



A GENERALIZATION OF NORMAL ALMOST CONTACT MANIFOLDS

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ABSTRACT. In this article, a new definition, called φ -normal, is introduced, which is a generalization of normal condition on almost contact manifolds. Then some examples of φ -normal almost contact manifolds that are not normal are presented, and a sufficient and necessary condition for equivalence of these two definitions in 3-dimensional almost contact manifolds is provided. In the end, it is proven that a φ -normal contact metric manifold is a Sasakian manifold.

1. Introduction

An almost contact structure on a $(2n + 1)$ -dimensional manifold M is a reduction of the structural group of the tangent bundle of M to the unitary group $U(n)$. If a manifold M has a contact structure, i.e., there exists a globally defined 1-form η on M such that $\eta \wedge d\eta^n$ never vanishes, then η induces an almost contact structure on M [3]. An almost contact manifold naturally endows with structure tensors (φ, ξ, η) , and there can be defined an almost complex structure J on $M \times R$ on which $J(X, f \frac{d}{dt}) = (\varphi(X) - f\xi, \eta(X) \frac{d}{dt})$, for vector field X on M and $f \in C^\infty(M \times R)$. If now J is integrable, we say that the almost contact structure is normal [7]. As the vanishing of the Nijenhuis torsion of J is a necessary and sufficient condition for integrability [4], this also corresponds to a necessary and sufficient condition for the Nijenhuis torsion of φ . In this way a Sasakian manifold [5], that is a normal contact metric manifold, is analogous to Kähler manifolds. Sasakian manifold is a member of a larger family of almost contact manifolds, called trans-Sasakian manifold, that contains Sasakian, α -Sasakian, Kenmotsu, β -Kenmotsu, and cosymplectic manifolds, that all are normal almost contact

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manifolds. Also by a result of Sasaki-Hsu [6], every Brieskorn manifold has a normal contact structure. In [8], Sato proved that if M admits a normal almost contact structure, then $\pi_2(M) = 0$ or M is homotopy equivalent to $S_1 \times S_2$, where $\pi_2(M)$ denotes the second homotopy group of M . Then Geiges completed the investigation begun by Sato and gave a classification of the closed 3-manifolds that admit normal contact forms or normal almost contact structures [2].

According to the description above, normality plays an important role in almost contact manifolds. This motivates us to introduce a new definition called φ -normal, as a generalization of the normal condition. In this paper, we replace the normal condition in almost contact manifolds with a weaker condition named by φ -normal, and demonstrate the validity of the new definition.

The paper is organized as follows: In Section 2, fundamental concepts and definitions essential for the study are presented, including the new definition of φ -normality for almost contact manifolds. In section Section 3, an example of a φ -normal almost contact manifold that is not normal is provided, and a necessary and sufficient condition for the equivalence of these two definitions in 3-dimensional almost contact manifolds is presented. In Section 4, the focus is on φ -normal contact metric manifolds, and it is proven that the normal and φ -normal properties are equivalent on the contact metric manifolds.

2. Preliminaries

In this paper, the notion $\mathfrak{X}(M)$ represents the Lie algebra of all vector fields on M .

A smooth manifold M^{2n+1} is referred to as an almost contact manifold ([1]) if there exist a $(1, 1)$ -tensor field φ , a 1-form η , and a vector field ξ satisfying the following conditions,

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1.$$

Moreover, an almost contact manifold M with the almost contact structure (φ, ξ, η) , together with the Riemann metric g , is called an almost contact metric manifold, if for every $X, Y \in \mathfrak{X}(M)$ the following equation holds,

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y).$$

In this case, the fundamental 2-form Φ is defined as,

$$\Phi(X, Y) = g(X, \varphi Y),$$

for every $X, Y \in \mathfrak{X}(M)$. An almost contact metric manifold is further referred to as a contact metric manifold if $\Phi = d\eta$. On a contact metric manifold $(M, \varphi, \xi, \eta, g)$, the following relation is satisfied ([1]),

$$(2.1) \quad (\nabla_X \varphi)Y + (\nabla_{\varphi X} \varphi)\varphi Y = 2g(X, Y)\xi - \eta(Y)(X + hX + \eta(X)\xi),$$

where the tensor field h on almost contact manifolds is defined by

$$h = \frac{1}{2}\mathcal{L}_\xi \varphi,$$

in which \mathcal{L}_ξ denotes the Lie differentiation in the direction of ξ . This can be expressed as,

$$2h(X) = [\xi, \varphi X] - \varphi[\xi, X].$$

It is proved that on a contact metric manifold, h is a symmetric operator, that means $g(hX, Y) = g(X, hY)$, and it anti-commutes with φ , meaning $\varphi \circ h = -h \circ \varphi$, additionally

$$(2.2) \quad \nabla_X \xi = -\varphi X - \varphi hX$$

holds.

A contact metric structure (φ, ξ, η, g) is called K -contact if, ξ is a Killing vector field, which is equivalent to $h = 0$ ([1]).

The Nijenhuis torsion $[T, T]$ of a tensor field T of type $(1, 1)$ is defined as a tensor field of type $(1, 2)$ given by

$$[T, T](X, Y) = T^2[X, Y] + [TX, TY] - T[TX, Y] - T[X, TY].$$

Almost contact manifolds, which are analogous to almost complex manifolds in a certain sense, are defined as manifolds with structural group $U(n) \times 1$ ([1]). They can be thought of as odd-dimensional analogues of almost complex manifolds. Let M be an almost contact manifold with structure tensors (φ, ξ, η) . The product $M \times R$ naturally carries an almost complex structure J defined by,

$$J(X, f \frac{d}{dt}) = (\varphi X - f\xi, \eta(X) \frac{d}{dt}).$$

If J is integrable, meaning $[J, J] = 0$, the almost contact structure (φ, ξ, η) is said to be normal. This follows that an almost contact structure (φ, ξ, η) is normal, if

$$(2.3) \quad [\varphi, \varphi](X, Y) + d\eta(X, Y)\xi = 0.$$

A Sasakian manifold is a normal contact metric manifold, and this is equivalent to the fact that $(M \times R, J)$ is a Kähler manifold [1]. It is known that the Sasakian condition can be expressed as an almost contact metric manifold satisfying the relation [1],

$$(\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)X.$$

An interesting property to note on Sasakian manifolds is immediate, namely $\varphi[\varphi, \varphi] = 0$. The question arises whether the converse is true: if in a contact metric manifold the equality $\varphi[\varphi, \varphi] = 0$ holds, can we deduce that the manifold is Sasakian?

Motivated by the so called property of the Sasakian manifolds, we introduce the new definition:

Definition 2.1. An almost contact structure (φ, ξ, η) , is said to be φ -normal, if $\varphi[\varphi, \varphi] = 0$.

By equality (2.3), and since $\varphi\xi = 0$, it is evident that normal almost contact manifolds are φ -normal, but the converse is not necessarily true.

3. φ -normal almost contact manifolds

In this section, we compare two notions, normal and φ -normal conditions on almost contact manifolds and provide a necessary and sufficient condition for the equivalence of these two notions.

It is evident that φ -normality is a generalization of normal condition. In the following, we give an example of a φ -normal almost contact manifold which is not normal.

Example 3.1. Consider $M = R^3$ with vector fields

$$e_1 = \frac{\partial}{\partial x}, \quad e_2 = \frac{\partial}{\partial y} + z \frac{\partial}{\partial z}, \quad e_3 = \frac{\partial}{\partial z}.$$

Then M carries an almost contact structure (φ, ξ, η) defined by

$$\begin{aligned} \xi &= e_3, \quad \eta = dz - zdy, \\ \varphi e_1 &= e_2, \quad \varphi e_2 = -e_1, \quad \varphi \xi = 0. \end{aligned}$$

Thus we have $\eta(e_1) = 0 = \eta(e_2)$, and $\eta(\xi) = 1$. It can be seen that $\eta \wedge d\eta = 0$, and thus η is not a contact form.

The following Lie brackets can easily be obtained,

$$[\xi, e_1] = 0 = [e_1, e_2], \quad [\xi, e_2] = \xi.$$

By a simple calculation one can see that $[\varphi, \varphi](e_i, e_j) = 0$ for each $i, j = 1, 2, 3$. Thus we have $\varphi[\varphi, \varphi] = 0$ and thus M is φ -normal. But M is not normal because,

$$\begin{aligned} [\varphi, \varphi](\xi, e_2) + 2d\eta(\xi, e_2)\xi &= -\eta[\xi, e_2]\xi \\ &= -\eta(\xi)\xi \\ &= -\xi \neq 0. \end{aligned}$$

As is known, in normal almost contact manifolds we have $h = 0$ [1], but such a role is not necessarily established in φ -normal almost contact manifolds, even in such a case, $\eta \circ h$ may not vanish. For example in the above example we have $2h(e_1) = \xi$ and thus $\eta \circ h(e_1) = \frac{1}{2}$. In the following, we show the importance of vanishing of $\eta \circ h$ in 3-dimensional φ -normal almost contact manifolds.

Proposition 3.2. The 3-dimensional φ -normal almost contact manifold (M, φ, ξ, η) is normal, if and only if $\eta \circ h = 0$.

Proof. If M is normal, then evidently $h = 0$, and thus $\eta \circ h = 0$.

Now suppose that M is φ -normal and $\eta \circ h = 0$, then

$$\begin{aligned} 0 &= \varphi[\varphi, \varphi](\xi, X) = \varphi(\varphi^2[\xi, X] - \varphi[\xi, \varphi X]) \\ &= -\varphi[\xi, X] + [\xi, \varphi X] - \eta([\xi, \varphi X])\xi \\ &= 2h(X) - 2d\eta(\xi, \varphi X)\xi. \end{aligned}$$

According to the given assumption we have $\eta \circ h = 0$, thus by the above equality, we get $d\eta(\xi, \varphi X) = 0$, and using again the equality $h(X) - d\eta(\xi, \varphi X)\xi = 0$ gives $h = 0$, and since $[\varphi, \varphi](\xi, X) = -2\varphi h(X)$, then we have $[\varphi, \varphi](\xi, X) = 0$, and thus

$$([\varphi, \varphi] + 2d\eta \otimes \xi)(\xi, X) = 0.$$

Now let's take the local φ -basis $\{X, \varphi X, \xi\}$ for M , then we have,

$$\begin{aligned} [\varphi, \varphi](X, \varphi X) &= [\varphi X, \varphi^2 X] + \varphi^2[X, \varphi X] - \varphi[X, \varphi^2 X] \\ &= -[\varphi X, X] + (\varphi X)(\eta(X))\xi + \eta(X)[\varphi X, \xi] \\ &\quad - [X, \varphi X] + \eta([X, \varphi X])\xi - \eta(X)\varphi[X, \xi] \\ &= \{(\varphi X)(\eta(X)) + \eta([X, \varphi X])\}\xi + \eta(X)([\varphi X, \xi] - \varphi[X, \xi]) \\ &= 2d\eta(\varphi X, X)\xi + 2\eta(X)h(X). \\ &= -2d\eta(X, \varphi X)\xi, \end{aligned}$$

which proves $([\varphi, \varphi] + 2d\eta \otimes \xi)(X, \varphi X) = 0$. Therefore we have $[\varphi, \varphi] + 2d\eta \otimes \xi = 0$, that means M is normal. □

In the following, we provide an example that shows that the above Proposition is not necessarily true for dimensions greater than 3.

Example 3.3. Consider $M = R^5$ with vector fields

$$e_1 = \frac{\partial}{\partial x^1}, \quad e_2 = \frac{\partial}{\partial x^2}, \quad e_3 = x^2 \frac{\partial}{\partial x^5} + \frac{\partial}{\partial x^3}, \quad e_4 = \frac{\partial}{\partial x^4}, \quad e_5 = \frac{\partial}{\partial x^5}.$$

Then M carries an almost contact structure (φ, ξ, η) defined by

$$\begin{aligned} \xi &= e_5, \quad \eta = dx^5 - x^2 dx^3, \\ \varphi e_1 &= e_2, \quad \varphi e_2 = -e_1, \quad \varphi e_3 = e_4, \quad \varphi e_4 = -e_3, \quad \varphi \xi = 0. \end{aligned}$$

We have $\eta(e_i) = 0$ for $i = 1, \dots, 4$, and $\eta(\xi) = 1$. It can be seen that $\eta \wedge (d\eta)^2 = 0$, and thus η is not a contact form.

It is evident that the only non vanishing Lie bracket is $[e_2, e_3] = \xi$.

Also by a simple calculation one can see that the only non vanishing terms of $[\varphi, \varphi](e_i, e_j)$ are

$$[\varphi, \varphi](e_1, e_4) = [\varphi(e_1), \varphi(e_4)] = -[e_2, e_3] = -\xi.$$

Thus we have $\varphi[\varphi, \varphi] = 0$ and this shows that M is φ -normal. But M is not normal because,

$$\begin{aligned} ([\varphi, \varphi] + 2d\eta \otimes \xi)(e_1, e_4) &= [\varphi, \varphi](e_1, e_4) + \eta([e_1, e_4])\xi, \\ &= -[e_2, e_3] \\ &= -\xi \neq 0. \end{aligned}$$

It is noteworthy that $h = 0$. Thus this is an example of a φ -normal almost contact manifold with $h = 0$, that is not normal.

4. φ -normal contact metric manifolds

In this section, the equivalence between normality and φ -normality on contact metric manifolds is established, showing that φ -normal contact metric manifolds are Sasakian manifolds. The proof involves some lemmas and propositions.

Lemma 4.1. *On a contact metric manifold $(M, \varphi, \xi, \eta, g)$, for every $X, Y \in \mathfrak{X}(M)$, the following equation holds,*

$$(4.1) \quad \frac{1}{2}\varphi[\varphi, \varphi](X, Y) = ((\nabla_X \varphi)Y - (\nabla_Y \varphi)X) + \eta(Y)X - \eta(X)Y.$$

Proof. By definition of $[\varphi, \varphi](X, Y)$, we have ([1], p.87):

$$[\varphi, \varphi](X, Y) = (\nabla_{\varphi X} \varphi)Y - (\nabla_{\varphi Y} \varphi)X - \varphi((\nabla_X \varphi)Y) + \varphi((\nabla_Y \varphi)X),$$

and then,

$$\begin{aligned} \varphi[\varphi, \varphi](X, Y) &= \varphi(\nabla_{\varphi X} \varphi)Y - \varphi(\nabla_{\varphi Y} \varphi)X + (\nabla_X \varphi)Y - (\nabla_Y \varphi)X \\ &\quad + (\eta(\nabla_X \varphi)Y - \eta(\nabla_Y \varphi)X)\xi \\ &= \varphi(\nabla_{\varphi X} \varphi)Y - \varphi(\nabla_{\varphi Y} \varphi)X + (\nabla_X \varphi)Y - (\nabla_Y \varphi)X \\ &\quad - g(\varphi Y, \nabla_X \xi) + g(\varphi X, \nabla_Y \xi). \end{aligned}$$

Using formula (2.2) and symmetrization of h , results the following formula,

$$(4.2) \quad \varphi[\varphi, \varphi](X, Y) = \varphi(\nabla_{\varphi X} \varphi)Y - \varphi(\nabla_{\varphi Y} \varphi)X + (\nabla_X \varphi)Y - (\nabla_Y \varphi)X.$$

On the other hand, using the equality

$$\varphi(\nabla_{\varphi X} \varphi)Y = -(\nabla_{\varphi X} \varphi)\varphi Y + ((\nabla_{\varphi X} \eta)Y)\xi + \eta(Y)\nabla_{\varphi X} \xi,$$

and changing the role of X and Y in the above equality, and subtracting the obtained one, from the above equality, we get

$$\begin{aligned} \varphi(\nabla_{\varphi X} \varphi)Y - \varphi(\nabla_{\varphi Y} \varphi)X &= -(\nabla_{\varphi X} \varphi)\varphi Y + ((\nabla_{\varphi X} \eta)Y)\xi + \eta(Y)\nabla_{\varphi X} \xi + (\nabla_{\varphi Y} \varphi)\varphi X \\ &\quad - ((\nabla_{\varphi Y} \eta)X)\xi - \eta(X)\nabla_{\varphi Y} \xi. \end{aligned}$$

Comparing the above equality with formula (2.1), we get

$$\begin{aligned} \varphi(\nabla_{\varphi X} \varphi)Y - \varphi(\nabla_{\varphi Y} \varphi)X &= (\nabla_X \varphi)Y + 2\eta(Y)X - g(hX, Y)\xi \\ &\quad - (\nabla_Y \varphi)X - 2\eta(X)Y + g(hY, X)\xi \\ &= (\nabla_X \varphi)Y - (\nabla_Y \varphi)X + 2\eta(Y)X - 2\eta(X)Y. \end{aligned}$$

By substituting the above equality into equation (4.2), we get the assertion. \square

Lemma 4.2. *On a contact metric manifold $(M, \varphi, \xi, \eta, g)$, for every $X, Y \in \mathfrak{X}(M)$, the following equation holds,*

$$(4.3) \quad \varphi[\varphi, \varphi](\varphi X, \varphi Y) = -\varphi[\varphi, \varphi](X, Y) + 2\eta(X)hY - 2\eta(Y)hX.$$

Proof. Using Lemma 4.1, and substituting X and Y by φX and φY respectively in formula (4.1), we have

$$\frac{1}{2}\varphi[\varphi, \varphi](\varphi X, \varphi Y) = (\nabla_{\varphi X}\varphi)\varphi Y - (\nabla_{\varphi Y}\varphi)\varphi X.$$

Adding the above equality to equation (4.1), and using (2.1), we obtain the desired result. \square

Now let's consider the effect of a weaker assumption than φ -normality, on contact metric manifolds.

Proposition 4.3. *A contact metric manifold is K-contact if and only if*

$$\varphi[\varphi, \varphi](X, Y) + \varphi[\varphi, \varphi](\varphi X, \varphi Y) = 0.$$

Proof. Let $(M, \varphi, \xi, \eta, g)$ be a contact metric manifold, if it is K -contact, that means $h = 0$, then Lemma 4.2 gives the assertion.

Now suppose that,

$$\varphi[\varphi, \varphi](X, Y) + \varphi[\varphi, \varphi](\varphi X, \varphi Y) = 0.$$

Using equation (4.3), we obtain,

$$\eta(X)hY - \eta(Y)hX = 0.$$

Substituting X by ξ , we obtain $h = 0$, which proves that M is K -contact. \square

In the following example, we provide a contact manifold that is not φ -normal, but satisfies the equality $\varphi[\varphi, \varphi](X, Y) + \varphi[\varphi, \varphi](\varphi X, \varphi Y) = 0$. This shows that the latter relationship is weaker than φ -normality.

Example 4.4. *Consider $M = R^5$ with vector fields*

$$e_1 = \frac{\partial}{\partial x^1}, e_2 = x^1 \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} - x^1 \frac{\partial}{\partial x^3}, e_3 = \frac{\partial}{\partial x^3}, e_4 = \frac{\partial}{\partial x^4} - x^5 \frac{\partial}{\partial x^3}, e_5 = \frac{\partial}{\partial x^5} - \frac{\partial}{\partial x^3}.$$

Then M carries an almost contact structure (φ, ξ, η) defined by

$$\begin{aligned} \xi &= e_3, \quad \eta = dx^3 + x^1 dx^2 + x^5 dx^4 + dx^5, \\ \varphi e_1 &= e_4, \quad \varphi e_2 = e_5, \quad \varphi e_3 = 0, \quad \varphi e_4 = -e_1, \quad \varphi e_5 = -e_2. \end{aligned}$$

We have $\eta(e_i) = 0, i = 1, \dots, 4$, and $\eta(\xi) = 1$. It can be observed that $\eta \wedge (d\eta)^2 \neq 0$, indicating that η is a contact form.

It is evident that the only non vanishing Lie brackets are $[e_1, e_2] = e_1 - \xi$, and $[e_4, e_5] = \xi$. Through a simple calculation, we find that the only non vanishing Nijenhuis torsions $[\varphi, \varphi](e_i, e_j)$ are,

$$[\varphi, \varphi](e_1, e_2) = \xi - e_1 = -[\varphi, \varphi](e_4, e_5), \quad [\varphi, \varphi](e_4, e_2) = e_4 = [\varphi, \varphi](e_1, e_5).$$

Thus we have,

$$\begin{aligned} \varphi[\varphi, \varphi](e_1, e_2) &= -e_4 = -\varphi[\varphi, \varphi](\varphi e_1, \varphi e_2) \\ \varphi[\varphi, \varphi](e_4, e_2) &= -e_1 = -\varphi[\varphi, \varphi](\varphi e_4, \varphi e_2). \end{aligned}$$

Therefore $\varphi[\varphi, \varphi] \neq 0$, but $\varphi[\varphi, \varphi] + \varphi[\varphi, \varphi](\varphi., \varphi.) = 0$.

A Sasakian manifold is known to be a normal contact metric manifold. Here we prove that the normal assumption can be replaced by the weaker assumption of being φ -normal.

Theorem 4.5. *A φ -normal contact metric manifold is Sasakian.*

Proof. By using the co-boundary formula for the exterior derivative d on 2-form Φ , we can express it as follows,

$$3d\Phi(X, Y, Z) = g(Z, (\nabla_Y\varphi)X - (\nabla_X\varphi)Y) + g(X, (\nabla_Z\varphi)Y),$$

considering that on contact metric manifolds, $d\Phi = 0$, we obtain the equation,

$$g(X, (\nabla_Z\varphi)Y) = g(Z, (\nabla_X\varphi)Y - (\nabla_Y\varphi)X).$$

Using Lemma 4.1, the above formula can be rewritten as,

$$g(X, (\nabla_Z\varphi)Y) = g(Z, \frac{1}{2}\varphi[\varphi, \varphi](X, Y) - \eta(Y)X + \eta(X)Y).$$

Since manifold is φ -normal, we have

$$\begin{aligned} g(X, (\nabla_Z\varphi)Y) &= g(Z, -\eta(Y)X + \eta(X)Y,) \\ &= g(X, -\eta(Y)Z + g(Z, Y)\xi). \end{aligned}$$

Consequently, we obtain the expression

$$(\nabla_Z\varphi)Y = g(Z, Y)\xi - \eta(Y)Z,$$

and this proves that M is Sasakian. □

In the following we provide an example of a contact manifold, but not contact metric. That is φ -normal, but not normal.

Example 4.6. *Consider $M = R^5$ with vector fields*

$$e_1 = \frac{\partial}{\partial x^1}, e_2 = \frac{\partial}{\partial x^2}, e_3 = x^2 \frac{\partial}{\partial x^5} + \frac{\partial}{\partial x^3}, e_4 = \frac{\partial}{\partial x^4} - (x^1 - x^5) \frac{\partial}{\partial x^5} + x^3 \frac{\partial}{\partial x^3}, e_5 = \frac{\partial}{\partial x^5}.$$

Then M carries an almost contact structure (φ, ξ, η) defined by

$$\begin{aligned} \xi &= e_5, \eta = dx^5 - x^2 dx^3 + (x^1 - x^5 + x^2 x^3) dx^4, \\ \varphi e_1 &= e_2, \varphi e_2 = -e_1, \varphi e_3 = e_4, \varphi e_4 = -e_3, \varphi \xi = 0. \end{aligned}$$

We have $\eta(e_i) = 0$ for $i = 1, \dots, 4$, and $\eta(\xi) = 1$. It can be seen that $\eta \wedge (d\eta)^2 \neq 0$, and thus η is a contact form.

It is evident that the only non vanishing Lie brackets are,

$$-[e_1, e_4] = [e_2, e_3] = -[e_4, e_5] = \xi, [e_3, e_4] = e_3.$$

Also by a simple calculation one can see that the only non vanishing Nijenhuis torsions $[\varphi, \varphi](e_i, e_j)$ are

$$[\varphi, \varphi](e_2, e_3) = \xi = -[\varphi, \varphi](e_1, e_4).$$

Thus we have $\varphi[\varphi, \varphi] = 0$ and this shows that M is φ -normal. But M is not normal because,

$$\begin{aligned}([\varphi, \varphi] + 2d\eta \otimes \xi)(e_4, e_5) &= [\varphi, \varphi](e_4, e_5) - \eta([e_4, e_5])\xi, \\ &= \eta(\xi)\xi \\ &= \xi \neq 0.\end{aligned}$$

REFERENCES

- [1] D. E. Blair, Riemannian geometry of contact and symplectic manifolds, Second edition, Progress in Mathematics, 203, Birkhäuser Boston, Ltd., Boston, MA, 2010.
- [2] H. Geiges, Normal contact structures on 3-manifolds, *Tohoku Math. J. (2)* **49** (1997), no. 3, 415–422.
- [3] J. W. Gray, Some global properties of contact structures, *Ann. of Math. (2)* **69** (1959) 421–450.
- [4] A. Newlander, and L. Nirenberg, Complex analytic coordinates in almost complex manifolds, *Ann. of Math.* **65** (1957) 391–404.
- [5] S. Sasaki, On differentiable manifolds with certain structures which are closely related to almost contact structure I, *Tohoku Math. J. (2)* (1960) 459–476.
- [6] S. Sasaki and C. J. Hsu, On a property of Brieskorn manifolds, *Tohoku Math. J. (2)* **28** (1976), no. 1, 67–78.
- [7] S. Sasaki and Y. Hatakeyama, On differentiable manifolds with certain structures which are closely related to almost contact structures II, *Tohoku Math. J. (2)* **13** (1961) 281–294.
- [8] H. Sato, Remarks concerning contact manifolds, *Tohoku Math. J.* **29** (1977), no. 4, 577–584.

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