



AN ALGEBRAIC PROOF OF THE CLASSIFICATION OF FIVE-DIMENSIONAL NILSOLITONS

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ABSTRACT. In 2002, using a variational method, Lauret classified five-dimensional nilsolitons. In this work, using the algebraic Ricci soliton equation, we obtain the same classification. We show that, among ten classes of five-dimensional connected and simply connected nilmanifolds, seven classes admit the Ricci soliton structure. Furthermore, we compute the derivation that satisfies the algebraic Ricci soliton equation in each case..

1. Introduction

Suppose that (M, g) is a complete Riemannian manifold and ric_g denotes its Ricci tensor. If for a real number c and a complete vector field X , the Riemannian metric g satisfies the equation

$$(1.1) \quad \text{ric}_g = cg + \mathcal{L}_X g,$$

then g is called a Ricci soliton. In this equation, if $c > 0$, $c = 0$ or $c < 0$, then g is named shrinking, steady, or expanding Ricci soliton.

Although the above definition is a generalization of Einstein metrics but the main motivation for considering Ricci solitons is the Ricci flow equation

$$(1.2) \quad \frac{\partial}{\partial t} g(t) = -2\text{ric}_{g(t)},$$

which was introduced by Hamilton in 1982 for a curve $g(t)$ of Riemannian metrics on a manifold M (see [2]). A Riemannian metric g is a Ricci soliton on M if and only if the following one-parameter

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family of Riemannian metrics is a solution of (1.2) (see [13] and [11]),

$$(1.3) \quad g_t = (-2ct + 1)\phi_t^* g,$$

where ϕ_t is a one-parameter group of diffeomorphisms. A very interesting case happens when we study Ricci solitons on Lie groups. Suppose that G is a Lie group, \mathfrak{g} is its Lie algebra and g is a left invariant Riemannian metric on G . If for a real number c and a derivation $D \in \text{Der}(\mathfrak{g})$, the (1, 1)-Ricci tensor Ric_g of g satisfies the equation

$$(1.4) \quad \text{Ric}_g = cId + D,$$

then g is called an algebraic Ricci soliton. Moreover, if G is a nilpotent Lie group then it is called a nilsoliton. It is shown that all algebraic Ricci solitons are Ricci soliton (for more details see [7] and [8]).

In 2001, Lauret proved that on nilpotent Lie groups a left-invariant Riemannian metric g is a Ricci soliton if and only if it is an algebraic Ricci soliton. In fact, for left-invariant Riemannian metrics on nilpotent Lie groups the equations (1.1) and (1.4) are equivalent (see [8] and [6]).

A solvable Lie group S , equipped with a left-invariant Riemannian metric, is termed a solvmanifold. Let $\mathfrak{n} := [\mathfrak{s}, \mathfrak{s}]$. A solvmanifold is called standard, if $\mathfrak{a} := \mathfrak{n}^\perp$ is abelian, and is termed rank-one if $\dim(\mathfrak{a}) = 1$. Consider $(\mathfrak{n}, \langle \cdot, \cdot \rangle)$ as a fixed inner product vector space of dimension n . We denote the real algebraic variety of all nilpotent Lie algebra structures on \mathfrak{n} by $\mathcal{N} \subset \bigwedge^2 \mathfrak{n}^* \otimes \mathfrak{n}$. For an element $\mu \in \mathcal{N}$, let \mathcal{N}_μ be the simply connected nilpotent Lie group equipped with the left-invariant Riemannian metric $g = \langle \cdot, \cdot \rangle$ and Lie algebra (\mathfrak{n}, μ) . In [9], Lauret demonstrated that any element $\mu \in \mathcal{N}$ can naturally be viewed as an $(n + 1)$ -dimensional rank-one solvmanifold S_μ , where S_μ is the unique rank-one metric solvable extension of \mathcal{N}_μ that satisfies the conditions to be Einstein. He characterized the Einstein spaces of the form S_μ as critical points of the functional $\tilde{s}c : \bigwedge^2 \mathfrak{n}^* \otimes \mathfrak{n} \rightarrow \mathbb{R}$ restricted to \mathcal{S} , the sphere in $\bigwedge^2 \mathfrak{n}^* \otimes \mathfrak{n}$ (see Theorem 1 of [9]). The critical points of this restricted functional, which is a polynomial of degree 4, contains the set of all nilsolitons. In this work, we utilize equations based on structural constants. This method is simpler than computing the critical points of the aforementioned functional. Additionally, this approach allows for the computation of the derivation associated with algebraic Ricci solitons. Furthermore, while the previous functional is limited to nilsolitons, the current equations apply not only to nilsolitons, but also to all algebraic Ricci solitons (for more details see [8, 11, 14–17] and [20], for a non-Riemannian (Finslerian) case see [19]).

Naturally, one can generalize the concept of algebraic Ricci soliton to homogeneous spaces. In 2014, Jablonski showed that any homogeneous Ricci soliton is an algebraic Ricci soliton (see [6]). So the classification of left-invariant Ricci solitons on Lie groups reduces to the classification of algebraic Ricci solitons on them.

While the classification of spaces with specific properties is a significant area of study in mathematics, the classification of five-dimensional nilsolitons holds particular importance. Specifically, since any Einstein solvmanifold is standard (see [12]), the classification of five-dimensional nilsolitons leads to the

classification of all six-dimensional Einstein solvmanifolds. Moreover, these spaces have numerous applications in physics (see [3] and [4]). Hervik, in [4], utilized nilsolitons to discover higher-dimensional black holes with Ricci nilsoliton horizons. He established that every n -dimensional nilpotent group (for $n \leq 6$) corresponds to a $(n + 2)$ -dimensional Ricci nilsoliton black hole solution. In this paper, we use the equations 1.6 and 1.7. These equations can use for any Lie group equipped with a left-invariant Riemannian metric. Using these equations and the classification of five-dimensional nilmanifolds given in [5] and [1], we classify five-dimensional nilsolitons. We see that our results are compatible with the results of [10]. So these equations are applicable for computation to classify algebraic Ricci solitons in general cases.

Suppose that g is a left-invariant Riemannian metric on a Lie group G , and α_{ijk} are the structural constants of the Lie algebra \mathfrak{g} , with respect to an orthonormal basis $\{E_1, \dots, E_n\}$, defined by the following equations:

$$(1.5) \quad [E_i, E_j] = \sum_{k=1}^n \alpha_{ijk} E_k.$$

In an earlier paper (see [18]), we proved that g is an algebraic Ricci soliton if and only if there exists a real number c such that, for any $t, p, q = 1, \dots, n$ the structural constants satisfy in the following equation:

$$(1.6) \quad \begin{aligned} c\alpha_{qpt} + \frac{1}{4} \sum_{i=1}^n \sum_{j=1}^n \sum_{r=1}^n 2\alpha_{rjj} & \left(\alpha_{iqt}(\alpha_{pri} + \alpha_{ipr} - \alpha_{rip}) - \alpha_{ipt}(\alpha_{qri} + \alpha_{iqr} - \alpha_{riq}) \right) \\ & + 2(\alpha_{rji} + \alpha_{irj} - \alpha_{jir})(\alpha_{ipt}\alpha_{qjr} - \alpha_{iqt}\alpha_{pjr}) \\ & + (\alpha_{jri} + \alpha_{ijr} - \alpha_{rij}) \left(\alpha_{ipt}(\alpha_{qjr} + \alpha_{rqj} - \alpha_{jrq}) - \alpha_{iqt}(\alpha_{pjr} + \alpha_{rpj} - \alpha_{jrp}) \right) \\ & - 2\alpha_{pqi}\alpha_{rjj}(\alpha_{irt} + \alpha_{tir} - \alpha_{rti}) + 2\alpha_{pqi}\alpha_{ijr}(\alpha_{rjt} + \alpha_{trj} - \alpha_{jtr}) \\ & + \alpha_{pqi}(\alpha_{ijr} + \alpha_{rij} - \alpha_{jri})(\alpha_{jrt} + \alpha_{tjr} - \alpha_{rtj}) = 0. \end{aligned}$$

In this case we showed that, for any $i = 1, \dots, n$, the derivation D satisfies in the following equation:

$$(1.7) \quad \begin{aligned} D(E_i) = -cE_i + \frac{1}{4} \sum_{l=1}^n \left\{ \sum_{j=1}^n \sum_{r=1}^n 2\alpha_{rjj}(\alpha_{irl} + \alpha_{lir} - \alpha_{rli}) \right. \\ \left. - 2\alpha_{ijr}(\alpha_{ril} + \alpha_{lrj} - \alpha_{jlr}) \right. \\ \left. - (\alpha_{ijr} + \alpha_{rij} - \alpha_{jri})(\alpha_{jrl} + \alpha_{ljr} - \alpha_{rlj}) \right\} E_l. \end{aligned}$$

In [18], using the above equations, we classified all algebraic Ricci solitons on three-dimensional Lie groups.

Using the equations (1.6) and (1.7) which is different from Lauret’s variational method, again, we classify all five-dimensional nilsolitons.

In this paper, a Riemannian nilmanifold means a connected and simply connected Riemannian manifold (M, g) , such that there exists a nilpotent Lie subgroup of its isometry group $I(M)$ which acts transitively on M . Wilson, in [21], proved that if M is a homogeneous Riemannian nilmanifold, then there is a unique nilpotent Lie subgroup of $I(M)$ which acts simply transitively on M . Also, he

showed that this Lie subgroup is a normal subgroup of $I(M)$. Thus, we can assume a homogeneous Riemannian nilmanifold as a nilpotent Lie group N equipped with a left-invariant Riemannian metric g .

In 2006, Homoloya and Kowalski, up to isometry, classified five-dimensional two-step nilpotent Lie groups equipped with left-invariant Riemannian metrics (see [5]). In a paper by Figula and Nagy in 2018, the classification of five-dimensional nilmanifolds has been completed, up to isometry, by the classification of five-dimensional nilpotent Lie algebras of nilpotency classes three and four equipped with inner products. It is important to note that this classification does not include the Lie algebras that are direct products of Lie algebras of lower dimensions (see [1]).

In the next section we will classify all left-invariant Ricci solitons on all ten classes of five-dimensional nilmanifolds.

2. The classification of five-dimensional nilsolitons

In [5], all five-dimensional two-step nilmanifolds up to isometry have been classified by Homoloya and Kowalski. Recently, in [1], Figula and Nagy have completed the classification of five-dimensional nilmanifolds up to isometry by the classification of five-dimensional nilpotent Lie groups of nilpotency class greater than two. In this section using formula (1.6) and the above classifications, we classify all algebraic Ricci solitons on simply connected five-dimensional nilmanifolds.

Suppose that N is an arbitrary simply connected five-dimensional nilmanifold then, up to isometry, its Lie algebra is one of the following ten cases. In all cases the set $\{E_1, \dots, E_5\}$ is an orthonormal basis for the Lie algebra and in any case we give only non-vanishing commutators. We use the notation $d(a_1, \dots, a_n)$ to denote the diagonal matrix with entries a_1, \dots, a_n .

In the cases of nilpotency class greater than two, we will consider in detail the cases 2.6 and 2.7, which are the most difficult cases.

2.1. Two-step nilpotent Lie algebra with one-dimensional center. Let N be the two-step nilpotent Lie group with one-dimensional center and \mathfrak{n} denotes its Lie algebra. By [5], the non-zero Lie brackets are as follows:

$$(2.1) \quad [E_1, E_2] = sE_5, \quad [E_3, E_4] = mE_5,$$

where $s \geq m > 0$. So, in this case, the structure constants with respect to the orthonormal basis $\{E_1, \dots, E_5\}$ are of the following forms:

$$\alpha_{125} = -\alpha_{215} = s, \quad \alpha_{345} = -\alpha_{435} = m.$$

Easily, using the formula (1.6), we see that \mathfrak{n} is an algebraic Ricci soliton, if and only if

$$\begin{cases} cs + \frac{1}{2}sm^2 + \frac{3}{2}s^3 = 0, \\ cm + \frac{1}{2}ms^2 + \frac{3}{2}m^3 = 0. \end{cases}$$

A direct computation shows that the above equations hold, if and only if $m = s$. So for the constant c we have $c = -2m^2$. Now, the equation (1.7) shows that for the matrix representation of D we have

$$D = d\left(\frac{3}{2}m^2, \frac{3}{2}m^2, \frac{3}{2}m^2, \frac{3}{2}m^2, 3m^2\right).$$

2.2. Two-step nilpotent Lie algebra with two-dimensional center. The second case is two-step nilpotent Lie group N with two-dimensional center. In [5], it is shown that there are real numbers $m \geq s > 0$ such that the non-zero Lie brackets are

$$(2.2) \quad [E_1, E_2] = mE_4, \quad [E_1, E_3] = sE_5.$$

In this case for the structure constants we have:

$$\alpha_{124} = -\alpha_{214} = m, \quad \alpha_{135} = -\alpha_{315} = s.$$

Now, the equation (1.6) shows that \mathfrak{n} satisfies the algebraic Ricci soliton equation (1.4), if and only if

$$\begin{cases} cm + \frac{3}{2}m^3 + \frac{1}{2}s^2m = 0, \\ cs + \frac{3}{2}s^3 + \frac{1}{2}m^2s = 0. \end{cases}$$

Easily we can see, for the solutions we have $m = s$ and $c = -2m^2$. Now, by equation (1.7), for the matrix representation of D in the basis $\{E_1, \dots, E_5\}$ we have

$$D = d\left(m^2, \frac{3}{2}m^2, \frac{3}{2}m^2, \frac{5}{2}m^2, \frac{5}{2}m^2\right).$$

2.3. Two-step nilpotent Lie algebra with three-dimensional center. The third Lie group which we have considered is the two-step nilpotent Lie group N with three-dimensional center. This is the last five-dimensional two-step nilpotent Lie group. Based on [5], the non-zero Lie bracket of this case is

$$(2.3) \quad [E_1, E_2] = mE_3,$$

where m is a positive real number. So, the non-zero structure constants are

$$\alpha_{123} = -\alpha_{213} = m.$$

It is easy to show that the equation (1.6) holds, if and only if $cm + \frac{2}{3}m^3 = 0$. The last equation shows that $c = -\frac{3}{2}m^2$. So, by (1.7), the representation of D in the basis $\{E_1, \dots, E_5\}$, is of the form

$$D = d\left(m^2, m^2, 2m^2, \frac{3}{2}m^2, \frac{3}{2}m^2\right).$$

2.4. Four-step nilpotent Lie algebra $\mathfrak{l}_{5,7}$, (case A). In [1], Figula and Nagy have proven that the five-dimensional nilpotent Lie algebras \mathfrak{n} of nilpotency class greater than two are of the forms $\mathfrak{l}_{5,7}$, $\mathfrak{l}_{5,6}$, $\mathfrak{l}_{5,5}$ and $\mathfrak{l}_{5,9}$. In this subsection and the next subsection, we study the necessary and sufficient conditions for the Lie algebra $\mathfrak{l}_{5,7}$ to be an algebraic Ricci soliton. By [1], the non-vanishing Lie brackets of $\mathfrak{l}_{5,7}$ are

$$(2.4) \quad [E_1, E_2] = mE_3 + sE_4 + uE_5, \quad [E_1, E_3] = vE_4 + wE_5, \quad [E_1, E_4] = xE_5,$$

where for the real numbers m, s, u, v, w, x we have two cases: (case A: $m, v, x > 0, s = 0$ and $w \geq 0$) and (case B: $m, v, x > 0$ and $s > 0$). In this subsection we study the case A. In the case of $\mathfrak{l}_{5,7}$ (in two cases A and B) for the structure constants we have:

$$\begin{aligned}\alpha_{123} &= -\alpha_{213} = m, & \alpha_{124} &= -\alpha_{214} = s, & \alpha_{125} &= -\alpha_{215} = u, \\ \alpha_{134} &= -\alpha_{314} = v, & \alpha_{135} &= -\alpha_{315} = w, & \alpha_{145} &= -\alpha_{415} = x.\end{aligned}$$

The equation (1.6) together with some computations shows that $\mathfrak{l}_{5,7}(\text{case A})$, satisfies the algebraic Ricci soliton equation (1.4), if and only if

$$x = m, \quad c = -2m^2, \quad u = w = s = 0, \quad \text{and} \quad v = \frac{2}{\sqrt{3}}m.$$

Now, using (1.7), the matrix representation of D is of the form

$$D = d\left(\frac{1}{3}m^2, \frac{3}{2}m^2, \frac{11}{6}m^2, \frac{13}{6}m^2, \frac{5}{2}m^2\right).$$

2.5. Four-step nilpotent Lie algebra $\mathfrak{l}_{5,7}$, (case B). As we mentioned above, in this case (case B) for the same structure constants of case A, we have $m, v, x > 0$ and $s > 0$. We see that the equation (1.6) deduces to a system of equations with no solution. So the Lie algebra $\mathfrak{l}_{5,7}$, (case B) does not admit algebraic Ricci soliton structure.

2.6. Four-step nilpotent Lie algebra $\mathfrak{l}_{5,6}$, (case A). During this subsection and the next subsection, we have considered the Lie algebra $\mathfrak{l}_{5,6}$. In two cases (cases A and B), by [1], the Lie brackets are

$$(2.5) \quad [E_1, E_2] = mE_3 + sE_4 + uE_5, \quad [E_1, E_3] = vE_4 + wE_5, \quad [E_1, E_4] = xE_5, \quad [E_2, E_3] = yE_5,$$

where m, s, u, v, w, x, y are real numbers such that $m, v, x, y \neq 0$. For the structure constants we have:

$$\begin{aligned}\alpha_{123} &= -\alpha_{213} = m, & \alpha_{124} &= -\alpha_{214} = s, & \alpha_{125} &= -\alpha_{215} = u, & \alpha_{134} &= -\alpha_{314} = v, \\ \alpha_{135} &= -\alpha_{315} = w, & \alpha_{145} &= -\alpha_{415} = x, & \alpha_{235} &= -\alpha_{325} = y.\end{aligned}$$

Similar to the Lie algebra $\mathfrak{l}_{5,7}$ we have two cases, case A and B. In the case A we have $s = 0$ and $w \geq 0$ and for the case B $s > 0$.

A direct computation, using (1.6), shows that the Lie algebra $\mathfrak{l}_{5,6}(\text{case A})$, is an algebraic Ricci soliton, if and only if (for two cases A and B)

$$\left\{ \begin{array}{l} myu = 0, \\ mvs + mwu + sxu = 0, \\ cm + \frac{3m}{2}(m^2 + s^2 + u^2) + \frac{x}{2}(xm - sw) = 0, \\ cs + \frac{3s}{2}(m^2 + s^2 + u^2 + v^2) + \frac{1}{2}(-mwx + w^2s + y^2s) + wvu = 0, \\ cu + \frac{3u}{2}(m^2 + s^2 + u^2 + w^2 + x^2 + y^2) + \frac{v^2u}{2} + vsw = 0, \\ uxv = 0, \\ mvs + mwu - \frac{vwx}{2} = 0, \\ cv + \frac{3v}{2}(s^2 + v^2 + w^2) + \frac{v}{2}(u^2 + y^2) + usw = 0, \\ cw + \frac{3w}{2}(u^2 + v^2 + w^2 + x^2 + y^2) + \frac{s}{2}(-mx + sw) + svu = 0, \\ umx = 0, \\ sxu + vxw - \frac{mvs}{2} = 0, \\ cx + \frac{3x}{2}(u^2 + w^2 + x^2) + \frac{1}{2}(m^2x - msw + xy^2) = 0, \\ mvu = 0, \\ mwu + sxu + vxw = 0, \\ suy + vwy = 0, \\ cy + \frac{3y}{2}(u^2 + w^2 + y^2) + \frac{y}{2}(s^2 + v^2 + x^2) = 0, \\ wyx - \frac{msy}{2} = 0, \\ uyx = 0. \end{array} \right.$$

A direct computation shows that the above equations hold, if and only if

$$u = w = s = 0, m = \pm\sqrt{\frac{3}{2}}x, y = \pm x, v = \pm\sqrt{\frac{3}{2}}x \text{ and } c = -\frac{11}{4}x^2.$$

The equation (1.7) shows that for the matrix representation of D we have

$$D = d\left(\frac{3}{4}x^2, \frac{3}{2}x^2, \frac{9}{4}x^2, 3x^2, \frac{15}{4}x^2\right).$$

2.7. Four-step nilpotent Lie algebra $\mathfrak{l}_{5,6}$, (case B). In the case B for the same structure constants of the above case, we have $s > 0$. Then we see that the system of equations defined by (1.6) has no solution. Therefore similar to the case $\mathfrak{l}_{5,7}$ (case B), the Lie algebra $\mathfrak{l}_{5,6}$ (case B), does not admit an algebraic Ricci soliton structure.

2.8. Three-step nilpotent Lie algebra $\mathfrak{l}_{5,5}$. This section is devoted to $\mathfrak{l}_{5,5}$ which is a three-step nilpotent Lie algebra with one-dimensional center. For the real numbers $s, u \geq 0$, and $m, v, w > 0$, the non-vanishing Lie brackets are as follows:

$$(2.6) \quad [E_1, E_2] = mE_4 + sE_5, \quad [E_1, E_3] = uE_5, \quad [E_1, E_4] = vE_5, \quad [E_2, E_3] = wE_5.$$

Hence for the structure constants we have:

$$\begin{aligned} \alpha_{124} = -\alpha_{214} = m, \quad \alpha_{125} = -\alpha_{215} = s, \quad \alpha_{135} = -\alpha_{315} = u, \\ \alpha_{145} = -\alpha_{145} = v, \quad \alpha_{235} = -\alpha_{325} = w. \end{aligned}$$

Now substituting the above structure constants in the equation (1.6), leads us to a system of equations with the following solution,

$$u = s = 0, v = m, w = \frac{\sqrt{2}}{2}m, c = -\frac{7}{4}m^2.$$

So the Lie algebra $\mathfrak{l}_{5,5}$ admits an algebraic Ricci soliton structure, if and only if the above equations hold. In this case, by (1.7), the matrix representation of D in the basis $\{E_1, \dots, E_5\}$, is

$$D = d\left(\frac{3}{4}m^2, m^2, \frac{3}{2}m^2, \frac{7}{4}m^2, \frac{5}{2}m^2\right).$$

2.9. Three-step nilpotent Lie algebra $\mathfrak{l}_{5,9}$, (case A). For the non-vanishing Lie brackets of this case we have:

$$(2.7) \quad [E_1, E_2] = mE_3 + sE_4 + uE_5, \quad [E_1, E_3] = vE_4, \quad [E_2, E_3] = wE_5,$$

where $m > 0$, $w > v > 0$ and $s, u \geq 0$. So we have:

$$\begin{aligned} \alpha_{123} = -\alpha_{213} = m, \quad \alpha_{124} = -\alpha_{214} = s, \quad \alpha_{125} = -\alpha_{215} = u, \\ \alpha_{134} = -\alpha_{314} = v, \quad \alpha_{235} = -\alpha_{325} = w. \end{aligned}$$

Easily we see that the system of equations which is defined by (1.6) has no solution. Therefore, the case A of the Lie algebra $\mathfrak{l}_{5,9}$ does not admit Algebraic Ricci soliton structure.

2.10. Three-step nilpotent Lie algebra $\mathfrak{l}_{5,9}$, (case B). In the case B of the Lie algebra $\mathfrak{l}_{5,9}$, for the same Lie algebra brackets, we have $m, v > 0$, $v = w$, $u = 0$ and $s \geq 0$. Then, the equation (1.6), leads us to the following system of equations:

$$\begin{cases} mvs = 0, \\ cm + \frac{3}{2}(m^3 + s^2m) = 0, \\ cs + \frac{3}{2}(s^3 + m^2s) + 2v^2s = 0, \\ cv + \frac{3}{2}s^2v + 2v^3 = 0, \\ cv + \frac{1}{2}s^2v + 2v^3 = 0. \end{cases}$$

The solution of the above system of equations is as follows:

$$s = 0, v = \frac{\sqrt{3}}{2}m, c = -\frac{3}{2}m^2.$$

So the case B of the Lie algebra $\mathfrak{l}_{5,9}$ admits an algebraic Ricci soliton structure with the derivation

$$D = d\left(\frac{5}{8}m^2, \frac{5}{8}m^2, \frac{5}{4}m^2, \frac{15}{8}m^2, \frac{15}{8}m^2\right).$$

Now, we summarize the above results and give their relations with Lauret classification (Theorem 5.1 of [10]) in the table 1.

Remark 2.1. *For the Lie groups of nilpotency class greater than two we have used the classification given in [1], and we only consider Lie algebras which are not direct products of Lie algebras of lower dimension. Therefore, class μ'_8 of Lauret classification does not appear in our classification.*

TABLE 1. The classification of five-dimensional nilsolitons

case	Lie algebra structure	conditions for parameters of Lie algebra structure	Ricci soliton	the constant c and the derivation D	equivalence class in Lauret classification
case 2.1	$[E_1, E_2] = sE_3,$ $[E_3, E_4] = mE_5$	$s \geq m > 0$	+, if $s = m$	$c = -2m^2,$ $D = d(\frac{3}{2}m^2, \frac{3}{2}m^2, \frac{3}{2}m^2, \frac{3}{2}m^2, 3m^2)$	μ'_4
case 2.2	$[E_1, E_2] = mE_4,$ $[E_1, E_3] = sE_5$	$m \geq s > 0$	+, if $s = m$	$c = -2m^2,$ $D = d(m^2, \frac{3}{2}m^2, \frac{3}{2}m^2, \frac{5}{2}m^2, \frac{5}{2}m^2)$	μ'_6
case 2.3	$[E_1, E_2] = mE_3$	$m > 0$	+	$c = -\frac{3}{2}m^2,$ $D = d(m^2, m^2, 2m^2, \frac{3}{2}m^2, \frac{3}{2}m^2)$	μ'_7
case 2.4	$[E_1, E_2] = mE_3 + sE_4 + uE_5,$ $[E_1, E_3] = vE_4 + wE_5,$ $[E_1, E_4] = xE_5$	$m, v, x > 0,$ $s = 0$ and $w \geq 0$	+, if $x = m,$ $u = w = 0$ and $s = 0$ and $v = \frac{2}{\sqrt{3}}m$	$c = -2m^2,$ $D = d(\frac{1}{3}m^2, \frac{3}{2}m^2, \frac{11}{6}m^2, \frac{13}{6}m^2, \frac{5}{2}m^2)$	μ'_1
case 2.5	"	$m, v, x > 0,$ $w \geq 0$ and $s > 0$	-	-	-
case 2.6	$[E_1, E_2] = mE_3 + sE_4 + uE_5,$ $[E_1, E_3] = vE_4 + wE_5,$ $[E_1, E_4] = xE_5,$ $[E_2, E_3] = yE_5$	$m, v, x, y \neq 0,$ $s = 0, w \geq 0$	+, if $u = \frac{w}{m} = \frac{s}{v} = 0 = \pm\sqrt{\frac{3}{2}}x,$ $y = \pm x$	$c = -\frac{11}{4}x^2,$ $D = d(\frac{3}{4}x^2, \frac{3}{2}x^2, \frac{9}{4}x^2, 3x^2, \frac{15}{4}x^2)$	μ'_2
case 2.7	"	$s > 0$	-	-	-
case 2.8	$[E_1, E_2] = mE_4 + sE_5,$ $[E_1, E_3] = uE_5,$ $[E_1, E_4] = vE_5,$ $[E_2, E_3] = wE_5$	$s, u \geq 0,$ $m, v, w > 0$	+, if $s = u = 0, v = m$ and $w = \frac{\sqrt{2}}{2}m$	$c = -\frac{7}{4}m^2,$ $D = d(\frac{3}{4}m^2, m^2, \frac{3}{2}m^2, \frac{7}{4}m^2, \frac{5}{2}m^2)$	μ'_3
case 2.9	$[E_1, E_2] = mE_3 + sE_4 + uE_5,$ $[E_1, E_3] = vE_4,$ $[E_2, E_3] = wE_5,$	$m > 0,$ $w > v > 0,$ $s, u \geq 0$	-	-	-
case 2.10	"	$m, v > 0,$ $w = v,$ $s \geq 0,$ $u = 0$	+, if $s = 0,$ $v = \frac{\sqrt{3}}{2}m$	$c = -\frac{3}{2}m^2,$ $D = d(\frac{5}{8}m^2, \frac{5}{8}m^2, \frac{5}{4}m^2, \frac{15}{8}m^2, \frac{15}{8}m^2)$	μ'_5

Remark 2.2. Based on the results given in table 1, there are some three and four-step nilmanifolds which are not nilsoliton.

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