



ON SOME PROPERTIES OF NEUTRAL SFS-SPACES

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ABSTRACT. One of the important problems of the operator algebras theory is the geometric characterization of state spaces of operator algebras. In this regard, in the mid-1980s, a paper by Friedman and Russo introduced facially symmetric spaces. The primary aim of this work was to provide the geometric characterization of predual spaces of JBW^* -triples that possess an algebraic structure. Many of the properties required in these characterizations are natural assumptions for the state spaces of physical systems. Such spaces are considered as a geometric model for the states of quantum mechanics.

In this paper, we show that if any indecomposable geometric tripotent of a neutral strongly facially symmetric space is a minimal geometric tripotent then any extreme point is a norm exposed point. Moreover, in an atomic neutral locally base normed strongly facially symmetric space any extreme point is a norm exposed point. We also prove that every real neutral strongly facially symmetric space with unitary tripotents is finite.

1. Introduction

In the early 1980s, the development of the theory of JB^* -triples began in the works of Kaup; in many respects, these triples are parallel to the function-analytical aspects of the theory of operator algebras [9, 10]. The JB^* -triples that are characterized by holomorphic properties of their unit balls form a wide class of Banach spaces based on ternary algebraic structures, which contains C^* -algebras, Hilbert spaces, and spaces of rectangular matrices. Many axioms used by Alfsen and Schultz show that there are unordered analogs of JB^* -triples [1]. In 1989, the paper [2] of Friedman and Russo was published, in which facially symmetric spaces were introduced; the main purpose for introducing

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these spaces was a geometric characterization of the predual spaces of Banach spaces admitting an algebraic structure. Many properties required in these characterizations are natural assumptions for state spaces of physical systems. These spaces are considered as a geometric model for states in quantum mechanics.

It was proved, in [6], that the predual space of a complex von Neumann algebra, and more general, JB^* -triples is a neutral strongly facially symmetric space. In [5], a geometric characterization was given for complex Hilbert spaces and complex spin factors, and also a description of JBW^* -triples of ranks 1 and 2 and of Cartan factors of types 1 and 4 was presented. Later, Friedman and Russo [4] obtained a description of atomic facially symmetric spaces and showed that a neutral strongly facially symmetric space is isometrically isomorphic to the predual space of one of the Cartan factors of types 1-6. Neal and Russo [12] found geometric conditions under which a facially symmetric space is isometric to the predual space of a complex JBW^* triple. In [8], a complete description of strongly facially symmetric spaces that are isometrically isomorphic to the predual space of an atomic commutative von Neumann algebra was obtained. In [7], it was proved that the predual space of the real part of a von Neumann algebra is a strongly facially symmetric space, if and only if this algebra is the direct sum of an Abelian algebra and an algebra of type I_2 . In [11], it was proved that the predual space of a JBW -algebra is a strongly facially symmetric space, if and only if the algebra is the direct sum of an Abelian algebra and an algebra of type I_2 .

In this paper, we show that if any non-decomposable geometric tripotent of an neutral strongly facially symmetric space is a minimal geometric tripotent, then any extreme point is norm exposed point. Moreover, in atomic neutral locally base normed strongly facially symmetric space, any extreme point is a norm exposed point. We also prove that every real neutral strongly facially symmetric space with unitary geometric tripotents is finite.

2. Strongly facially symmetric spaces

Let Z be a real or complex normed space. We say that elements $f, g \in Z$ are orthogonal and write $f \diamond g$, if $\|f + g\| = \|f - g\| = \|f\| + \|g\|$. We say subsets $S, T \subset Z$ are orthogonal, and write $S \diamond T$, if $f \diamond g$ for all $(f, g) \in S \times T$. For a subset S of Z , we put $S^\diamond = \{f \in Z : f \diamond g \forall g \in S\}$; the set S^\diamond is called the orthogonal complement of S . A convex subset F of the unit ball $Z_1 = \{f \in Z : \|f\| \leq 1\}$ is called a face, if the relation $\lambda g + (1 - \lambda)h \in F$, where $g, h \in Z_1$, $\lambda \in (0, 1)$, implies $g, h \in F$. A face F of the unit ball is said to be norm exposed, if $F_u = \{f \in Z_1 : u(f) = 1\}$, for some $u \in Z^*$ with $\|u\| = 1$. An element $u \in Z^*$ is called a projective unit, if $\|u\| = 1$ and $u(g) = 0$ for all $g \in F_u^\diamond$.

A norm exposed face F_u in Z_1 is called a symmetric face, if there exists a linear isometry S_u from Z to Z such that $S_u^2 = I$ whose fixed point set coincides with the topological direct sum of the closure $\overline{\text{sp}}F_u$ of the linear hull of the face F_u and its orthogonal complement F_u^\diamond , i.e., with $\overline{\text{sp}}F_u \oplus F_u^\diamond$.

A space Z is said weakly facially symmetric (WFS), if each norm exposed face in Z_1 is symmetric.

For each symmetric face F_u , contractive projections $P_k(F_u)$, $k = 0, 1, 2$ on Z are defined as follows. First, $P_1(F_u) = (I - S_u)/2$ is the projection on the eigenspace corresponding to the eigenvalue -1 of the

symmetry S_u . Next, $P_2(F_u)$ and $P_0(F_u)$ are defined as projections of Z onto $\overline{sp}F_u$ and F_u^\diamond , respectively; i.e., $P_2(F_u) + P_0(F_u) = (I + S_u)/2$. The projections $P_k(F_u)$ are called geometric Peirce projections.

A WFS space Z is said to be strongly facially symmetric (SFS), if for each norm exposed face F_u of Z_1 and each $y \in U$ satisfying the conditions $\|y\| = 1$ and $F_u \subset F_y$, we have $S_u^*y = y$, where S_u is the symmetry corresponding F_u .

A projective unit $u \in Z^*$ is called geometric tripotent, if F_u is a symmetric face and $S_u^*u = u$ for the symmetry S_u corresponding to F_u . By \mathcal{GT} and \mathcal{SF} we denote the sets of all geometric tripotents and symmetric faces, respectively; the correspondence $\mathcal{GT} \ni u \mapsto F_u \in \mathcal{SF}$ is one-to-one [2, Proposition 1.6]. In [13], a necessary and sufficient condition was found when the elements of the dual space for a reflexive SFS-space are geometric tripotents.

A contractive projection Q on Z is said to be neutral, if $\|Qf\| = \|f\|$ implies $Qf = f$ for each $f \in Z$. A space Z is said to be neutral, iff for each symmetric face F_u , the projection $P_2(u)$ corresponding to the symmetry S_u is neutral.

Instructive examples of neutral complex strongly facially symmetric spaces are Hilbert spaces, the preduals of von Neumann algebras or JBW^* -algebras, and more generally, the preduals of complex JBW^* -triples. Moreover, geometric tripotents correspond to nonzero partial isometries of von Neumann algebras and tripotents in a JBW^* -triples (see [5]).

For each geometric tripotent u from the dual WFS space Z , we denote the geometric Peirce projections by $P_k(u) = P_k(F_u)$, $k = 0, 1, 2$.

We set

$$U = Z^*, \quad U_1 = Z_1^*, \quad Z_k(u) = Z_k(F_u) = P_k(u)Z, \quad U_k(u) = P_k(F_u) = P_k(u)^*Z^*.$$

The geometric Peirce decomposition

$$Z = Z_2(u) + Z_1(u) + Z_0(u), \quad U = U_2(u) + U_1(u) + U_0(u)$$

holds. Geometric tripotents u and v are said to be orthogonal, if $u \in U_0(v)$ (which implies $v \in U_0(u)$) or, equivalently, $u \pm v \in \mathcal{GT}$ (see [3, Lemma 2.5]). More generally, elements a and b of U are said to be orthogonal, denoted $a \diamond b$, if one of them belongs to $U_2(u)$ and the other belongs to $U_0(u)$ for some geometric tripotent u .

In the neutral SFS-space Z for $f \neq 0$ by v_f denote a single geometric tripotent v for which $\langle f, v \rangle = \|f\|$ and $\langle v, \{f\}^\diamond \rangle = 0$. If $f, g \in Z$, then $f \diamond g$ if and only if $v_f \diamond v_g$.

3. Extreme points of the neutral SFS-spaces

Let Z be a strongly facially symmetric space. A symmetric face F_u of a unit ball Z_1 has the Jordan decomposition property (JD), if for every nonzero $f \in sp_{\mathbb{R}}F_u$, where $sp_{\mathbb{R}}F_u$ is a real linear shell of F_u , there exist such $g, h \in \mathbb{R}^+F_u$ that $f = g - h$ and $g \diamond h$ or, equivalently, $(sp_{\mathbb{R}}F_u)_1 \subset co(F_u \cup F_{-u})$ (see [12, Lemma 2.3]).

Definition 3.1 ([12]). *A strongly facially symmetric space Z has the property (JD), if every symmetric face has the Jordan decomposition property. In this case, we say that Z is locally base normed.*

Definition 3.2. A normed space Z is said to be atomic, if every symmetric face of Z_1 has an extreme point.

Definition 3.3 ([5]). A neutral strongly facially symmetric space Z is said to satisfy property (PE), if every extreme point of Z_1 is a norm exposed point.

A geometric tripotent u is called minimal, if $\dim U_2(u) = 1$. By M denote the set of all minimal geometric tripotents. In a weakly facially symmetric space for $u \in M$ every face F_u is a singleton, that is, to each $u \in M$ there corresponds a unique norm-exposed point of Z_1 (see [5, Proposition 2.4]).

A geometric tripotent u is called indecomposable, if for $v \in \mathcal{GT}$ from $v \leq u$, it follows $v = u$. By I we denote the set of all indecomposable geometric tripotents. Since every minimal geometric tripotent is indecomposable, it follows that $M \subset I$. Naturally, the question arises whether the reverse inclusion is true. In all known examples of SFS-spaces, the condition $M = I$ always holds. In [5, proposition 2.9], a positive answer to this question is given in the special case when a neutral atomic strongly facially symmetric space has the property (PE). Further, in the case of neutral strongly facially symmetric spaces with the property (JD) such an answer was obtained in [12, proposition 2.2]. In [14] it was shown that a reflexive strongly facially symmetric space of rank 1 has the property (PE) and $M = I$, and in [17] it was proved that every finite-dimensional real neutral strongly facially symmetric space has the properties (PE), $M = I$, (JD).

In the work of Friedman and Russo, the following problem was formulated (see [2, Problem 3]):

Problem. Let Z be a weakly facially symmetric space. Is every extreme point of Z_1 a norm-exposed point?

The next theorem gives a positive answer to this problem.

Theorem 3.4. Let Z be an neutral strongly facially symmetric space with the property $M = I$. Then Z has the property (PE).

Proof. Let $f \in Z_1$ be an extreme point. According to [2, Theorema 4.3], it follows that there is the smallest norm exposed face F_u containing f , where $u = u_f$. We show that u is an indecomposable geometric tripotent.

Suppose it does not hold. Let $u \in \mathcal{GT} \setminus I$, then there exists a geometric tripotent ν such that $\nu \leq u$ and $\nu \neq u$. Suppose first that $f \in \overline{sp}F_\nu \oplus F_\nu^\circ$. Then

$$(3.1) \quad \begin{aligned} f &= \|P_2(\nu)f\| \frac{P_2(\nu)f}{\|P_2(\nu)f\|} + \|P_0(\nu)f\| \frac{P_0(\nu)f}{\|P_0(\nu)f\|}, \\ \|P_2(\nu)f\| + \|P_0(\nu)f\| &= \|P_2(\nu)f + P_0(\nu)f\| = \|f\| = 1. \end{aligned}$$

Since f is an extreme point and $P_2(\nu)P_0(\nu) = 0$, by (3.1), it follows that or $f = P_2(\nu)f$, or $f = P_0(\nu)f$.

Let us assume $f = P_2(\nu)f$. Then

$$1 = \langle f, u \rangle = \langle P_2(\nu)f, u \rangle = \langle f, P_2(\nu)^*u \rangle = \langle f, \nu \rangle$$

Therefore, $f \in F_v$. This contradicts the fact that F_u is the smallest symmetric face containing f . So, $f \neq P_2(v)f$.

Let $f = P_0(v)f$. Then by [2, Theorema 4.2] we have

$$1 = \langle f, u \rangle = \langle P_0(v)f, u \rangle = \langle f, P_0(v)^*(v + u - v) \rangle = \langle f, u - v \rangle$$

Thus, $f \in F_{u-v}$. This contradicts the fact that F_u is the smallest symmetric face containing f . So, $f \neq P_0(v)f$. Hence, $f \notin \overline{sp}F_v \oplus F_v^\circ$. Then $f \in F_u \setminus \overline{sp}F_v \oplus F_v^\circ$.

On the subspace $sp\{f\} \oplus \overline{sp}F_v \oplus F_v^\circ$, we define a functional x by

$$x(\lambda f + g) = \lambda, \lambda \in \mathbb{C}, g \in \overline{sp}F_v \oplus F_v^\circ.$$

By the Hahn-Banach theorem, x admits an extension to Z with the same norm; we denote this extension by the same symbol x . We have

$$(3.2) \quad \|x\| = 1, x(f) = 1, x|_{\overline{sp}F_v \oplus F_v^\circ} = 0.$$

Take geometric tripotent $w \in Z^*$ such that $F_x = F_w$. Then by virtue of (3.2) we have that

$$(3.3) \quad w(f) = 1, w|_{\overline{sp}F_v \oplus F_v^\circ} = 0.$$

Next, assume that $F_u \cap F_w = F_u$, then $F_v \subset F_w$, which contradicts (3.3). Assume that $F_u \cap F_w \neq F_u$. Then there exists a geometric tripotent e such that $F_e = F_u \cap F_w$. Since $F_u \cap F_w \neq F_u$, then $e < u$. Since $f \in F_e$, this contradicts the fact that F_u is the smallest norm exposed face containing f . So, $u \in I$. $u \in M$ derives from the assumption that $M = I$. Hence, by virtue of [5, proposition 2.4] for every $u \in M$, the face F consists of a single point, i.e. $F_u = \{f\}$. This means that f is a norm exposed point. □

Using Theorem 4.4 and [5, Proposition 2.9] we get to the following result.

Corollary 3.5. *Let Z be an atomic neutral strongly facially symmetric space. Then the properties (PE) and $M = I$ are equivalent.*

In [12, Proposition 2.2], it is shown that in a neutral locally base normed strongly facially symmetric space, the property $M = I$ holds. Therefore, from Corollary 3.5 we can conclude the following.

Corollary 3.6. *If Z is an atomic neutral locally base normed strongly facially symmetric space, then every extreme point of Z_1 is a norm-exposed point.*

4. Finite real neutral SFS-spaces

Let Z be a neutral strongly facially symmetric space.

A geometric tripotent u is called

- 1) maximal if $U_0(u) = \{0\}$;
- 2) complete if $U_2(u) = U$;
- 3) unitary if $Z_1 = \text{co}(F_u \cup F_{-u})$.

Proposition 4.1. *Let Z be a neutral strongly facially symmetric space and $\alpha \in \mathbb{C}, |\alpha| = 1$.*

- 1) If $u \in I$, then $\alpha u \in I$.
- 2) If $u \in M$, then $\alpha u \in M$.
- 3) If u is a maximal geometric tripotent, then αu is a maximal geometric tripotent.
- 4) If u a complete geometric tripotent, then αu is a complete geometric tripotent.
- 5) If u a unitary geometric tripotent, then αu is a unitary geometric tripotent.

Proof. 1) Let $u \in I$ and $\alpha \in \mathbb{C}$, $|\alpha| = 1$. Then by [15, Lemma 3.5] it follows that αu is a geometric tripotent. Now we will show the indecomposability of geometric tripotents αu . Let us assume that $\alpha u \notin I$, i.e. there is a geometric tripotent ν such that $\nu \leq \alpha u$. Then according to [15, Proposition 3.6] we have $\bar{\alpha}\nu \leq u$. Since u is indecomposable, it follows that $\bar{\alpha}\nu = u$. Therefore, $\nu = \alpha u$. This means that αu is an indecomposable geometric tripotent.

2) Let $u \in M$ and $\alpha \in \mathbb{C}$, $|\alpha| = 1$. Then by [15, Lemma 3.5] it follows that αu is a geometric tripotent and $F_{\alpha u} = \alpha F_u$. Therefore,

$$Z_2(\alpha u) = \overline{sp}F_{\alpha u} = \overline{sp}\alpha F_u = \overline{sp}F_u = Z_2(u).$$

Thus, $U_2(\alpha u) = U_2(u)$. Hence, $\dim U_2(\alpha u) = 1$, i.e. $\alpha u \in M$ for all $\alpha \in \mathbb{C}$, $|\alpha| = 1$.

3) Let u be a maximal geometric tripotent, and let $\alpha \in \mathbb{C}$, $|\alpha| = 1$. Then by [15, Lemma 3.5] it follows that αu is a geometric tripotent and $F_{\alpha u} = \alpha F_u$. Therefore, by [15, Lemma 3.5] we have $F_{\alpha u}^\diamond = \alpha F_u^\diamond = 0$, i.e. $U_0(u) = \{0\}$. Hence, αu is a maximal geometric tripotent.

4) Let u be a complete geometric tripotent and $\alpha \in \mathbb{C}$, $|\alpha| = 1$. Then by [15, Lemma 3.5] it follows that αu is a geometric tripotent and $F_{\alpha u} = \alpha F_u$. So,

$$Z_2(\alpha u) = \overline{sp}F_{\alpha u} = \overline{sp}\alpha F_u = \overline{sp}F_u = Z_2(u) = Z,$$

i.e. αu is a complete geometric tripotent.

5) Let u be a complete geometric tripotent and $\alpha \in \mathbb{C}$, $|\alpha| = 1$. Then by [15, Lemma 3.5] it follows that αu is a geometric tripotent and $F_{\alpha u} = \alpha F_u$. Hence,

$$\text{co}(F_{\alpha u} \cup F_{-\alpha u}) = \text{co}(\alpha F_u \cup \alpha F_{-u}) = \alpha \text{co}(F_u \cup F_{-u}) = \alpha Z_1 = Z_1,$$

i.e. αu is a unitary geometric tripotent. □

Remark 4.2. *Every unitary geometric tripotent is complete. Although in all examples of SFS-spaces known to us, every complete geometric tripotent is unitary, but in the general case the validity of this statement remains open.*

Proposition 4.3. *Let Z be a real locally base normed neutral SFS-space. Then the sets of complete geometric tripotents and unitary geometric tripotents coincide.*

Proof. It is known that every unitary geometric tripotent is a complete geometric tripotent. Let's show the opposite. Let u be a complete geometric tripotent. Since Z is a real neutral locally basic normed SFS space, then by [12, Lemma 2.3] we have

$$(4.1) \quad (sp_{\mathbb{R}}F_u)_1 \subset \text{co}(F_u \cup F_{-u}),$$

On the other hand, u is a complete geometric tripotent, then $(sp_{\mathbb{R}}F_u)_1 = Z_1$. Therefore, from (4.1) it follows that $Z_1 \subset co(F_u \cup F_{-u})$. Hence, $Z_1 = co(F_u \cup F_{-u})$, i.e. u is a unitary geometric tripotent. \square

A geometric tripotent w is finite, if any maximal geometric tripotent $u \in U_2(w)$ is already complete in $U_2(w)$. The SFS-space Z itself is finite, if any geometric tripotent in U is finite.

In [15] it was proved that every finite-dimensional real neutral strongly facially symmetric space is finite.

Theorem 4.4. *Let Z be a real neutral SFS-space with unitary geometric tripotent. Then Z is finite.*

Proof. Let v be a unitary geometric tripotent (therefore, complete, i.e. $U_2(v) = U$) and u is a maximal geometric tripotent in $U_2(v)$. Then by [16, Lemma 3.4], there exist mutually orthogonal geometric tripotents $w, \bar{w} \leq v$ such that $u = w - \bar{w}$. Since u is a maximal geometric tripotent, then by [16, Lemma 3.5] we get that $w + \bar{w}$ is a maximal geometric tripotent.

On the other hand, if $w, \bar{w} \leq v$ are mutually orthogonal geometric tripotents, then by [3, Lemma 2.1] we obtain that $F_w, F_{\bar{w}} \subset F_{w+\bar{w}}$, consequently $w, \bar{w} \leq w + \bar{w}$. Since, $L_v = \{\nu \in \mathcal{GT} : \nu \leq v\} \cup \{0\}$ is a complete orthomodular lattice (see [2, Proposition 4.5]), then $w + \bar{w} \leq v$. Therefore, by the maximality of $w + \bar{w}$ it follows that $v = w + \bar{w}$.

Let $f \in F_{w+\bar{w}}$. Then $\|P_2(w - \bar{w})f\| \leq \|f\|$ and, in view of [2, Lemma 4.2] we obtain

$$\begin{aligned} \langle P_2(w - \bar{w})f, w + \bar{w} \rangle &= \langle f, P_2(w - \bar{w})^*(w + \bar{w}) \rangle \\ &= \langle f, P_2(w - \bar{w})^*w \rangle + \langle f, P_2(w - \bar{w})^*\bar{w} \rangle = \langle f, w + \bar{w} \rangle = 1 \end{aligned}$$

Consequently, $\|P_2(w - \bar{w})f\| = \|f\|$ and from the neutrality of space Z we have that $P_2(w - \bar{w})f = f$. Thus, $P_2(w - \bar{w})g = g$ for all $g \in \overline{sp}F_v$. Then $\overline{sp}F_u = Z$, i.e. u is complete in $U_2(v)$. Therefore, v is finite geometric tripotent. It is known that (see [15, Lemma 4.2(b)]) if $v \in \mathcal{GT}$ is a finite geometric tripotent, then any geometric tripotent in $U_2(v)$ is also finite in U . Then any geometric tripotent finite in U . Hence, Z is finite. \square

Example 4.5. *Let's consider the space $l_1(n)$ with the norm $\|x\| = \sum_{i=1}^n |x_i|$, $x = (x_i) \in l_1(n)$, which is a neutral SFS-space. In this space, there exists a unitary geometric tripotent $e \in l_{\infty}(n) \cong (l_1(n))^*$, $e = (\varepsilon_1, \dots, \varepsilon_n)$, $\varepsilon_i \in \{-1, 1\}$, $i = \overline{1, n}$, and the symmetric face corresponding to e has the following form*

$$F_e = \left\{ x \in l_1(n) : \sum_{i=1}^n \varepsilon_i x_i = 1, \varepsilon_i x_i \geq 0, i = \overline{1, n} \right\}.$$

Then, according to the Theorem 4.4, the space $l_1(n)$ is finite.

More generally, consider a measure space (Ω, Σ, μ) with measure μ having the direct sum property, i.e. there is a family $\{\Omega_i\}_{i \in J} \subset \Sigma, 0 < \mu(\Omega_i) < \infty, i \in J$, such that for any $A \in \Sigma$ with $\mu(\Omega_i) < \infty$, there exist a countable subset $J_0 \subset J$ and a set B of zero measure such that $A = \bigcup_{i \in J_0} (A \cap \Omega_i) \cup B$.

Let $L_1(\Omega, \Sigma, \mu)$ be the space of all (classes of) real integrable functions on (Ω, Σ, μ) . According to Theorem 4.4 the space $L_1(\Omega, \Sigma, \mu)$ with the norm

$$\|f\| = \int_{\Omega} |f(t)| d\mu(t), \quad f \in L_1(\Omega, \Sigma, \mu),$$

is a finite neutral SFS-space with unitary geometric tripotent $e = \tilde{\chi}_A - \tilde{\chi}_{\Omega \setminus A} \in L^\infty(\Omega, \Sigma, \mu) \cong L_1(\Omega, \Sigma, \mu)^*$ for some $A \in \Sigma$, where $\tilde{\chi}_A$ is the class containing the indicator function of the set $A \in \Sigma$ and the symmetric face corresponding to the geometric tripotent e has the following form:

$$F_e = \left\{ f \in L_1(\Omega, \Sigma, \mu) : \|f\| = 1, \int_{\Omega} e(t)f(t)d\mu(t) = 1 \right\}.$$

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