison regarding the dominating number and the total dominating number between the graphs $\Gamma'_2(\mathcal{M}(X, \mathcal{A}))$ and G_2.

Theorem 4.5. *$dt(G_2) \leq dt(\Gamma'_2(\mathcal{M}(X, \mathcal{A})))$.*

Proof. Let $D \subset \mathscr{D}$ be a dominating set in Γ'_2. Let $V' = \{1_A \in V : 1_A \sim f \text{ for some } f \in D\}$. Clearly, $|V'| \leq |D|$ and $D \cap V \subset V'$. To show that V' is a dominating set in G_2, let $1_B \in V \setminus V'$. If $1_B \in D$, then $1_B \in V'$, which is not. So, $1_B \in \mathscr{D} \setminus D$. Since D is a dominating set in Γ'_2, there exists $f \in D$ such that $f, 1_B$ are adjacent in Γ'_2. Let $1_A \in V'$ such that $1_A \sim f$. By Theorem 3.1, $1_A, 1_B$ are adjacent in Γ'_2 and hence they are adjacent in G_2. Therefore, V' is a dominating set in G_2. Now $dt(G_2) \leq |V'| \leq |D|$ and this holds for every dominating set D in Γ'_2. Consequently, $dt(G_2) \leq dt(\Gamma'_2)$. □

Lemma 4.6. *Let $D \subset \mathscr{D}$.*

1. *Every total dominating set in G_2 is also a total dominating set in $\Gamma'_2(\mathcal{M}(X, \mathcal{A}))$.*
2. *If D is a total dominating set in $\Gamma'_2(\mathcal{M}(X, \mathcal{A}))$, then there exists a total dominating set D' in G_2 such that $|D| \geq |D'|$.*

Theorem 4.7. *The total dominating number of G_2 and $\Gamma'_2(\mathcal{M}(X, \mathcal{A}))$ are equal.*

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In the same way, we consider the induced subgraph G of $\Gamma(\mathcal{M}(X,\mathcal{A}))$ having the same set of vertices V. Similarly, we can show that, each equivalence classes $[1_A]$ is a stable set in $\Gamma(\mathcal{M}(X,\mathcal{A}))$ and for any two distinct equivalence classes $[1_A]$ and $[1_B]$, either $[1_A]\sqcup[1_B]$ is a stable set in $\Gamma(\mathcal{M}(X,\mathcal{A}))$ or they make a complete bipartite subgraph of $\Gamma(\mathcal{M}(X,\mathcal{A}))$. Hence, $\Gamma(\mathcal{M}(X,\mathcal{A}))$ is a $|V|$-partite graph. This observation leads to the following question:

Question 4.8. *Are $\Gamma(\mathcal{M}(X,\mathcal{A}))$ and $\Gamma_2'(\mathcal{M}(X,\mathcal{A}))$ isomorphic as graph?*

Before answering this question, we observe that the corresponding special subgraphs are isomorphic.

Theorem 4.9. *G and G_2 are graph isomorphic.*

Proof. Let us define a function $\psi : V \to V$ as follows $\psi(1_A) = 1_{\psi(A)}$ where $1_{\psi(A)} \sim 1_{X\setminus A}$. It can be checked that ψ is a bijection which preserves the adjacency relation between G and G_2. □

We now deduce a sufficient condition for the graphs $\Gamma(\mathcal{M}(X,\mathcal{A})))$ and $\Gamma_2'(\mathcal{M}(X,\mathcal{A}))$ to be isomorphic.

Theorem 4.10. *If $|[1_A]| = |[1_{X\setminus A}]|$ for each $1_A \in V$, then $\Gamma(\mathcal{M}(X,\mathcal{A}))$ and $\Gamma_2'(\mathcal{M}(X,\mathcal{A}))$ are graph isomorphic.*

Proof. Since $|[1_A]| = |[1_{X\setminus A}]| = |[1_{\psi(A)}]|$ for each $1_A \in V$, there exists a bijection $\phi_A : [1_A] \to [1_{\psi(A)}]$. Define $\phi : \mathscr{D}(\mathcal{M}(X,\mathcal{A})) \to \mathscr{D}(\mathcal{M}(X,\mathcal{A}))$ by $\phi(f) = \phi_A(f)$, whenever $f \sim 1_A$. Then ϕ is a bijective map. Let f, g be adjacent in $\Gamma(\mathcal{M}(X,\mathcal{A}))$ and $1_A, 1_B \in V$ be such that $f \sim 1_A$ and $g \sim 1_B$. Since f, g are adjacent, $1_A, 1_B$ are adjacent in G. Therefore, $\psi(1_A), \psi(1_B)$ are adjacent in G_2. By the definition of ϕ, $\phi(f) \sim \psi(1_A)$ and $\phi(g) \sim \psi(1_B)$. Therefore, $\phi(f), \phi(g)$ are adjacent in $\Gamma_2'(\mathcal{M}(X,\mathcal{A}))$. Similarly, if f, g are adjacent in $\Gamma_2'(\mathcal{M}(X,\mathcal{A}))$, then $\phi^{-1}(f), \phi^{-1}(g)$ are adjacent in $\Gamma(\mathcal{M}(X,\mathcal{A}))$. Hence, $\Gamma(\mathcal{M}(X,\mathcal{A}))$ and $\Gamma_2'(\mathcal{M}(X,\mathcal{A}))$ are graph isomorphic. □

We now show that this condition holds in the Lebesgue measure space $(\mathbb{R}, \mathcal{S}, \ell)$.

Lemma 4.11. *Let $A \in \mathcal{S}$ be such that A is compact in $\mathbb{R}$ (with respect to the usual topology). Then there exist $a, b \in A$ such that $\ell(A \cap [a,b]) = \ell(A)$.*

Proof. Since A is compact in $\mathbb{R}$, A is closed and $A \subset [x,y]$ for some $x, y \in \mathbb{R}$. Note that, $\ell(A) \leq y - x$, finite. Let $B = \{z \in [a,b] : \ell(A \cap [z,y]) = \ell(A)\}$ and $a = supB$. Since $x \in B$, $B \neq \emptyset$ and hence such a exists and $x \leq a < y$. If possible let $a \notin A$, then a is not a limit point of A. There exists $\delta > 0$ such that $(a-\delta, a+\delta) \cap A = \emptyset \ldots (1)$. Therefore $\ell((a-\delta, a+\delta) \cap A) = 0$. Since a is the supremum of the set B, there exists $a' \in B$ such that $a - \delta < a' < a$. Now $a' \in B \implies \ell(A \cap [a', y]) = \ell(A)$. Therefore $\ell(A \cap [x, a')) = 0$. This along with (1) implies $\ell(A \cap [x, a+\delta)) = 0 \implies \ell(A \cap [a+\delta, y]) = \ell(A)$, i.e., $a + \delta \in B$, which contradicts that $a = supB$. Therefore $a \in A$. Also $[x,a) = \bigcup\limits_{n=1}^{\infty}[x, a - \frac{1}{n}] \implies \ell(A \cap [x,a)) \leq \sum\limits_{n=1}^{\infty} \ell(A \cap [x, a - \frac{1}{n}])$. Since $a = supB$, $\ell(A \cap [x, a - \frac{1}{n}]) = 0$ for all n and hence $\ell(A \cap [x,a)) = 0 \implies \ell(A \cap [a,y]) = \ell(A)$. Similarly, if $b = inf\{z \in [a,y] : \ell(A \cap [a,z]) = \ell(A)\}$, then $b \in A$ and $\ell(A \cap [a,b]) = \ell(A)$. □

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Lemma 4.12. *Let* $A \in \mathcal{S}$ *and* $\ell(A) > 0$. *Then for every* $r \in [0, \ell(A)]$, *there exists* $A_r \in \mathcal{S}$ *such that* $A_r \subset A$ *and* $\ell(A_r) = r$.

Proof. We consider the function $\psi : \mathbb{R} \to [0, \ell(A)]$ given by $\psi(x) = \ell(A \cap (-\infty, x])$. Since $|\psi(x) - \psi(y)| \leq |x - y|$ for all $x, y \in \mathbb{R}$, ψ is continuous on $\mathbb{R}$. Also $\lim\limits_{x \to -\infty} \psi(x) = 0$ and $\lim\limits_{x \to \infty} \psi(x) = \ell(A)$. Therefore ψ takes every values in between $[0, \ell(A)]$, i.e., for each $r \in [0, \ell(A)]$, there exists $x_r \in \mathbb{R}$ such that $\psi(x_r) = r$, i.e., $\ell(A \cap (-\infty, x_r]) = r$. □

Remark 4.13. *Let* A *be a compact set in the Lemma 3.12. Then we can consider the continuous map* ψ *defined from* $[a, b]$ *to* $[0, \ell(A)]$ *for some* $a, b \in A$ *which is given by* $\psi(x) = \ell(A \cap [a, x])$. *Also for each* $r \in [0, \ell(A)]$, *there exists* $y_r \in A \cap [a, b]$ *such that* $\ell(A \cap [a, y_r]) = r$.

Theorem 4.14. *If* $\ell(A) > 0$ *for some* $A \in \mathcal{S}$, *then* A *contains an uncountable set of measure zero.*

Proof. Without loss of generality let $\ell(A \cap [0, 1]) > 0$. By the regularity of ℓ, there exists a compact set K in $\mathbb{R}$ such that $\ell(K) > 0$ and $K \subset A \cap [0, 1]$. By Lemma 3.11, there exists $a, b \in K \cap [0, 1]$ such that $\ell(K \cap [a, b]) = \ell(K)$. By Remark 3.13, there exist $x_{11}, x_{12} \in K \cap [a, b]$ such that $\ell(K \cap [a, x_{11}]) = \frac{1}{3}\ell(K)$ and $\ell(K \cap [a, x_{12}]) = \frac{2}{3}\ell(K)$. Clearly, $a < x_{11} < x_{12} < b$. Let $I_{11} = K \cap [a, x_{11}]$, $I_{12} = K \cap [x_{11}, x_{12}]$ and $I_{13} = K \cap [x_{12}, b]$. Then $\ell(I_{11}) = \frac{1}{3}\ell(K) = \ell(I_{12}) = \ell(I_{13})$, i.e., we trisect $K \cap [a, b]$ into three parts I_{11}, I_{12}, I_{13} with equal measure. Let $E_1 = I_{11} \sqcup I_{13}$. Then $E_1 \in \mathcal{S}$ and $\ell(E_1) = \frac{2}{3}\ell(K)$. Similarly, we trisect I_{11} into $I_{21} = K \cap [a, x_{21}]$, $I_{22} = K \cap [x_{21}, x_{22}]$ and $I_{23} = K \cap [x_{22}, x_{11}]$ with equal measure, where $x_{21}, x_{22} \in K \cap [a, b]$. We also trisect I_{13} into $I_{24} = K \cap [x_{12}, x_{23}]$, $I_{25} = K \cap [x_{23}, x_{24}]$ and $I_{26} = K \cap [x_{24}, b]$ with equal measure, where $x_{23}, x_{24} \in K \cap [a, b]$. Let $E_2 = (I_{21} \sqcup I_{23}) \sqcup (I_{24} \sqcup I_{26})$. Then $E_2 \in \mathcal{S}$ and $\ell(E_2) = (\frac{2}{3})^4 \ell(K)$. Continuing this process, at n-th stage, we get $E_n \in \mathcal{S}$ which is the union of 2^n-many disjoint measurable sets $I_{n1}, \cdots, I_{n2^n}$ each of which has measure $\frac{1}{3^n}\ell(K)$. Thus, $\ell(E_n) = (\frac{2}{3})^{2^n} \ell(K)$. Let $E = \bigcap\limits_{n=1}^{\infty} E_n$. Then $E \in \mathcal{S}$ and $\ell(E) = \lim\limits_{n \to \infty} \ell(E_n) = \ell(K). \lim\limits_{n \to \infty} (\frac{2}{3})^{2^n} = 0$. Note that for every chain of Lebesgue measurable sets of the form I_{mn}, $x_{mn} \in \bigcap I_{mn}$ and $\bigcap I_{mn} \subset \bigcap\limits_{n=1}^{\infty} E_n \subset E \implies E \neq \emptyset$. It only remains to show that E is uncountable. If possible let E be countable, say $E = \{x_n : n \in \mathbb{N}\}$. Since $x_1 \in E$, $x_1 \in E_1 \implies$ there exists $k \in \{1, 3\}$ such that $x_1 \notin I_{1k_1}$. Similarly $x_2 \in E_2 \implies x_2 \notin I_{2k_2}$ where I_{2k_2} is one of the measurable sets in E_2 such that $I_{2k_2} \subset I_{1k_1}$. Thus for each $\in \mathbb{N}$, there exists some $k_n \in \{1, 2, ..., 2^n\}$ such that $x_n \notin I_{nk_n}$ and $I_{nk_n} \subset I_{(n-1)k_{n-1}}$. Since $I_{nk_n} \subset I_{(n-1)k_{n-1}}$ for all $n \in \mathbb{N}$, $\bigcap\limits_{n=1}^{\infty} I_{nk_n} \neq \emptyset$, say $x \in \bigcap\limits_{n=1}^{\infty} I_{nk_n}$. Then $x \in E$ and $x \notin \{x_1, x_2, ..., x_n, ...\} = E$, a contradiction. Hence, E is an uncountable set. □

Corollary 4.15. *If* $\ell(A) > 0$ *for* $A \in \mathcal{S}$, *then* A *contains exactly* $2^{\mathfrak{c}}$*-many Lebesgue measurable sets.*

As the collection $\mathcal{S}$ of all Lebesgue measurable sets on $\mathbb{R}$ contains $2^{\mathfrak{c}}$-many distinct Lebesgue measurable sets, $|\mathcal{M}(\mathbb{R}, \mathcal{S})| = 2^{\mathfrak{c}}$. So, for each $f \in \mathscr{D}(\mathcal{M}(\mathbb{R}, \mathcal{S}))$, $|[f]| \leq |\mathcal{M}(\mathbb{R}, \mathcal{S})| = 2^{\mathfrak{c}}$.

Theorem 4.16. *For each* $f \in \mathscr{D}(\mathcal{M}(\mathbb{R}, \mathcal{S}))$, $|[f]| = 2^{\mathfrak{c}}$.

Proof. Let $f \in \mathscr{D}(\mathcal{M}(\mathbb{R}, \mathcal{S}))$. Then $\ell(X \setminus Z(f)) > 0$. By Corollary 3.15, $X \setminus Z(f)$ contains exactly $2^{\mathfrak{c}}$-many Lebesgue measurable sets, say $\{A_\alpha : \alpha \in \Lambda\}$. For each $\alpha \in \Lambda$, define $f_\alpha : \mathbb{R} \to \mathbb{R}$ by

$$f_\alpha(x) = \begin{cases} 0 & \text{if } x \in Z(f) \\ 1 & \text{if } x \in A_\alpha \\ 2 & \text{otherwise} \end{cases}$$

Clearly, $f_\alpha \in [f]$ for all $\alpha \in \Lambda$ and for distinct $\alpha, \beta \in \Lambda$, $f_\alpha \neq f_\beta$. Therefore, $|[f]| \geq 2^{\mathfrak{c}}$. □

By Theorem 3.10, the zero-divisor graph and the co-maximal graph on $\mathcal{M}(\mathbb{R}, \mathcal{S})$ are isomorphic. However, the sufficient condition of Theorem 3.10 is not true for an arbitrary measure space. In particular, for the counting measure space $(X, \mathcal{P}(X), c)$, we establish that these graphs are not isomorphic, whenever X is an uncountable set.

Lemma 4.17. *For each* $f \in \mathcal{M}(X, \mathcal{P}(X))$, $|[f]| = |\mathbb{R}^{|X \setminus Z(f)|}|$.

Proof. Let $X \setminus Z(f) = \{x_\alpha : \alpha \in \Lambda\}$. Define $\eta : [f] \to (\mathbb{R} \setminus \{0\})^{|\Lambda|}$ by $\eta(g) = \Pi_{\alpha \in \Lambda}(g(x_\alpha))$. Then it can be verified that η is a bijection and hence, $|[f]| = |(\mathbb{R} \setminus \{0\})^{|\Lambda|}| = |\mathbb{R}^{|X \setminus Z(f)|}|$. □

Let $f \in \mathscr{D}(\mathcal{M}(X, \mathcal{P}(X)))$. By Theorem 2.17 it follows that, $ecc(f) = 2$ in $\Gamma_2'(\mathcal{M}(X, \mathcal{P}(X)))$ if and only if $Z(f)$ is a singleton set. In a similar manner one can prove that, $ecc(f) = 2$ in $\Gamma(\mathcal{M}(X, \mathcal{P}(X)))$ if and only if $X \setminus Z(f)$ is a singleton set. These observations lead to the following important lemma.

Lemma 4.18. *If* $\Gamma(\mathcal{M}(X, \mathcal{P}(X)))$ *and* $\Gamma_2'(\mathcal{M}(X, \mathcal{P}(X)))$ *are graph isomorphic, then for each* $x \in X$ *there exists* $y \in X$ *such that* $|[1_x]| = |[1_{X \setminus \{y\}}]|$.

Proof. Let $\psi : \Gamma \to \Gamma_2'$ be a graph isomorphism. Fix an $x \in X$. Let $f \in [1_x]$. Then $X \setminus Z(f) = \{x\} \implies ecc(f) = 2$ in Γ. Since ψ is a graph isomorphism, $ecc(\psi(f)) = 2$ in $\Gamma_2' \implies Z(\psi(f))$ is a singleton set, i.e., $Z(\psi(f)) = \{y\}$ for some $y \in X$. Therefore, $\psi(f) \in [1_{X \setminus \{y\}}]$. Conversely let $g \in [1_{X \setminus \{y\}}]$. Then $Z(g) = \{y\} \implies ecc(g) = 2$ in $\Gamma_2' \implies ecc(\psi^{-1}(g)) = 2$ in $\Gamma \implies X \setminus Z(\psi^{-1}(g))$ is a singleton set, i.e., $X \setminus Z(\psi^{-1}(g)) = \{x'\}$ for some $x' \in X$. If $x \neq x'$, then $1_x, \psi^{-1}(g)$ are adjacent in Γ. Therefore their images under ψ are adjacent in Γ_2', i.e., $\psi(1_x), g$ are adjacent in Γ_2'. But $1_x \in [1_x] \implies \psi(1_x) \in [1_{X \setminus \{y\}}]$, by our definition. Therefore, $Z(\psi(1_x)) = \{y\} = Z(g) \implies \psi(1_x), g$ are not adjacent in Γ_2', which is a contradiction. Therefore, $X \setminus Z(\psi^{-1}(g)) = \{x\} \implies \psi^{-1}(g) \in [1_x]$. Hence, the restriction map $\psi : [1_x] \to [1_{X \setminus \{y\}}]$ is a bijection, i.e., $|[1_x]| = |[1_{X \setminus \{y\}}]|$. □

Theorem 4.19. $\Gamma(\mathcal{M}(X, \mathcal{P}(X)))$ *and* $\Gamma_2'(\mathcal{M}(X, \mathcal{P}(X)))$ *are graph isomorphic if and only if* X *is atmost countable.*

Proof. If X is atmost countable, then for any $f \in \mathscr{D}$, $|X \setminus Z(f)|$ is countable and hence by Lemma 3.17, $|[f]| = |\mathbb{R}^{|X \setminus Z(f)|}| = |\mathbb{R}| = \mathfrak{c}$, i.e., $|[f]| = |[g]|$ for all $f, g \in \mathscr{D}$. Therefore by Theorem 3.10, Γ and Γ_2' are isomorphic. Conversely let X be uncountable. Then for any $x, y \in X$, $X \setminus \{x\}$ is uncountable $\implies |\mathbb{R}^{|X \setminus \{x\}|}| > |\mathbb{R}^{|\{y\}|}|$, i.e., $|[1_{X \setminus \{x\}}]| > |[1_y]|$. By Lemma 3.18, Γ and Γ_2' are not graph isomorphic. □

5. The annihilator graph $AG(\mathcal{M}(X,\mathcal{A}))$ of $\mathcal{M}(X,\mathcal{A})$

Let $f \in \mathcal{M}(X,\mathcal{A})$. The annihilator ideal of f in the almost everywhere sense is $ann(f) = \{g \in \mathcal{M}(X,\mathcal{A}) : f \cdot g \equiv 0 \text{ almost everywhere on } X\} = \{g \in \mathcal{M}(X,\mathcal{A}) : \mu(X \setminus Z(f) \cap X \setminus Z(g)) = 0\}$. Clearly $ann(f) \cup ann(g) \subset ann(f \cdot g)$ for any $f, g \in \mathcal{M}(X,\mathcal{A})$. The annihilator graph $AG(\mathcal{M}(X,\mathcal{A}))$ (in short AG) of $\mathcal{M}(X,\mathcal{A})$ defined on the set $\mathscr{D}(\mathcal{M}(X,\mathcal{A}))$ and two vertices f, g are adjacent if and only if $ann(f) \cup ann(g) \subsetneqq ann(f \cdot g)$. To express the adjacency relation between two vertices in a measure-theoretic framework, we require the following lemma.

Lemma 5.1. *Let $f, g \in \mathcal{M}(X,\mathcal{A})$. Then $ann(f) \subset ann(g)$ if and only if $\mu(Z(f) \setminus Z(g)) = 0$.*

Proof. Let $h \in ann(f)$, i.e., $h \cdot f = 0$ on $X \setminus F$ for some $F \in \mathcal{A}$ with $\mu(F) = 0$. Therefore, $X \setminus Z(h) \cap X \setminus F \subset Z(f) \implies X \setminus Z(h) \cap X \setminus ((Z(f) \setminus Z(g)) \cup F) \subset Z(g)$. If $\mu(Z(f) \setminus Z(g)) = 0$, then $h \cdot g \equiv 0$ almost everywhere on X, i.e., $h \in ann(g)$. Conversely, if $\mu(Z(f) \setminus Z(g)) > 0$, then $1_{Z(f)\setminus Z(g)} \in ann(f) \setminus ann(g)$. □

Theorem 5.2. *Let $f, g \in \mathcal{M}(X,\mathcal{A})$. Then f, g are adjacent in $AG(\mathcal{M}(X,\mathcal{A}))$ if and only if $\mu(Z(f) \setminus Z(g)) > 0$ and $\mu(Z(g) \setminus Z(f)) > 0$.*

The next theorem determines a condition for the existence of a third vertex which is adjacent to a given pair of vertices in $AG(\mathcal{M}(X,\mathcal{A}))$.

Theorem 5.3. *Let f, g be distinct vertices in $AG(\mathcal{M}(X,\mathcal{A}))$.*

1. *If $\mu(X \setminus Z(f) \cap X \setminus Z(g)) > 0$ or $\mu(Z(f) \cap Z(g)) > 0$, then there is a vertex adjacent to both f, g.*
2. *Let $\mu(X \setminus Z(f) \cap X \setminus Z(g)) = 0 = \mu(Z(f) \cap Z(g))$. Then there exists a vertex adjacent to both f, g if and only if $Z(f)$ and $Z(g)$ are not atoms.*

Proof. It is easy to check that $\mu(X \setminus Z(f) \cap X \setminus Z(g)) > 0$ or $\mu(Z(f) \cap Z(g)) > 0$ implies that $1_{Z(f)\cup Z(g)}$ or $1_{Z(f)\cap Z(g)}$ is adjacent to both f, g in AG respectively. Let $\mu(X \setminus Z(f) \cap X \setminus Z(g)) = 0 = \mu(Z(f) \cap Z(g))$. Without loss of generality, let $Z(f) = X \setminus Z(g)$. If both $Z(f)$ and $Z(g)$ are not atoms, then there exist $A_1, A_2, B_1, B_2 \in \mathcal{A}$ each with positive measure such that $Z(f) = A_1 \sqcup A_2$ and $Z(g) = B_1 \sqcup B_2$. Clearly, $1_{A_1 \cup B_1} \in \mathscr{D}$ and is adjacent to both of f and g in AG. Conversely, let h be adjacent to f and g in AG. Then $\mu(Z(f) \setminus Z(h)) > 0$ and $\mu(Z(h) \setminus Z(g)) > 0$. Now $Z(f) = (Z(f) \cap Z(h)) \sqcup (Z(f) \cap X \setminus Z(h)) = (Z(h) \setminus Z(g)) \sqcup (Z(f) \setminus Z(h))$, as $Z(f) = X \setminus Z(g)$. Therefore, $Z(f)$ is not an atom, and similarly, $Z(g)$ is also not an atom. □

Corollary 5.4. *For any two distinct non-adjacent vertices f, g in $AG(\mathcal{M}(X,\mathcal{A}))$, $d(f,g) = 2$ and hence, the diameter of $AG(\mathcal{M}(X,\mathcal{A}))$ is 2. Also, the eccentricity of every vertex in it is 2.*

We now compare the co-maximal graph and the zero-divisor graph of $\mathcal{M}(X,\mathcal{A})$ with the annihilator graph of $\mathcal{M}(X,\mathcal{A})$ and further deduce the conditions for their equality.

Theorem 5.5. *The following statements are true:*

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(1) $\Gamma'_2(\mathcal{M}(X,\mathcal{A}))$ *is a subgraph of* $AG(\mathcal{M}(X,\mathcal{A}))$.
(2) $\Gamma(\mathcal{M}(X,\mathcal{A}))$ *is a subgraph of* $AG(\mathcal{M}(X,\mathcal{A}))$.

Proof. We already have $\mathscr{D}$ as the vertex set of all the three graphs.

(1) Let $f,g \in \mathscr{D}$ be adjacent in Γ'_2, i.e., $\mu(Z(f)\cap Z(g))=0$. Now, $Z(f)=(Z(f)\setminus Z(g))\sqcup(Z(f)\cap Z(g)) \implies \mu(Z(f)\setminus Z(g))=\mu(Z(f))>0$. Similarly, $\mu(Z(g)\setminus Z(f))>0$. Hence f,g are adjacent in AG.
(2) Analogous to the above proof. □

Theorem 5.6. *$AG(\mathcal{M}(X,\mathcal{A}))$ is a complete bipartite graph if and only if X can be partitioned into two atoms.*

The proof can be done by modifying the arguments made in Theorem 2.11 in appropriate manner.

Theorem 5.7. *The following statements are equivalent:*

(1) $\Gamma(\mathcal{M}(X,\mathcal{A}))=AG(\mathcal{M}(X,\mathcal{A}))$;
(2) $\Gamma'_2(\mathcal{M}(X,\mathcal{A}))=AG(\mathcal{M}(X,\mathcal{A}))$;
(3) *X can be partitioned into two atoms.*

Theorem 5.8. *A vertex f in $AG(\mathcal{M}(X,\mathcal{A}))$ is a vertex of a triangle if and only if either $Z(f)$ or $X\setminus Z(f)$ is not an atom.*

Proof. If $X\setminus Z(f)$ is not an atom, then by Theorem 2.14, f is a vertex of triangle in Γ'_2. By Theorem 4.5(1), f is a vertex of a triangle in AG. Again, if $Z(f)$ is not an atom, then there exists $A_1,A_2\in\mathcal{A}$ with $\mu(A_1),\mu(A_2)>0$ such that $Z(f)=A_1\sqcup A_2$. Clearly, $1_{A_1},1_{A_2}\in\mathscr{D}$ and $f-1_{A_1}-1_{A_2}-f$ is a triangle in Γ. By Theorem 4.5(2), $f-1_{A_1}-1_{A_2}-f$ is a triangle in AG. If both $Z(f)$ and $X\setminus Z(f)$ are atoms, then by Theorem 4.6, AG is complete bipartite and hence, f can not be a vertex of a triangle. □

Corollary 5.9. *$AG(\mathcal{M}(X,\mathcal{A}))$ is triangulated if and only if X can not be partitioned into two atoms. Therefore, the girth of $AG(\mathcal{M}(X,\mathcal{A}))$ is given by*

$$gr(AG(\mathcal{M}(X,\mathcal{A})))=\begin{cases}4 & \textit{if } X \textit{ can be partitioned into two atoms}\\ 3 & \textit{otherwise}\end{cases}$$

The next result directly follows from Theorem 4.3.

Theorem 5.10. *Let $f,g\in\mathscr{D}(\mathcal{M}(X,\mathcal{A}))$. Then $f\perp g$ in $AG(\mathcal{M}(X,\mathcal{A}))$ if and only if $\mu(X\setminus Z(f)\cap X\setminus Z(g))=0=\mu(Z(f)\cap Z(g))$ and either $Z(f)$ or $Z(g)$ is an atom.*

Corollary 5.11. *An edge $f-g$ in $AG(\mathcal{M}(X,\mathcal{A}))$ is an edge of a triangle if and only if $f\not\perp g$.*

If X can be partitioned into two atoms, then $AG(\mathcal{M}(X,\mathcal{A}))$ is never hypertriangulated. Let X can not be partitioned into two atoms. Then for any $A\in\mathcal{A}$ with $\mu(A)>0$ and $\mu(X\setminus A)>0$, either A or $X\setminus A$ is not an atom. Considering this condition, we calculate when $AG(\mathcal{M}(X,\mathcal{A}))$ is hypertriangulated.

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Theorem 5.12. *$AG(\mathcal{M}(X,\mathcal{A}))$ is hypertriangulated if and only if μ is non-atomic.*

Proof. If μ is non-atomic, then for any pair of vertices f, g in AG, $f \not\perp g$ (by Theorem 4.10), because $Z(f)$ and $Z(g)$ are not atoms. Consequently, by Corollary 4.11, AG is hypertriangulated. Conversely, let $A \in \mathcal{A}$ be an atom. As X is not an atom, $\mu(X \setminus A) > 0$ and so, $1_A, 1_{X\setminus A} \in \mathscr{D}$. By Theorem 4.10, $1_A \perp 1_{X\setminus A}$. Therefore, $1_A - 1_{X\setminus A}$ is not an edge of a triangle in AG. $\square$

For a given vertex, the existence of an orthogonal complement is given below, which follows from Theorem 4.10.

Theorem 5.13. *A vertex f in $AG(\mathcal{M}(X,\mathcal{A}))$ has an orthogonal complement if and only if either $Z(f)$ or $X \setminus Z(f)$ is an atom.*

We observe that, if X is partitioned into three atoms, then for each $f \in \mathscr{D}(\mathcal{M}(X,\mathcal{A}))$, either $Z(f)$ or $X \setminus Z(f)$ is an atom. Using this fact, we prove the following result.

Theorem 5.14. *$AG(\mathcal{M}(X,\mathcal{A}))$ is complemented if and only if X can be partitioned either into two atoms or into three atoms.*

Proof. If X is partitioned into two atoms, then by Theorem 4.6, AG is a complete bipartite graph and hence AG is complemented. If X is partitioned into three atoms, then for each vertex f, either $Z(f)$ or $X \setminus Z(f)$ is an atom, and so by Theorem 4.13, each $f \in \mathscr{D}$ has an orthogonal complement. Consequently, AG is complemented. If X contains no atoms, then for any pair $f, g \in \mathscr{D}$, $f \not\perp g$. Hence, AG is not complemented. Let X contain an atom A, and X can not be partitioned into two or three atoms. Then $\mu(X \setminus A) > 0$, otherwise X will be an atom. By our hypothesis, $X \setminus A$ is not an atom. Then $X \setminus A = B_1 \sqcup B_2$, where $\mu(B_1), \mu(B_2) > 0$. Again by our hypothesis, one of B_1, B_2 is not an atom. Without loss of generality, let B_1 be not an atom. Then $B_1 = C_1 \sqcup C_2$, where $\mu(C_1), \mu(C_2) > 0$. Now consider $f \in \mathscr{D}$ such that $Z(f) = C_1 \sqcup B_2$. Then $X \setminus Z(f) = C_2 \sqcup A$. Therefore, both $Z(f), X \setminus Z(f)$ are not atoms. By Theorem 4.13, $f \in \mathscr{D}$ has no orthogonal complement. Consequently, AG is not complemented. $\square$

Theorem 5.15. *$AG(\mathcal{M}(X,\mathcal{A}))$ is complemented if and only if $AG(\mathcal{M}(X,\mathcal{A}))$ is uniquely complemented.*

The dominating number of $AG(\mathcal{M}(X,\mathcal{A}))$ is given below.

Theorem 5.16. $dt(AG(\mathcal{M}(X,\mathcal{A}))) = 2$.

Proof. Clearly, $dt(AG) > 1$. Fix an $A \in \mathcal{A}$ with $\mu(A), \mu(X \setminus A) > 0$. We claim that $\{1_A, 1_{X\setminus A}\}$ is a dominating set in AG. Suppose $f \in \mathscr{D}$ is not adjacent to $1_{X\setminus A}$ in AG, i.e., either $\mu(Z(f) \setminus A) = 0$ or $\mu(A \setminus Z(f)) = 0$. Using Theorem 4.5, we can prove that f is adjacent to 1_A in AG. Thus, $\{1_A, 1_{X\setminus A}\}$ is a dominating set in AG proving $dt(AG) = 2$. $\square$

The dominating set $\{1_A, 1_{X\setminus A}\}$ is a total dominating set in $AG(\mathcal{M}(X,\mathcal{A}))$. Therefore, the following result is immediate.

Corollary 5.17. $dt_t(AG(\mathcal{M}(X,\mathcal{A}))) = 2$.

6. The weakly zero-divisor graph $W\Gamma(\mathcal{M}(X,\mathcal{A}))$ of $\mathcal{M}(X,\mathcal{A})$

The weakly zero-divisor graph of $\mathcal{M}(X,\mathcal{A})$ is denoted by $W\Gamma(\mathcal{M}(X,\mathcal{A}))$ (in short $W\Gamma$) whose set of vertices is $\mathscr{D}(\mathcal{M}(X,\mathcal{A}))$ and two vertices f,g are adjacent, if there exist $h_1 \in ann(f)\cap\mathscr{D}(\mathcal{M}(X,\mathcal{A}))$ and $h_2 \in ann(g)\cap\mathscr{D}(\mathcal{M}(X,\mathcal{A}))$ such that $|h_1 \cdot h_2 \equiv 0$ a.e. on X. On using Lemma 4.1, we deduce the adjacency relation in measure theoretic way.

Theorem 6.1. *Let $f,g \in \mathscr{D}(\mathcal{M}(X,\mathcal{A}))$. In $W\Gamma(\mathcal{M}(X,\mathcal{A}))$,*

1. *if $\mu(Z(f)\triangle Z(g)) > 0$, then f,g are adjacent;*
2. *if $\mu(Z(f)\triangle Z(g)) = 0$ and $Z(f)$ is not an atom, then f,g are adjacent;*
3. *if $\mu(Z(f)\triangle Z(g)) = 0$ and $Z(f)$ is an atom, then f,g are not adjacent.*

We omit the proof because it can be accomplished using standard measure-theoretic arguments.

Corollary 6.2. *A vertex f is self-adjacent in $W\Gamma(\mathcal{M}(X,\mathcal{A}))$ if and only if $Z(f)$ is not an atom.*

To make $W\Gamma(\mathcal{M}(X,\mathcal{A}))$ a simple graph, we redefined the vertex set. Let $\mathscr{D}'(\mathcal{M}(X,\mathcal{A})) = \{f \in \mathscr{D}(\mathcal{M}(X,\mathcal{A})) : Z(f)$ is an atom$\}$ be the new vertex set of $W\Gamma(\mathcal{M}(X,\mathcal{A}))$. If μ is non-atomic, then $\mathcal{A}$ does not contain any atom and hence $W\Gamma(\mathcal{M}(X,\mathcal{A}))$ is an empty graph. For the non-emptiness of $W\Gamma(\mathcal{M}(X,\mathcal{A}))$, we must consider μ as not non-atomic. Then two vertices f,g are adjacent in $W\Gamma(\mathcal{M}(X,\mathcal{A}))$ if and only if $\mu(Z(f)\triangle Z(g)) > 0$. Consider the equivalence relation $\sim$ on $\mathscr{D}'(\mathcal{M}(X,\mathcal{A}))$ by: "$f \sim g$ if and only if $\mu(Z(f)\triangle Z(g)) = 0$". For each $f \in \mathscr{D}'(\mathcal{M}(X,\mathcal{A}))$, let $[f]$ denote the equivalence class of f under the relation $\sim$ on $\mathscr{D}'(\mathcal{M}(X,\mathcal{A}))$. Thus, for each $f \in \mathscr{D}'(\mathcal{M}(X,\mathcal{A}))$,

1. $[f]$ is a stable set in $W\Gamma(\mathcal{M}(X,\mathcal{A}))$.
2. if $[f],[g]$ are distinct classes, then $[f]\sqcup[g]$ is a complete bipartite subgraph of $W\Gamma(\mathcal{M}(X,\mathcal{A}))$.

Let W be a collection of distinct class representatives under $\sim$ on $\mathscr{D}'(\mathcal{M}(X,\mathcal{A}))$. Two atoms $A,B \in \mathcal{A}$ are said to be distinct, if $\mu(A\triangle B) > 0$. Therefore, $|W|$ is the number of distinct atoms in $\mathcal{A}$. Hence, the following theorem is immediate.

Theorem 6.3. *$W\Gamma(\mathcal{M}(X,\mathcal{A}))$ is a complete $|W|$-partite graph.*

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