



## INNER AMENABILITY OF CERTAIN LAU ALGEBRAS ASSOCIATED TO DISCRETE CROSSED PRODUCTS

M. R. GHANEI

ABSTRACT. For a discrete group  $\Gamma$ , a Hopf von Neumann algebra  $(\mathfrak{M}, \Delta)$  and a  $W^*$ -dynamical system  $(\mathfrak{M}, \Gamma, \alpha)$  such that  $(\alpha_s \otimes \alpha_s) \circ \Delta = \Delta \circ \alpha_s$ , we show that the crossed product  $\mathfrak{M} \rtimes_{\alpha} \Gamma$  with a comultiplication is a Hopf von Neumann algebra. Furthermore, we prove that the inner amenability of the predual  $\mathfrak{M}_*$  is equivalent to the inner amenability of  $(\mathfrak{M} \rtimes_{\alpha} \Gamma)_*$ . Finally, we conclude that if the action  $\alpha : \Gamma \rightarrow \text{Aut}(\ell^{\infty}(\Gamma))$  is defined by  $\alpha_s(f)(t) = f(s^{-1}ts)$ , then the inner amenability of discrete group  $\Gamma$  is equivalent to the inner amenability of  $(\ell^{\infty}(\Gamma) \rtimes_{\alpha} \Gamma)_*$ .

### 1. Introduction

In the field of mathematical analysis, Lau [5] established the foundation for investigating a category of Banach algebras initially termed as  $F$ -algebras. Subsequently, Pier [9] adopted the term "Lau algebras" for this classification. A Lau algebra, denoted as  $\mathcal{L}$ , is a complex Banach algebra that serves as the predual of a von Neumann algebra  $\mathfrak{M}$ , with the identity element 1. This category of Banach algebras contains both the group algebras and the measure algebras of locally compact groups, as mentioned in [5, 6]. Moreover, the predual of a Hopf von Neumann algebra, specifically the quantum group algebra  $L^1(\mathbb{G})$  associated with a locally compact quantum group  $\mathbb{G}$  is contained in the class of Lau algebras.

Lau defined the notion of left amenability of these algebras in [5]. A Lau algebra  $\mathcal{L}$  is *left amenable* if there is a mean  $M \in \mathcal{L}^{**}$ , i.e.  $\|M\| = M(1) = 1$ , such that  $M(f \cdot a) = 1(a)M(f)$  for all  $a \in \mathcal{L}$

---

Communicated by Massoud Amini

MSC(2020): Primary: 43A07; Secondary: 46L10.

Keywords: Discrete crossed product,  $W^*$ -dynamical systems, Hopf von Neumann algebras, inner amenability of Lau algebras

Received: 29 July 2024, Accepted: 16 November 2024.

DOI: <https://dx.doi.org/10.30504/jims.2024.470441.1197>

and  $f \in \mathcal{L}^*$ . Equivalently,  $a \odot M = 1(a)M$ , where the first Arens multiplication  $\odot$  is defined by the equations

$$\langle F \odot H, f \rangle = \langle F, Hf \rangle, \quad \langle Hf, a \rangle = \langle H, f \cdot a \rangle, \quad \langle f \cdot a, b \rangle = \langle f, ab \rangle$$

for all  $F, H \in \mathcal{L}^{**}$ ,  $f \in \mathcal{L}^*$ , and  $a, b \in \mathcal{L}$ . For more details about this notion, we refer the reader to [10, 11]. It is worth noting that for any Banach space  $X$ , the duality between  $X$  and its dual Banach space  $X^*$  is written as  $\langle \phi, \xi \rangle$  or  $\phi(\xi)$  for all  $\xi \in X$  and  $\phi \in X^*$ .

Nasr-Isfahani [8] introduced and studied inner amenability for Lau algebras. Indeed, if  $\mathcal{L}$  is a Lau algebra, then a mean  $M \in \mathcal{L}^{**}$  is said to be *inner invariant mean* for  $\mathcal{L}$ , if  $M(f \cdot a) = M(a \cdot f)$ , or equivalently  $a \odot M = M \odot a$  for all  $a \in \mathcal{L}$ . The Lau algebra  $\mathcal{L}$  is called *inner amenable*, if there is an inner invariant mean in  $\mathcal{L}^{**}$ ; see also [1, 3].

In the context of Lau algebras, an element  $E$  in  $\mathcal{L}^{**}$  is called *mixed identity*, if it satisfies  $a \odot E = E \odot a = a$  for every  $a \in \mathcal{L}$ . Notably, a simple observation reveals that  $E \in \mathcal{L}^{**}$  qualifies as a mixed identity, if and only if it serves as a weak\* cluster point of a bounded approximate identity within  $\mathcal{L}$ . Note that, if the Lau algebra  $\mathcal{L}$  has a bounded approximate identity, then it is inner amenable. Consequently, we define a *strictly inner invariant mean* for  $\mathcal{L}$  as an inner invariant mean that avoids the mixed identity status. Hence, our focus on the exploration of inner amenability shifts to cases where a strictly inner invariant mean for  $\mathcal{L}$  exists.

Recall that a pair  $(\mathfrak{M}, \Delta)$  is a *Hopf von Neumann algebra* when  $\mathfrak{M}$  is a von Neumann algebra and  $\Delta : \mathfrak{M} \rightarrow \mathfrak{M} \bar{\otimes} \mathfrak{M}$  is a co-multiplication; i.e., a normal, unital, and \*-homomorphism satisfying

$$(\text{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \text{id}) \circ \Delta,$$

where  $\text{id}$  is the identity map on  $\mathfrak{M}$  and  $\bar{\otimes}$  denote the von Neumann algebra tensor product. For a Hopf von Neumann algebra  $(\mathfrak{M}, \Delta)$ , one can define a product  $*$  on  $\mathfrak{M}_*$ , turning it into Banach algebra, by  $\langle f * g, x \rangle = \langle f \otimes g, \Delta x \rangle$  for all  $f, g \in \mathfrak{M}_*$  and  $x \in \mathfrak{M}$ . Moreover, the module actions of  $\mathfrak{M}_*$  on  $\mathfrak{M}$  are given by

$$f \cdot x := (\text{id} \otimes f)(\Delta x) \quad \text{and} \quad x \cdot f := (f \otimes \text{id})(\Delta x),$$

for all  $f \in \mathfrak{M}_*$  and  $x \in \mathfrak{M}$ .

It is worthwhile to mention that there exists a Hopf von Neumann algebra  $\mathbb{H}_a := (L^\infty(G), \Delta_a)$  obtained from a locally compact group  $G$ , where  $L^\infty(G)$  is the usual Lebesgue space as defined in [4] and the co-multiplication  $\Delta_a : L^\infty(G) \rightarrow L^\infty(G) \bar{\otimes} L^\infty(G) = L^\infty(G \times G)$  is defined by  $\Delta_a f(r, t) = f(rt)$  for all  $f \in L^\infty(G)$  and  $r, t \in G$ .

In this paper, by using a discrete group  $\Gamma$  and a  $W^*$ -dynamical systems  $(\mathfrak{M}, \Gamma, \alpha)$ , we show that if  $(\mathfrak{M}, \Delta)$  is a Hopf von Neumann algebra and  $(\alpha_s \otimes \alpha_s) \circ \Delta = \Delta \circ \alpha_s$ , then the crossed product  $(\mathfrak{M} \rtimes_\alpha \Gamma)$  is Hopf von Neumann algebra and we prove an equivalence between the inner amenability of the Lau algebra  $\mathfrak{M}_*$  and the Lau algebra  $(\mathfrak{M} \rtimes_\alpha \Gamma)_*$ ; see Theorem 2.2. At the end of the paper we show that if the action of  $\alpha : \Gamma \rightarrow \text{Aut}(L^\infty(\Gamma))$  is defined by  $\alpha_s(f)(t) = f(s^{-1}ts)$ , then the inner amenability of  $\Gamma$  is equivalent to the inner amenability of  $(L^\infty(\Gamma) \rtimes_\alpha \Gamma)_*$ ; see Corollary 2.4.

### 2. Inner amenability and crossed product

A discrete  $W^*$ -dynamical system is defined as a triple  $(\mathfrak{M}, \Gamma, \alpha)$ , where  $\mathfrak{M}$  is a von Neumann algebra,  $\Gamma$  is a discrete group and  $\alpha : \Gamma \rightarrow \text{Aut}(\mathfrak{M})$  is a homomorphism of  $\Gamma$  into the group of  $*$ -automorphisms of  $\mathfrak{M}$ .

If we denote the Hilbert space on which the von Neumann algebra  $\mathfrak{M}$  acts as  $H$ , then a faithful normal  $*$ -representation  $\pi : \mathfrak{M} \rightarrow B(\ell^2(\Gamma, H))$  and a unitary representation  $\lambda : \Gamma \rightarrow B(\ell^2(\Gamma, H))$  can be defined as follows:

$$\pi(x)\xi(s) = \alpha_{s^{-1}}(x)\xi(s), \quad \lambda_s\xi(t) = \xi(s^{-1}t)$$

for all  $x \in \mathfrak{M}$ ,  $s, t \in \Gamma$  and  $\xi \in \ell^2(\Gamma, H)$ , where  $B(\ell^2(\Gamma, H))$  is the algebra of bounded linear operators on the Hilbert space  $\ell^2(\Gamma, H)$ . Additionally, the pair  $(\pi, \lambda)$  forms a covariant representation of  $(\mathfrak{M}, \Gamma, \alpha)$ , in the sense that  $\pi(\alpha_s(x)) = \lambda_s\pi(x)\lambda_s^*$  for all  $x \in \mathfrak{M}$  and  $s \in \Gamma$ .

The von Neumann algebra generated by  $\pi(\mathfrak{M}) \cup \lambda(\Gamma)$  is termed the *crossed product* of  $\mathfrak{M}$  by  $\Gamma$  and is denoted by  $\mathfrak{M} \rtimes_{\alpha} \Gamma$ . It is worth noting that  $\mathfrak{M} \rtimes_{\alpha} \Gamma$  includes an isometric copy of  $\mathfrak{M}$  and the group von Neumann algebra  $VN(\Gamma)$  of  $\Gamma$  and

$$\mathfrak{M} \rtimes_{\alpha} \Gamma = \text{span}\{\pi(x)\lambda_s : x \in \mathfrak{M}, s \in \Gamma\}^{\overline{w^*}}.$$

For simplicity, we will use the same notation for copies of elements, denoting both  $x \in \mathfrak{M}$  and  $s \in \Gamma$  by  $x$  and  $s$ , respectively. Henceforth, we will simplify notation and represent generating elements of  $\mathfrak{M} \rtimes_{\alpha} \Gamma$  as  $xs$  for all  $x \in \mathfrak{M}$  and  $s \in \Gamma$ , subject to the relations  $xs := \pi(x)\lambda_s$  and  $sx := \lambda_s\pi(x)$ . Therefore,

$$sx = \lambda_s\pi(x)\lambda_{s^{-1}}\lambda_s = \pi(\alpha_s(x))\lambda_s = \alpha_s(x)s.$$

Moreover, it is clear that  $xs = \lambda_s\pi(\alpha_{s^{-1}}(x)) = s\alpha_{s^{-1}}(x)$ .

We denote by  $\ell^{\infty}(\Gamma; \mathfrak{M})$  (resp.  $K(\Gamma; \mathfrak{M})$ ) the space of all bounded (resp. with compact support) functions  $u$  of  $\Gamma$  into  $\mathfrak{M}$  with the norm  $\|u\|_{\infty} = \sup_{s \in \Gamma} \|u(s)\|$ .

Let  $(\mathfrak{M}, \Gamma, \alpha)$  be a discrete  $W^*$ -dynamical system. Define in  $K(\Gamma; \mathfrak{M})$  the involutive normed algebra structure by

$$x * y(t) = \sum_{s \in \Gamma} x(s)\alpha_s(y(s^{-1}t)), \quad x^*(s) = \alpha_s(x(s^{-1})^*), \quad \|x\|_1 = \sum_{s \in \Gamma} \|x(s)\|.$$

For each  $z \in K(\Gamma; \mathfrak{M})$ , define  $\hat{\pi}_{\alpha}(z) \in \mathfrak{M} \rtimes_{\alpha} \Gamma$  by

$$\hat{\pi}_{\alpha}(z) = \sum_{s \in \Gamma} \pi_{\alpha}(z(s))\lambda_s = \sum_{s \in \Gamma} z(s)s.$$

Then, it is not hard to see that  $\hat{\pi}_{\alpha}$  is a faithful  $*$ -representation of  $K(\Gamma; \mathfrak{M})$  into  $\mathfrak{M} \rtimes_{\alpha} \Gamma$  and the image of  $K(\Gamma; \mathfrak{M})$  by  $\hat{\pi}_{\alpha}$  is a  $\sigma$ -weakly dense  $*$ -subalgebra of  $\mathfrak{M} \rtimes_{\alpha} \Gamma$ .

The elements of  $(\mathfrak{M} \rtimes_{\alpha} \Gamma)_*$  can be identified with functions  $u : \Gamma \rightarrow \mathfrak{M}_*$  such that  $u(s)(x) = \tilde{u}(xs)$  for  $s \in \Gamma$  and  $x \in \mathfrak{M}$ , where  $\tilde{u} \in (\mathfrak{M} \rtimes_{\alpha} \Gamma)_*$  takes the form:  $\tilde{u}(T) = \sum_{n=1}^{\infty} \langle T\xi_n, \eta_n \rangle$  for some  $(\xi_n), (\eta_n) \subseteq \ell^2(\Gamma, H)$ , satisfying  $\sum_{n=1}^{\infty} \|\xi_n\|^2 < \infty, \sum_{n=1}^{\infty} \|\eta_n\|^2 < \infty$  and  $T \in \mathfrak{M} \rtimes_{\alpha} \Gamma$ .

We denote by  $B_\alpha(\Gamma; \mathfrak{M}_*)$  the set of all functions  $u$  of  $\Gamma$  into  $\mathfrak{M}_*$  such that  $\|u\| < +\infty$ , where

$$(2.1) \quad \|u\| = \sup_{\|\hat{\pi}_\alpha(z)\| \leq 1, z \in K(\Gamma; \mathfrak{M})} \left| \sum_{s \in \Gamma} \langle z(s), u(s) \rangle \right| = \sup_{\|\hat{\pi}_\alpha(z)\| \leq 1, z \in K(\Gamma; \mathfrak{M})} |\langle \hat{\pi}_\alpha(z), \tilde{u} \rangle|.$$

Then, (2.1) defines a norm on  $B_\alpha(\Gamma; \mathfrak{M}_*)$  and  $B_\alpha(\Gamma; \mathfrak{M}_*)$  is a Banach space under this norm. Moreover,

$$\|u\|_\infty \leq \|u\|,$$

so that, in particular,  $B_\alpha(\Gamma; \mathfrak{M}_*)$  is contained in  $\ell^\infty(\Gamma; \mathfrak{M}_*)$ ; see [2].

We will focus, in particular, on the preduals of Hopf von Neumann algebras arising from the crossed product constructions. Insight into their structure can be found in [2, 7], where the framework of "Fourier spaces" for crossed products has been described.

A discrete Hopf von Neumann algebra dynamical system is a quadruple  $(\mathfrak{M}, \Delta, \Gamma, \alpha)$  such that  $(\mathfrak{M}, \Gamma, \alpha)$  is a discrete  $W^*$ -dynamical system,  $(\mathfrak{M}, \Delta)$  is a Hopf von Neumann algebra and  $(\alpha_s \otimes \alpha_s)\Delta = \Delta \circ \alpha_s$  for all  $s \in \Gamma$ .

**Lemma 2.1.** *Let  $(\mathfrak{M}, \Delta, \Gamma, \alpha)$  be a discrete Hopf von Neumann algebra dynamical system. Then  $\mathfrak{M} \rtimes_\alpha \Gamma$  is a Hopf von Neumann algebra.*

*Proof.* Define  $\tilde{\Delta} : \mathfrak{M} \rtimes_\alpha \Gamma \rightarrow \mathfrak{M} \rtimes_\alpha \Gamma \bar{\otimes} \mathfrak{M} \rtimes_\alpha \Gamma$  by  $\tilde{\Delta}(xs) = \Delta(x)(s \otimes s)$  for all  $x \in \mathfrak{M}$  and  $s \in \Gamma$ . Thus, for every  $x, y \in \mathfrak{M}$  and  $s, t \in \Gamma$  we have

$$\begin{aligned} \tilde{\Delta}(xsyt) &= \tilde{\Delta}(x\alpha_s(y)st) \\ &= \Delta(x)\Delta(\alpha_s(y))(st \otimes st) \\ &= \Delta(x)(\alpha_s \otimes \alpha_s)(\Delta(y))(st \otimes st) \\ &= \Delta(x)(\alpha_s \otimes \alpha_s)(\Delta(y))(s \otimes s)(s^{-1} \otimes s^{-1})(st \otimes st) \\ &= \Delta(x)(s \otimes s)\Delta(y)(s^{-1} \otimes s^{-1})(st \otimes st) \\ &= \tilde{\Delta}(xs)\tilde{\Delta}(yt). \end{aligned}$$

Moreover, by using  $sx = \alpha_s(x)s$  we have

$$\tilde{\Delta}(sx) = \tilde{\Delta}(\alpha_s(x)s) = \Delta(\alpha_s(x))(s \otimes s) = (\alpha_s \otimes \alpha_s)(\Delta(x))(s \otimes s) = (s \otimes s)\Delta(x).$$

Consequently,

$$\begin{aligned} \tilde{\Delta}(sxt) &= \tilde{\Delta}(st\alpha_{t^{-1}}(x)y) \\ &= (st \otimes st)\Delta(\alpha_{t^{-1}}(x)y) \\ &= (s \otimes s)(t \otimes t)(\alpha_{t^{-1}} \otimes \alpha_{t^{-1}})(\Delta(x))\Delta(y) \\ &= (s \otimes s)\Delta(x)(t \otimes t)\Delta(y) = \tilde{\Delta}(sx)\tilde{\Delta}(ty). \end{aligned}$$

Hence, we conclude that  $\tilde{\Delta}(sxt) = \tilde{\Delta}(sx)\tilde{\Delta}(ty)$ . Moreover,

$$\tilde{\Delta}(sxyt) = \tilde{\Delta}(\alpha_s(x)syt) = \tilde{\Delta}(\alpha_s(x)s)\tilde{\Delta}(yt) = \tilde{\Delta}(sx)\tilde{\Delta}(yt).$$

Thus, we conclude that  $\tilde{\Delta}$  is a  $*$ -representation. Suppose that  $\tilde{u}, \tilde{v} \in (\mathfrak{M} \rtimes_{\alpha} \Gamma)_*$ . For every  $s \in \Gamma$  we define  $w(s) = u(s) * v(s)$ , then

$$\begin{aligned} \|\tilde{w}\| &= \sup_{\|\hat{\pi}_{\alpha}(z)\| \leq 1, z \in K(\Gamma, \mathfrak{M})} \left| \sum_{s \in \Gamma} \langle z(s), w(s) \rangle \right| \\ &= \sup_{\|\hat{\pi}_{\alpha}(z)\| \leq 1, z \in K(\Gamma, \mathfrak{M})} \left| \sum_{s \in \Gamma} \langle \Delta(z(s)), u(s) \otimes v(s) \rangle \right| \\ &= \sup_{\|\hat{\pi}_{\alpha}(z)\| \leq 1, z \in K(\Gamma, \mathfrak{M})} \left| \langle \tilde{\Delta}(\sum_{s \in \Gamma} z(s)s), \tilde{u} \otimes \tilde{v} \rangle \right| \\ &\leq \|\tilde{u}\| \|\tilde{v}\| \sup_{\|\hat{\pi}_{\alpha}(z)\| \leq 1, z \in K(\Gamma, \mathfrak{M})} \left\| \sum_{s \in \Gamma} z(s)s \right\| \leq \|\tilde{u}\| \|\tilde{v}\|. \end{aligned}$$

Therefore,  $\tilde{w} \in (\mathfrak{M} \rtimes_{\alpha} \Gamma)_*$ . Now, suppose that  $x_i s_i \rightarrow 0$  in the weak\* topology  $\sigma(\mathfrak{M} \rtimes_{\alpha} \Gamma, (\mathfrak{M} \rtimes_{\alpha} \Gamma)_*)$ . So,

$$\begin{aligned} \lim_i \langle \tilde{\Delta}(x_i s_i), \tilde{u} \otimes \tilde{v} \rangle &= \lim_i \langle \Delta(x_i)(s_i \otimes s_i), \tilde{u} \otimes \tilde{v} \rangle = \lim_i \langle \Delta(x_i), u(s_i) \otimes v(s_i) \rangle \\ &= \lim_i \langle x_i, u(s_i) * v(s_i) \rangle = \lim_i \langle x_i, w(s_i) \rangle \\ &= \lim_i \langle x_i s_i, \tilde{w} \rangle = 0. \end{aligned}$$

Therefore,  $\tilde{\Delta}$  is a normal  $*$ -representation. Moreover,

$$\begin{aligned} (\tilde{\Delta} \otimes \text{id})\tilde{\Delta}(xs) &= (\tilde{\Delta} \otimes \text{id})(\Delta(x)(s \otimes s)) \\ &= ((\Delta \otimes \text{id})\Delta(x))(s \otimes s \otimes s) \\ &= ((\text{id} \otimes \Delta)\Delta(x))(s \otimes s \otimes s) \\ &= (\text{id} \otimes \tilde{\Delta})\tilde{\Delta}(xs). \end{aligned}$$

Thus,  $(\tilde{\Delta} \otimes \text{id})\tilde{\Delta} = (\text{id} \otimes \tilde{\Delta})\tilde{\Delta}$ . In summary, we have proved that  $(\mathfrak{M} \rtimes_{\alpha} \Gamma, \tilde{\Delta})$  forms a Hopf von Neumann algebra. □

**Theorem 2.2.** *Let  $(\mathfrak{M}, \Delta, \Gamma, \alpha)$  be a discrete Hopf von Neumann algebra dynamical system. Then  $\mathfrak{M}_*$  is inner amenable, if and only if  $(\mathfrak{M} \rtimes_{\alpha} \Gamma)_*$  is inner amenable.*

*Proof.* Suppose that  $F : \mathfrak{M} \rightarrow \mathbb{C}$  is an inner invariant mean and  $\psi : \mathfrak{M} \rtimes_{\alpha} \Gamma \rightarrow \mathbb{C}$  is a state. Define  $P : \mathfrak{M} \rtimes_{\alpha} \Gamma \rightarrow \mathbb{C}$  by

$$P(xs) = F(x)\psi(\lambda_s) \quad (x \in \mathfrak{M}, s \in \Gamma).$$

If  $\tilde{u} \in (\mathfrak{M} \rtimes_{\alpha} \Gamma)_*$ , then the following equalities hold:

$$\begin{aligned} \tilde{u} \cdot xs &= (\text{id} \otimes \tilde{u})\tilde{\Delta}(xs) = (\text{id} \otimes \tilde{u})(\Delta(x)(s \otimes s)) \\ &= ((\text{id} \otimes u(s))\Delta(x))s = (u(s) \cdot x)s. \end{aligned}$$

In the same way,  $xs \cdot \tilde{u} = (x \cdot u(s))s$ . Consequently,

$$\begin{aligned} P(\tilde{u} \cdot xs) &= P((u(s) \cdot x)s) = F(u(s) \cdot x)(\psi(\lambda_s)) \\ &= F(x \cdot u(s))\psi(\lambda_s) = P((x \cdot u(s))s) = P(xs \cdot \tilde{u}). \end{aligned}$$

Moreover,

$$\begin{aligned} \tilde{u} \cdot sx &= (\text{id} \otimes \tilde{u})\tilde{\Delta}(sx) = (\text{id} \otimes \tilde{u})((s \otimes s)(\Delta(x))) \\ &= s(\text{id} \otimes u(s))(\Delta(x)) = s(u(s) \cdot x). \end{aligned}$$

Similarly,  $sx \cdot \tilde{u} = s(x \cdot u(s))$ . Furthermore,

$$\begin{aligned} \alpha_s(x \cdot u(s)) &= \alpha_s((u(s) \otimes \text{id})(\Delta(x))) = (u(s) \circ \alpha_{s-1} \otimes \text{id})(\alpha_s \otimes \alpha_s)(\Delta(x)) \\ &= (u(s) \circ \alpha_{s-1} \otimes \text{id})\Delta(\alpha_s(x)) = \alpha_s(x) \cdot u(s) \circ \alpha_{s-1}. \end{aligned}$$

In the same way,  $\alpha_s(u(s) \cdot x) = u(s) \circ \alpha_{s-1} \cdot \alpha_s(x)$ . Thus,

$$\begin{aligned} P(\tilde{u} \cdot sx) &= P(s(u(s) \cdot x)) = P(\alpha_s(u(s) \cdot x)s) \\ &= F(\alpha_s(u(s) \cdot x))\psi(\lambda_s) = F(u(s) \circ \alpha_{s-1} \cdot \alpha_s(x))\psi(\lambda_s) \\ &= F(\alpha_s(x) \cdot u(s) \circ \alpha_{s-1})\psi(\lambda_s) = F(\alpha_s(x \cdot u(s)))\psi(\lambda_s) \\ &= P(\alpha_s(x \cdot u(s))s) = P(s(x \cdot u(s))) = P(sx \cdot \tilde{u}). \end{aligned}$$

So,  $(\mathfrak{M} \rtimes_{\alpha} \Gamma)_*$  is inner amenable.

Conversely, suppose that  $P : \mathfrak{M} \rtimes_{\alpha} \Gamma \rightarrow \mathbb{C}$  is an inner invariant mean. Define  $F : \mathfrak{M} \rightarrow \mathbb{C}$  by  $F(x) = P(xe)$ , where  $e$  is the identity element of  $\Gamma$ . It is known that if  $f \in \mathfrak{M}_*$ , then  $f = \sum_{n=1}^{\infty} \omega_{h_n, k_n}$  for some sequences  $(h_n), (k_n) \subseteq H$  with  $\sum_{n=1}^{\infty} \|h_n\|^2 < \infty$ ,  $\sum_{n=1}^{\infty} \|k_n\|^2 < \infty$  and  $\omega_{h_n, k_n}(x) = \langle xh_n, k_n \rangle$  for all  $x \in \mathfrak{M}$ .

Since  $\mathfrak{M} \rtimes_{\alpha} \Gamma$  is a von Neumann algebra defined on the Hilbert space  $\ell^2(\Gamma, H) = \ell^2(\Gamma) \otimes H$ , it can be deduced that by setting  $\tilde{u} = \sum_{n=1}^{\infty} \omega_{\delta_e \otimes h_n, \delta_e \otimes k_n}$ , we ensure that  $\tilde{u}$  belongs to the predual space  $(\mathfrak{M} \rtimes_{\alpha} \Gamma)_*$ . Hence,  $u(e) = f$  and consequently,

$$F(f \cdot x) = P((u(e) \cdot x)e) = P(\tilde{u} \cdot xe) = P(xe \cdot \tilde{u}) = P((x \cdot u(e))e) = F(x \cdot f).$$

Therefore,  $\mathfrak{M}_*$  is inner amenable. □

**Corollary 2.3.** *Let  $(\mathfrak{M}, \Delta, \Gamma, \alpha)$  be a discrete Hopf von Neumann algebra dynamical system.  $\mathfrak{M}_*$  is strictly inner amenable, if and only if  $(\mathfrak{M} \rtimes_{\alpha} \Gamma)_*$  is strictly inner amenable.*

*Proof.* Suppose that  $F : \mathfrak{M} \rightarrow \mathbb{C}$  is a strictly inner invariant mean. If  $\varphi \in \mathfrak{M}_*$ , then there is a  $\tilde{u}_{\varphi} \in (\mathfrak{M} \rtimes_{\alpha} \Gamma)_*$  such that  $u_{\varphi}(e) = \varphi$ . Define  $P : \mathfrak{M} \rtimes_{\alpha} \Gamma \rightarrow \mathbb{C}$  by  $P(xs) = F(x)\psi(\lambda_s)$  for all  $x \in \mathfrak{M}$  and  $s \in \Gamma$ . Then  $P$  is an inner invariant mean. If  $P : \mathfrak{M} \rtimes_{\alpha} \Gamma \rightarrow \mathbb{C}$  is a mixed identity in  $(\mathfrak{M} \rtimes_{\alpha} \Gamma)^*$ ,

then

$$\begin{aligned} F(x \cdot \varphi) &= P((x \cdot \varphi)e) = P((x \cdot u_\varphi(e))e) \\ &= P(xe \cdot \tilde{u}_\varphi) = \langle u_\varphi(e), x \rangle \\ &= \langle \varphi, x \rangle, \end{aligned}$$

for all  $\varphi \in \mathfrak{M}_*$  and  $x \in \mathfrak{M}$ . Similarly,  $F(\varphi \cdot x) = \langle \varphi, x \rangle$ . Therefore,  $F$  is a mixed identity in  $\mathfrak{M}^*$  leading to a contradiction.

Conversely, assume  $P : \mathfrak{M} \rtimes_\alpha \Gamma \rightarrow \mathbb{C}$  is a strictly inner invariant mean. Define  $F : \mathfrak{M} \rightarrow \mathbb{C}$  by  $F(x) = P(xe)$  for all  $x \in \mathfrak{M}$ . If  $\varphi \in \mathfrak{M}_*$  and  $x \in \mathfrak{M}$ , then

$$\begin{aligned} F(\varphi \cdot x) &= F(u_\varphi(e) \cdot x) \\ &= P((u_\varphi(e) \cdot x)e) \\ &= P(\tilde{u}_\varphi \cdot xe) \\ &\neq \langle u_\varphi(e), x \rangle = \langle \varphi, x \rangle. \end{aligned}$$

Therefore,  $\mathfrak{M}_*$  is strictly inner amenable. □

Let us recall that the concept of an inner amenable discrete group  $\Gamma$ . A discrete group  $\Gamma$  is called *inner amenable*, if there exists a mean  $M \neq \delta_e$  on  $\ell^\infty(\Gamma)$  such that  $M(\ell_t f) = M(r_t f)$  holds for all  $t \in \Gamma$  and  $f \in \ell^\infty(\Gamma)$ . Here,  $\delta_e(f) = f(e)$ , the functions  $\ell_t f$  and  $r_t f$  in  $\ell^\infty(\Gamma)$  are defined by  $\ell_t f(s) = f(ts)$  and  $r_t f(s) = f(st)$  for all  $s, t \in \Gamma$  and  $f \in \ell^\infty(\Gamma)$ .

Moreover, a mean  $M \neq \delta_e$  on  $\ell^\infty(\Gamma)$  satisfies  $M(\ell_t f) = M(r_t f)$  for all  $t \in \Gamma$  and  $f \in \ell^\infty(\Gamma)$ , if and only if  $M(f \cdot \varphi) = M(\varphi \cdot f)$  for all  $\varphi \in \ell^1(\Gamma)$  and  $f \in \ell^\infty(\Gamma)$ . This equivalence arises from the expressions

$$f \cdot \varphi = \sum_{t \in \Gamma} \varphi(t) \ell_t f \quad \text{and} \quad \varphi \cdot f = \sum_{t \in \Gamma} \varphi(t) r_t f.$$

Define  $\alpha : \Gamma \rightarrow \text{Aut}(\ell^\infty(\Gamma))$  by  $\alpha_s(f)(t) = f(s^{-1}ts)$  for all  $s, t \in \Gamma$ . On the other hand,  $(\ell^\infty(\Gamma), \Delta_a)$  is a Hopf von Neumann algebra, where  $\Delta_a(f)(s, t) = f(st)$ . Then it is easy to see that  $(\alpha_s \otimes \alpha_s) \circ \Delta_a = \Delta_a \circ \alpha_s$ . Consequently, we can deduce the following corollary.

**Corollary 2.4.** *The discrete group  $\Gamma$  is inner amenable, if and only if  $(\ell^\infty(\Gamma) \rtimes_\alpha \Gamma)_*$  is inner amenable.*

### Acknowledgments

This research was in part supported by a grant from IPM (No. 1403460322). Moreover, the author is grateful to the referee for certain comments and corrections towards the improvement of the paper.

### REFERENCES

- [1] H. R. Ebrahimi Vishki and A. R. Khoddami, Character inner amenability of certain Banach algebras, *Colloq. Math.* **122** (2011), no. 2, 225–232.
- [2] M. Fujita, Banach algebra structure in Fourier spaces and generalization of harmonic analysis on locally compact groups, *J. Math. Soc. Japan* **31** (1979), no. 1, 53–67.

- [3] M. R. Ghanei and M. Nemati, On the amenability of a class of Banach algebras with application to measure algebra, *Math. Slovaca* **69** (2019), no. 5, 1177–1184.
- [4] E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis I*, Springer-Verlag, Berlin, 1970.
- [5] A. T. -M. Lau, Analysis on a class of Banach algebras with application to harmonic analysis on locally compact groups and semigroups, *Fund. Math.* **118** (1983), no. 3, 161–175.
- [6] A. T. -M. Lau, Uniformly continuous functionals on Banach algebras, *Colloq. Math.* **51** (1987), no. 1, 195–205.
- [7] A. McKee, Weak amenability for dynamical systems, *Studia Math.* **258** (2021), no. 1, 53–70.
- [8] R. Nasr-Isfahani, Inner amenability of Lau algebras, *Arch. Math. (Brno)* **37** (2001), no. 1, 45–55.
- [9] J. P. Pier, *Amenable Banach algebras*, Pitman research notes in mathematical series, **172**, Longeman, Essex, 1988.
- [10] M. Ramezanzpour and H. R. Ebrahimi Vishki, Reiter's properties for the actions of locally compact quantum groups on von Neumann algebras, *Bull. Iranian Math. Soc.* **36** (2010), no. 2, 1–17.
- [11] Z. Ruan, Amenability of Hopf von Neumann algebras and Kac algebras, *J. Funct. Anal.* **139** (1996), no. 2, 466–499.

**Mohammad Reza Ghanei**

Department of Mathematics, Khansar Campus, University of Isfahan, Isfahan, Iran.

School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P. O. Box 19395- 5746, Tehran, Iran.

Email: [m.r.ghanei@khc.ui.ac.ir](mailto:m.r.ghanei@khc.ui.ac.ir) [mrg.ghanei@gmail.com](mailto:mrg.ghanei@gmail.com)