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ANALYSIS OF A CERTAIN PSEUDO q-CALCULUS AND ITS APPLICATIONS IN INTEGRAL INEQUALITIES

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ABSTRACT. We define three new operators which, in exceptional cases, are reduced to q-integral/derivative q_a-integral/derivative and ${}^bq$-integral/derivative operators. Fundamental properties emerge among these specific q-operators, like fractional calculus. By utilizing these three newly introduced operators, we can prove various inequalities, the most notable of which are the Chebyshev and Hermite-Hadamard types. Consequently, these new operators provide a generalized approach to many problems in classical inequalities using classical fractional calculus.

1. Introduction

The q-fractional operator (quantum fractional calculus) was introduced by Jackson [19, 20]. This operator has tremendous potential for applications in several fields, such as optimal control problems, q-transform analysis, and solutions of the q-difference, q-fractional integral inequalities, etc. [4, 6]. Al-Salam collaborated with Verma in the early 1970s on the q-fractional Leibniz rule [5–7]. They discovered q-analogues of parts of a series of papers by Osler on fractional calculus, whose first paper appeared in 1970. Recent developments in q-fractional calculus have naturally followed DIFdelthe developments advancements in the theory of q-series and orthogonal polynomials by several mathematicians including Askey and Andrews [8, 9, 15]. Interestingly, most of the results established in the fields of classical differential calculus and classical fractional calculus, like q-type Leibniz rules, q-Mittag-Leffler functions, q-Laplace transform, q-Mellin transform, q-Taylor series, q-Newton series, q-Hypergeometric series, q-Riemann-Liouville fractional operators, q-Caputo fractional derivatives,

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and various existence and uniqueness theorems, and many others deal with generalization of differentiation and integration to q-fractional order were able to derive with q-fractional calculus operator. Recently, some fractional integral inequalities have been obtained using the different kinds of q- integral. The interested reader is referred to [13, 24, 30, 40, 41] and references therein for more information and other extensions of Hermite-Hadamard inequality.

Pseudo-analysis, as one of the nonlinear analysis tools, generalizes the classical analysis, and is applicable in various fields, including partial differential equations, probability and measure theory, fuzzy logics, functional equations, variational calculus, semiring theory, functional analysis, control theory, and optimization, as in [3, 18, 34, 35, 38, 45]. Hence, pseudo-analysis is a vital issue with high capability in different fields. Recently, some researchers have proposed pseudo-analysis in the subject of inequalities such as Chebyshev type inequality and Hermite-Hadamard type inequality in a semiring $([\alpha,\beta],\oplus,\odot)$ [2, 18, 50]. For further information, we recommend that readers consult the papers [1, 2, 26, 32, 33, 36]. In 2017, the classical fractional operators were extended to pseudo fractional operators by Babakhani et al. [12]. Some researchers have made significant advancements in this field [11, 31, 43, 44].

In this paper, we introduce the q-integral/derivative. We define three new operators, which in exceptional cases, reduce to q-integral/derivative q_α-integral/derivative and ${}^{\beta}q$-integral/derivative operators of order $q \in (0,1)$ using a generator g on a semiring $([\alpha,\beta],\oplus,\odot)$ which pseudo analysis descript in this space. We also establish some properties for these operators. Then, using these three newly introduced operators, we investigate various inequalities, the most notable of which are the Chebyshev and Hermite-Hadamard inequality types. First, the definitions and results mentioned below are required throughout this manuscript.

Theorem 1.1 ([14], Chebyshev's Inequality). *If $f_1, f_2 : [\alpha,\beta] \to \mathbb{R}^+$ are nonnegative integrable functions on $[\alpha,\beta]$ which are all either monotonic increasing or monotonic decreasing then,*

$$\int_\alpha^\beta f_1(t)f_2(t)\mathrm{d}t \le (\beta-\alpha)^{n-1}\int_\alpha^\beta f_1(t)\mathrm{d}t\int_\alpha^\beta f_2(t)\mathrm{d}t. \tag{1.1}$$

Theorem 1.2 (Hermite-Hadamard Inequality). *If $\varrho : [\alpha,\beta] \to \mathbb{R}$ is a convex function, then the following holds:*

$$\varrho\left(\frac{\alpha+\beta}{2}\right) \le \frac{1}{\beta-\alpha}\int_\alpha^\beta \varrho(t)dt \le \frac{\varrho(\alpha)+\varrho(\beta)}{2}.$$

Moreover, the inequalities are reversed if f is concave.

Definition 1.3. *A function $\varrho : [\alpha,\beta] \to \mathbb{R}$ is said to be convex on $[\alpha,\beta]$ if for all $t, s \in [\alpha,\beta]$ and all $0 \le \eta \le 1$*

$$\varrho\big(\eta t + (1-\eta)s\big) \le \eta\varrho(t) + (1-\eta)\varrho(s).$$

Definition 1.4 ([28, 48]). *Suppose that $f_2 : [0,1] \to [0,\infty)$ and $f_1 : [\alpha,\beta] \subseteq \mathbb{R} \to [0,\infty)$ be two functions. The function f_1 is called an f_2-convex function whenever the inequality*

$$f_1\big(\eta t + (1-\eta)s\big) \le f_2(\eta)f_1(t) + f_2(1-\eta)f(s)$$

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holds for all $t, s \in [\alpha, \beta]$ and all $\eta \in (0, 1)$.

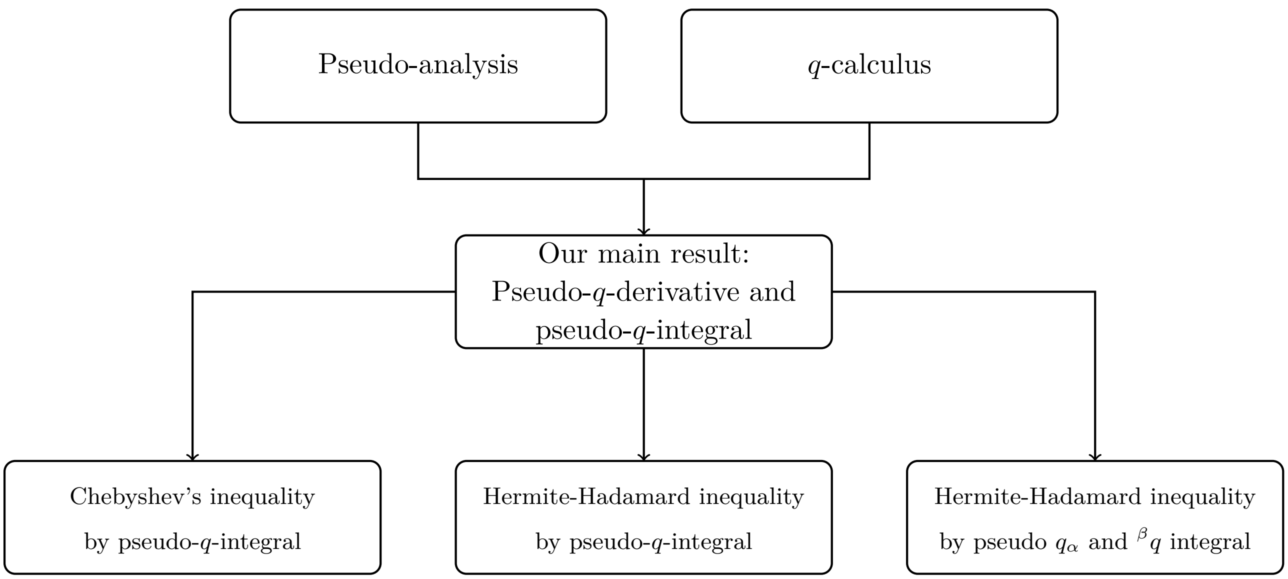
Two exceptional cases are as follows: If $f_2(t) = t$ is the identity function, then the notion of f_2-convex function f_1 coincides with the well-known concept of a convex function. Furthermore, if $\theta \in (0, 1]$ and $f_2(t) = t^\theta$, then the f_2-convex function f_1 is called an θ-convex function.

Definition 1.5 ([29, 49])**.** *Suppose that $f_2 : [0, 1] \to [0, \infty)$ and $f_1 : [\alpha, \beta] \subseteq \mathbb{R} \to [0, \infty)$ be two functions. The function f_1 is called a modified f_2-convex function whenever the inequality*

$$f_1\big(\eta t + (1-\eta)s\big) \leq f_2(\eta) f(t) + \big(1 - f_2(\eta)\big) f_1(s)$$

holds for all $t, s \in [\alpha, \beta]$ and all $\eta \in (0, 1)$.

Similarly, two exceptional cases are as follows: If $f_2(t) = t$ is the identity function, then the notion of modified f_2-convex function f_1 also coincides with the well-known concept of a convex function. Moreover, if $\theta \in [0, 1]$ and $h(t) = t^\theta$, then the modified f_2-convex function f_1 is called an $(\theta, 1)$-convex function.



2. Preliminaries

Our main result is included in the third section of this manuscript. For this purpose, we require two essential tools, pseudo-analysis, and q-calculus, which are described in Sections 2.1 and 2.2, respectively

2.1. Pseudo-Analysis.

This section gives some primary results of pseudo-operations, pseudo-analysis and pseudo-additive measures, and integrals [21, 25, 27, 33, 38, 39].

Let $[\alpha, \beta]$ be a closed (in some cases, can be considered semiclosed) subinterval of $[-\infty, \infty]$. The full order on $[\alpha, \beta]$ will be denoted by $\preceq$.

Definition 2.1 ([38])**.** *A binary operation $\oplus$ on $[\alpha, \beta]$ is pseudo-addition, if it is commutative, non-decreasing (for $\preceq$), continuous, associative, and with a zero (neutral) element denoted by $\mathbf{0}$. Let $[\alpha, \beta]_+ = \{t \,|\, t \in [\alpha, \beta], \mathbf{0} \preceq t\}$. A binary operation $\odot$ on $[\alpha, \beta]$ is pseudo-multiplication, if it is*

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commutative, positively non-decreasing, i.e., $t \preceq s$ implies that $t \odot u \preceq s \odot u$ for all $u \in [\alpha, \beta]_+$, associative and with a unit element $\mathbf{1} \in [\alpha, \beta]$, i.e., for each $t \in [\alpha, \beta], \mathbf{1} \odot t = t$. Also, $\mathbf{0} \odot t = \mathbf{0}$ and that $\odot$ is distributive over $\oplus$, i.e.,

$$t \odot (s \oplus u) = (t \odot s) \oplus (t \odot u).$$

The structure $([\alpha, \beta], \oplus, \odot)$ is a semiring [25]. Generator g will be denoted by $\mathbb{R}_g$, respectively, by $\mathbf{0}_g$ and $\mathbf{1}_g$ [21]. The generator function $g : \mathbb{R}_g \to \mathbb{R}$ is monotone (increasing or decreasing). The order relation in $\mathbb{R}_g$, denoted by $\leq_g$, satisfies the following:

$$t \leq_g s \longleftrightarrow t \ominus s \leq_g \mathbf{0}_g.$$

If $t \leq_g s$, we can also write it as $s \geq_g t$. If $t \leq_g s$ and $t \neq s$, we shall write it as $t <_g s$, or equivalently, $s >_g t$.

Definition 2.2 ([39]). *Let Ω be a non-empty set and $\mathcal{F}$ be a σ-algebra of subsets of a set Ω. A set function $m : \mathcal{F} \to [\alpha, \beta]$ is called a σ-$\oplus$-measure if it satisfies the following conditions:*

(1) $m(\emptyset) = \mathbf{0}$;

(2) $m\left(\bigcup_{j=1}^{\infty} F_j\right) = \bigoplus_{j=1}^{\infty} m(F_j)$ *holds for any sequence $\{F_j\}_{j \in \mathbb{N}}$ of pairwise disjoint sets from $\mathcal{F}$.*

Definition 2.3 ([39]). *Let $g : [a, b] \to [0, \infty]$ be a monotone and continuous function. Then the pseudo-operations $\oplus$ and $\odot$ are given by*

$$x \oplus y = g^{-1}(g(x) + g(y)) \quad \text{and} \quad x \odot y = g^{-1}(g(x)\, g(y))$$

for all $x, y \in [a, b]$.

If the pseudo-addition's zero element is a, we will consider increasing generators with $g(\alpha) = 0$ and $g(\beta) = \infty$. If the pseudo-addition's zero element is β, then we will consider decreasing generators with $g(\beta) = 0$ and $g(\alpha) = \infty$.

Definition 2.4 ([27]). *Let g be a generator of a pseudo-addition $\oplus$ on the interval $[-\infty, +\infty]$. Binary operation $\ominus$ and $\oslash$ on $[-\infty, +\infty]$ defined as follows:*

$$t \ominus s = g^{-1}(g(t) - g(s)) \quad t \oslash s = g^{-1}\left(\frac{g(t)}{g(s)}\right).$$

If expressions $g(t) - g(s)$ and $\frac{g(t)}{g(s)}$ have sense are said to be the pseudo-subtraction and pseudo-division consistent with the pseudo-addition $\oplus$.

Definition 2.5 ([21]). *Let $g : \mathbb{R}_g \to \mathbb{R}$ be a generator function which is monotone (increasing or decreasing). Then the pseudo-scalra product is given as*

$$k \odot t = g(kg(t))$$

for all $t \in \mathbb{R}_g$ and $k \in \mathbb{R}$.

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Definition 2.6 ([39]). *Let pseudo-operations $\oplus$ and $\odot$ be defined by a monotone and continuous function $g : [\alpha, \beta] \to [0, \infty]$. The g-integral for a measurable function $\varrho : [\gamma, \delta] \to [\alpha, \beta]$ is given by*

$$\int_{[\gamma,\delta]}^{\oplus} \varrho \odot \mathrm{d}t = g^{-1}\left(\int_{\gamma}^{\delta} g(\varrho(t))\mathrm{d}t\right).$$

Definition 2.7 ([33]). *Let g be the additive generator of the strict pseudo-addition $\oplus$ on $[\alpha, \beta]$ such that g is continuously differentiable on (α, β). The corresponding pseudo-multiplication $\odot$ is $w \odot z = g^{-1}(g(w)g(z))$. If the function ϱ is differentiable on (γ, δ) and has the same monotonicity as the function g, then the g-derivative of ϱ at the point $x \in (\gamma, \delta)$ is defined by*

$$\frac{\mathrm{d}^{\oplus}\varrho(t)}{\mathrm{d}t} = g^{-1}\left(\frac{\mathrm{d}}{\mathrm{d}t} g(\varrho(t))\right).$$

Also, if there exists the k-g-derivative of ϱ, then

$$\frac{\mathrm{d}^{(k)\oplus}\varrho(t)}{\mathrm{d}t} = g^{-1}\left(\frac{\mathrm{d}^k}{\mathrm{d}t^k} g(\varrho(t))\right).$$

Definition 2.8 ([27]). *Let $g : [-\infty, +\infty] \to [-\infty, +\infty]$ be a continuous, strictly increasing, and odd function such that $g(0) = \mathbf{0}$, $g(1) = \mathbf{1}$, $g(+\infty) = +\infty$. The system of pseudo-arithmetical operations $\{\oplus, \ominus, \odot, \oslash\}$ generated by this function is the consistent system.*

2.2. **q-Calculus.** In this section, we remind concepts of q-derivative/integral, q_α-derivative/integral and ${}^{\beta}q$-derivative/integral. These operators are generalized to a specific q-derivative/integral using the pesudo-analysis in the next section, named pesudo-q-derivative/integral.

Definition 2.9 ([23]). *The q-derivative of a real function $\varrho(t)$ is defined as*

$$D_q\varrho(t) = \frac{\varrho(qt) - \varrho(t)}{qt - t} \tag{2.1}$$

with $q \in (0, 1)$. The q-derivative calculates the rise of $\varrho(t)$ over the interval (qt, t). As such, it is numerically equal to the slope of the line that passes through points $(qt, \varrho(qt))$ and $(t, \varrho(t))$.

Definition 2.10 ([23]). *Suppose $0 < \alpha < \beta$. The definite q-integral is defined as*

$$\int_0^{\beta} \varrho(t)\mathrm{d_q}t = (1 - q)\beta \sum_{j=0}^{\infty} q^j \varrho(q^j \beta) \tag{2.2}$$

and

$$\int_{\alpha}^{\beta} \varrho(t)\mathrm{d_q}t = \int_0^{\beta} \varrho(t)\mathrm{d_q}t - \int_0^{\alpha} \varrho(t)\mathrm{d_q}t.$$

The definite q-integral defined above is too general for our purpose of studying inequalities. For example, if $\varrho(t) \geq 0$, it is not necessarily true that $\int_{\alpha}^{\beta} \varrho(t)\mathrm{d_q}t \geq 0$.

From now on, we will use a particular type of definite q-integral, which we will call the restricted definite q-integral.

Definition 2.11 ([17]). *Let $0 < q < 1$, $\beta > 0$, and $k \in \mathbb{Z}^+$. The restricted q-integral is $\int_{\beta q^k}^{\beta} \varrho(t)\mathrm{d_q}t$. In addition to $\varrho(t)$, the restricted definite q-integral depends on q, β, and k.*

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From now on, throughout the paper, we will use the following notations:

$$\gamma_j = \beta q^j, \quad j \in \{0, 1, ..., k\}, \quad \alpha = \gamma_k = \beta q^k.$$

The following formula readily follows from (1.3) and (1.4):

$$\int_\alpha^\beta \varrho(t) \mathrm{d_q} t = \int_{\beta q^k}^\beta \varrho(t) \mathrm{d_q} t = (1-q)\beta \sum_{j=0}^{k-1} q^j \varrho(\beta q^j) = (1-q) \sum_{j=0}^{k-1} \gamma_j \varrho(\gamma_j).$$

Note that the restricted integral $\int_\alpha^\beta \varrho(t)\mathrm{d_q}t$ is just a finite sum, so no questions about convergency arise. It is easy to check that:

$$\int_\alpha^\beta D_q \varrho \mathrm{d_q} t = \varrho(\beta) - \varrho(\alpha).$$

Obviously, if $\varrho(t) \geq g(t)$ on $[\alpha, \beta]$, then $\int_\alpha^\beta \varrho(t)\mathrm{d_q}t \geq \int_\alpha^\beta g(t)\mathrm{d_q}t$. If $0 < j < k$, then

$$\int_\alpha^\beta \varrho(t) \mathrm{d_q} t = \int_\alpha^{\gamma_j} \varrho(t) \mathrm{d_q} t + \int_{\gamma_j}^\beta \varrho(t) \mathrm{d_q} t.$$

The following is the formula for the q-integration by parts [23]:

$$\int_\alpha^\beta \varrho(t)(D_q g)(t) \mathrm{d_q} t = \varrho(\beta) g(\beta) - \varrho(\alpha) g(\alpha) - \int_\alpha^\beta g(qt)(D_q \varrho)(t) \mathrm{d_q} t.$$

The usual Riemann integral can be considered as a limit of the restricted definite q-integral in the following way. Since $\alpha = \beta q^k$, it follows that $q = (\frac{\alpha}{\beta})^{\frac{1}{k}}$. Fix α and β, and suppose that $k \to \infty$ (hence $q \to 1$). Then $\int_\alpha^\beta \varrho(t)\mathrm{d_q}t \to \int_\alpha^\beta \varrho(t)\mathrm{d}t$ and so ϱ is Riemann integrable on $[\alpha, \beta]$.

Definition 2.12 ([23]). *$\varrho : [\alpha, \beta] \to \mathbb{R}$ is called q-convex on $[\alpha, \beta]$, if $D_q^2 \varrho \geq 0$ on $[\alpha, \beta]$.*

It is easy to see that $\varrho(t)$ is q-convex, if and only if : $q\varrho(t) - (1+q)\varrho(qt) + \varrho(q^2 t) \geq 0$ for all $t \in [\alpha, \beta]$ and $q^2 t \in [\alpha, \beta]$. If ϱ is convex on $[\alpha, \beta]$, then it is also q-convex on $[\alpha, \beta]$.

Example 2.13. *Consider the function $\varrho = \sin(4t)$. Clearly, ϱ is not a convex function on $\left[\frac{\pi}{2}, \pi\right]$, but ϱ is $\frac{1}{2}$-convex.*

A generalization of the q-derivative (2.1) was introduced in [46,47]. Here, the notion of q_α-derivative is extended to an analogous notion, namely, the q^β-derivative.

Definition 2.14 ([13]). *Suppose that $\varrho : [\alpha, \beta] \to \mathbb{R}$ is a continuous function and $q \in (0,1)$. The q_α-derivative of ϱ at a point $t \in (\alpha, \beta]$ is defined by*

$$ {}_\alpha d_q \varrho(t) = \frac{\varrho(t) - \varrho(qt + (1-q)\alpha)}{(1-q)(t-\alpha)}. \tag{2.3}$$

If $t = \alpha$, then the q_α-derivative of ϱ at α is defined by

$$ {}_\alpha d_q \varrho(\alpha) = \lim_{t \to \alpha} {}_\alpha d_q \varrho(t),$$

provided that the limit exists and it is finite.

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Analogously, the q^β-derivative of ϱ at a point $t \in [\alpha, \beta)$ is defined by

$$ {}^{\beta}d_q\varrho(t) = \frac{\varrho(t) - \varrho(qt + (1-q)\beta)}{(1-q)(t-\beta)}. \tag{2.4}$$

If $t = \beta$, then the q^β-derivative of f at β is defined by

$$ {}^{\beta}d_q\varrho(\beta) = \lim_{t\to\beta} {}^{\beta}d_q\varrho(t),$$

provided that the limit exists and it is finite.

Remark 2.15 ([13])**.** *If we set $\alpha = 0$ in (2.3) or $t = \beta$ in (2.4), then both the notions of q_0-derivative and q^0-derivetive coincide with the familiar notion of the q-derivative of ϱ at a point $t \in [\alpha, \beta]$, which was defined in Definition 2.9.*

Motivated by the notion of a q_α-definite integral, which is defined in [46, 47], one can define the notion of a q^β-definite integral as follows:

Definition 2.16 ([13])**.** *Suppose that $\varrho : [\alpha, \beta] \to \mathbb{R}$ is a continuous function and $q \in (0, 1)$. The q_α-definite integral of ϱ on the interval $[\alpha, \beta]$ is defined by*

$$\int_\alpha^\xi \varrho(t)\ {}_\alpha\mathrm{d}_q t = (1-q)(\xi-\alpha)\sum_{j=0}^{\infty} q^j \varrho\big(q^j\xi + (1-q^j)\alpha\big) \tag{2.5}$$

for all $\xi \in [\alpha, \beta]$.

Analogously, the q^β-definite integral of ϱ on the interval $[\alpha, \beta]$ is defined by

$$\int_\xi^\beta \varrho(t)\ {}^\beta\mathrm{d}_q t = (1-q)(\beta-\xi)\sum_{j=0}^{\infty} q^j \varrho\big(q^j\xi + (1-q^j)\beta\big) \tag{2.6}$$

for all $\xi \in [\alpha, \beta]$.

Remark 2.17 ([13])**.** *If we set $\alpha = 0$ in (2.5) or $\xi = \beta$ in (2.6), then both the notions of q_0-definite integral and q^0-definite integral coincide with the familiar notion of the classical q-integral of ϱ on the interval $[\alpha, \beta]$, which was defined in Definition 2.10.*

Also, if we set $\alpha = 0$ and $\xi = \beta = 1$ in (2.5), then we obtain

$$\int_0^1 \varrho(t)\ {}_0\mathrm{d}_q t = (1-q)\sum_{j=0}^{\infty} q^j \varrho(q^j).$$

Analogously, if we set $\beta = 1$ and $\xi = \alpha = 0$ in (2.6), then we obtain

$$\int_0^1 \varrho(t)\ {}^1\mathrm{d}_q t = (1-q)\sum_{j=0}^{\infty} q^j \varrho(1-q^j).$$

Remark 2.18 ([13])**.** *Suppose that $f_1 : [\alpha, \beta] \to \mathbb{R}$ is a continuous function and $q \in (0, 1)$. From [22,46] we have*

$$ {}_\alpha d_q \int_\alpha^\xi f_1(t)\ {}_\alpha\mathrm{d}_q t = f_1(\xi) - f_1(\alpha)$$

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and

$$\int_{\gamma}^{\xi} {}_{\alpha}d_q f_1(t)\ {}_{\alpha}\mathrm{d}_q t = f_1(\xi) - f_1(\gamma)$$

for all $\gamma \in (\alpha, \xi)$. Arguments similar to that appeared in [46] yield

$${}^{\beta}d_q \int_{\xi}^{\beta} f_1(t)\ {}^{\beta}\mathrm{d}_q t = f_1(\beta) - f_1(\xi)$$

and

$$\int_{\xi}^{\gamma} {}^{\beta}d_q f_1(t)\ {}^{\beta}\mathrm{d}_q t = f_1(\gamma) - f_1(\xi)$$

for all $\gamma \in (\xi, \beta)$. Furthermore, if $f_2 : [\alpha, \beta] \to \mathbb{R}$ is a continuous function and c is an arbitrary real number, then we have

$$\int_{\alpha}^{\xi} \big[(cf_1)(t) + f_2(t)\big]\ {}_{\alpha}\mathrm{d}_q t = c\int_{\alpha}^{\xi} f_1(t)\ {}_{\alpha}\mathrm{d}_q t + \int_{\alpha}^{\xi} f_2(t)\ {}_{\alpha}\mathrm{d}_q t$$

and

$$\int_{\xi}^{\beta} \big[(cf_1)(t) + f_2(t)\big]\ {}^{\beta}\mathrm{d}_q t = c\int_{\xi}^{\beta} f_1(t)\ {}^{\beta}\mathrm{d}_q t + \int_{\xi}^{\beta} f_2(t)\ {}^{\beta}\mathrm{d}_q t.$$

Hence, whenever $f_1 \le f_2$, that is, $f_1(t) \le f_2(t)$ for all $t \in [\alpha, \beta]$, we find

$$\int_{\alpha}^{\xi} f_1(t)\ {}_{\alpha}\mathrm{d}_q t \le \int_{\alpha}^{\xi} f_2(t)\ {}_{\alpha}\mathrm{d}_q t, \quad \int_{\xi}^{\beta} f_1(t)\ {}^{\beta}\mathrm{d}_q t \le \int_{\xi}^{\beta} f_2(t)\ {}^{\beta}\mathrm{d}_q t.$$

Several examples are constructed for the notions defined in Definitions 2.14 and 2.16. Here, we do not intend to provide any examples. However, the interested readers may see, e.g., [13, Examples 9 and 10].

3. Pseudo-q-Derivative and Pseudo-q-Integral

In this section, as mentioned before at the beginning of section 2.2, the three operators of the previous section are extended to three new operators using pseudo analysis.

Definition 3.1. *Let $\varrho : [\alpha, \beta] \to \mathbb{R}$ be a continuous function, $q \in (0, 1)$, and let f be q-differentiable on $[\alpha, \beta]$. Assume further that $g : [\alpha, \beta] \to [0, \infty]$ is a monotone function such that either $g(\alpha) = 0$ or $g(\beta) = 0$. Then we have*

$$D_q^{\oplus}\varrho(t) = g^{-1}\Big(D_q\Big(g\big(\varrho(t)\big)\Big)\Big) = g^{-1}\left(\frac{g(\varrho(qt)) - g(\varrho(t))}{qt - \alpha}\right). \tag{3.1}$$

Moreover, if ϱ is q_α-derivative and $t \in (\alpha, \beta]$, then we define the q_α-g-derivative of f as

$${}_{\alpha}d_q^{\oplus}\varrho(t) = g^{-1}\Big({}_{\alpha}d_q g\big(\varrho(t)\big)\Big) = g^{-1}\left(\frac{g(\varrho(t)) - g(\varrho(qt + (1-q)\alpha)}{(1-q)(t-\alpha)}\right). \tag{3.2}$$

Analogously, the g-q^{β}-derivative of ϱ at a point $t \in [\alpha, \beta)$ is defined by

$${}^{\beta}d_q^{\oplus} = g^{-1}\Big({}^{\beta}d_q\Big(g\big(\varrho(t)\big)\Big)\Big) = g^{-1}\left(\frac{g(\varrho(t)) - g(\varrho(qt) + (1-q)\beta)}{(1-q)(t-\beta)}\right). \tag{3.3}$$

Theorem 3.2. *Let f_1 and f_2 be two q-g-differentiable functions on (γ, δ) with the values in $[\alpha, \beta]$. Then, for any arbitrary $c \in [\alpha, \beta]$ we have*

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(i) $D_q^{\oplus}(f_1 \oplus f_2) = D_q^{\oplus} f_1 \oplus D_q^{\oplus} f_2$;
(ii) $D_q^{\oplus}(c \otimes f_1) = c \otimes D_q^{\oplus} f_1$;
(iii) $D_q^{\oplus} c = 0$.

Theorem 3.3. *Let f_1 and f_2 be two g-q-differentiable functions on $[\alpha, \beta]$. Then we have*

$$D_q^{\oplus}\big(f_1(t) \odot f_2(t)\big) = D_q^{\oplus} f_1(t) \odot f_2(t) \oplus f_1(qt) \odot D_q^{\oplus} f_2(t). \tag{3.4}$$

Proof. Note first that

$$\begin{aligned} D_q^{\oplus}\big(f_1(t) \odot f_2(t)\big) &= g^{-1}\Big(D_q g\big(f_1(t) \odot f_2(t)\big)\Big) \\ &= g^{-1}\Bigg(D_q g\bigg(g^{-1}\Big(g\big(f_1(t)\big)g\big(f_2(t)\big)\Big)\bigg)\Bigg) \\ &= g^{-1}\Big(D_q g\big(f_1(t)\big)D_q f_1(t) g\big(f_2(t)\big) + g\big(f_1(qt)\big)D_q g\big(f_2(t)\big)D_q f_2(t)\Big). \end{aligned}$$

Then we have

$$\begin{aligned} &\big(D_q^{\oplus} f_1(t) \odot f_2(t)\big) \oplus \big(f_1(qt) \odot D_q^{\oplus} f_2(t)\big) \\ &= g^{-1}\Big(g\big(D_q^{\oplus} f_1(t) \odot f_2(t)\big) + g\big(f_1(qt) \odot D_q^{\oplus} f_2(t)\big)\Big) \\ &= g^{-1}\Bigg(g\bigg(g^{-1}\Big(g\big(D_q^{\oplus} f_1(t)\big)g\big(f_2(t)\big)\Big)\bigg) + g\bigg(g^{-1}\Big(g\big(f_1(qt)\big)g\big(D_q^{\oplus} f_2(t)\big)\Big)\bigg)\Bigg) \\ &= g^{-1}\Bigg(g\bigg(g^{-1}\Big(D_q\big(f_1(t)\big)\Big)\bigg)g\big(f_2(t)\big) + g\big(f_1(qt)\big)g\bigg(g^{-1}\Big(D_q\big(f_2(t)\big)\Big)\bigg)\Bigg) \\ &= g^{-1}\Big(D_q g\big(f_1(t)\big)D_q f_1(t) g\big(f_2(t)\big) + g\big(f_1(qt)\big)D_q g\big(f_2(t)\big)D_q\big(f_2(t)\big)\Big). \end{aligned}$$

□

Theorem 3.4. *If $f_2(t) \neq f_2(qt)$ and both are non-zero, then*

$$D_q^{\oplus}(f_1 \oslash f_2)(t) = D_q^{\oplus} f_1(t) \odot f_2(t) \ominus f_1(t) D_q^{\oplus} f_2(t) \oslash f_2(t) \odot f_2(qt).$$

Proof. From the left side, we have

$$\begin{aligned} D_q^{\oplus}(f_1 \oslash f_2)(t) &= g^{-1}\big(D_q g(f_1 \oslash f_2)(t)\big) \\ &= g^{-1}\Bigg(D_q g\bigg(g^{-1}\left(\frac{g(f_1(t))}{g(f_2(t))}\right)\bigg)\Bigg) \\ &= g^{-1}\left(\frac{D_q g(f_1(t))D_q(f_1(t))g(f_2(t)) - D_q g(f_2(t))D_q(f_2(t))g(f_1(t))}{g(f_2(t))g(f_2(qt))}\right). \end{aligned}$$

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And from the right side we have

$$\begin{aligned}
&D_q^{\oplus}(f_1(t)) \odot f_2(t) \ominus f_1(t) \odot D_q^{\oplus} f_2(t) \oslash f_2(t) \odot f_2(qt)\\
&\qquad = g^{-1}\left(\frac{g(D_q^{\oplus}(f_1(t)) \odot f_2(t) \ominus f_1(t) \odot D_q^{\oplus} f_2(t)}{g(f_2(t) \odot f_2(qt))}\right)\\
&\qquad = g^{-1}\left(\frac{g(g^{-1}(g(D_q^{\oplus} f_1(t) \odot f_2(t)) - g(f_1(t) \odot D_q^{\oplus} f_2(t))))}{g(g^{-1}(g(f_2(t))g(f_2(qt))))}\right)\\
&\qquad = g^{-1}\left(\frac{g(g^{-1}(g(g^{-1}(D_q g(f_1(t)) \odot f_2(t)))) - g(g^{-1}(g(f_1(t))g(D_q^{\oplus} f_2(t))))}{g(f_2(t))g(f_2(qt))}\right)\\
&\qquad = g^{-1}\left(\frac{D_q g(f_1(t))g(f_2(t)) - g(f_1(t))D_q g(f_2(t))}{g(f_2(t))g(f_2(qt))}\right)\\
&\qquad = g^{-1}\left(\frac{D_q g(f_1(t))D_q f_1(t)g(f_2(t)) - D_q g(f_2(t))D_q f_2(t)g(f_1(t))}{g(f_2(t))g(f_2(qt))}\right). \qquad \square
\end{aligned}$$

Theorem 3.5. *Let $f_1, f_2 : [\alpha, \beta] \to \mathbb{R}$ be two continuous functions and $q \in (0,1)$. Then we have*

(i) ${}_{\alpha}d_q^{\oplus}(f_1 \oplus f_2) = {}_{\alpha}d_q^{\oplus} f_1 \oplus {}_{\alpha}d_q^{\oplus} f_2$;
(ii) ${}^{\beta}d_q^{\oplus}(f_1 \oplus f_2) = {}^{\beta}d_q^{\oplus} f_1 \oplus {}^{\beta}d_q^{\oplus} f_2$;
(iii) ${}_{\alpha}d_q^{\oplus}(c \otimes f_1) = c \otimes {}_{\alpha}d_q^{\oplus} f_1$;
(v) ${}^{\beta}d_q^{\oplus}(c \otimes f_2) = c \otimes {}^{\beta}d_q^{\oplus} f_2$.

Proof. It can be done by utilizing Definition 3.1 and conducting some elementary calculations similar to Theorems 3.3 and 3.4. $\square$

Theorem 3.6. *Let $f_1, f_2 : [\alpha, \beta] \to \mathbb{R}$ be two g-q_α-differentiable functions on $[a, b]$. Then we have*

$$\begin{aligned}
{}_{\alpha}D_q^{\oplus}(f_1 \odot f_2)(t) &= f_1(t) \odot {}_{\alpha}D_q^{\oplus} f_2(t) \oplus f_2\left(qt + (1-q)\alpha\right) \odot {}_{\alpha}D_q^{\oplus} f_1(t)\\
&= f_2(t) \odot {}_{\alpha}D_q^{\oplus} f_1(t) \oplus f_1\left(qt + (1-q)\alpha\right) \odot {}_{\alpha}D_q^{\oplus} f_2(t). \qquad (3.5)
\end{aligned}$$

Proof. Since f_1 and f_2 are continuous and $q \in (0,1)$, we have

$$\begin{aligned}
{}_{\alpha}D_q^{\oplus}(f_1 \odot f_2)(t) &= g^{-1}\left(\frac{g(f_1 \odot f_2)(t) - g\left(f_1\left(qt + (1-q)\alpha\right)\right) \odot f_2\left(qt + (1-q)\alpha\right)}{(1-q)(t-\alpha)}\right)\\
&= g^{-1}\left(\frac{g\left(f_1(t)\right) g\left(f_2(t)\right) - g\left(f_1\left(qt + (1-q)\alpha\right)\right) g\left(f_2\left(qt + (1-q)\alpha\right)\right)}{(1-q)(t-\alpha)}\right)\\
&= g^{-1}\left(\frac{\begin{array}{l} g\left(f_1(t)\right) g\left(f_2(t)\right) - g\left(f_1(t)\right) g\left(f_2\left(qt + (1-q)a\right)\right)\\ \quad + g\left(f_1(t)\right) g\left(f_2\left(qt + (1-q)\alpha\right)\right)\\ \qquad - g\left(f_1\left(qt + (1-q)\alpha\right)\right) g\left(f_2\left(qt + (1-q)\alpha\right)\right)\end{array}}{(1-q)(t-\alpha)}\right)\\
&= g^{-1}\left(g\left(f_1(t)\right)\left(\frac{g\left(f_2(t)\right) - g\left(f_2\left(qt + (1-q)\alpha\right)\right)}{(1-q)(t-\alpha)}\right)\right)\\
&\quad + g\left(f_2\left(qt + (1-q)\alpha\right)\right)\left(\frac{g\left(f_1(t)\right) - g\left(f_1\left(qt + (1-q)\alpha\right)\right)}{(1-q)(t-\alpha)}\right)\\
&= g^{-1}\left(g\left(f_1(t)\right) {}_{\alpha}D_q^{\oplus} f_2(t) + g\left(f_2\left(qt + (1-q)\alpha\right)\right)\left({}_{\alpha}D_q^{\oplus} f_1(t)\right)\right)\\
&= f_1(t) \odot {}_{\alpha}D_q^{\oplus} f_2(t) \oplus f_2\left(qt + (1-q)\alpha\right) \odot \left({}_{\alpha}D_q^{\oplus} f_2(t)\right). \qquad \square
\end{aligned}$$

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Remark 3.7. *The proof of the second equality in Equation (3.5) is similar by interchanging the functions f_1 and f_2. Also, the proof of Theorem 3.6 for ${}^{\beta}D_q^{\oplus}$ is similar, and thus, is omitted.*

Theorem 3.8. *Assume that $f_1, f_2 : [\alpha, \beta] \to \mathbb{R}$ are two g-q_α-differentiable functions on $[\alpha, \beta]$. Then*

$$ {}_{\alpha}D_q^{\oplus}(f_1 \oslash f_2)(t) = f_2(t) \odot {}_{\alpha}D_q^{\oplus} f_1(t) \ominus f_1(t) \odot {}_{\alpha}D_q^{\oplus} f_2(t) \oslash f_2(t) \odot f_2\left(qt + (1-q)\alpha\right). $$

Proof. Since f_1 and f_2 are continuous and $q \in (0,1)$, we have

$$ \begin{aligned} {}_{\alpha}D_q^{\oplus}(f_1 \oslash f_2)(t) &= g^{-1}\left(\frac{g(f_1 \oslash f_2)(t) - g\left((f_1 \oslash f_2)\left(qt + (1-q)\alpha\right)\right)}{(1-q)(t-\alpha)}\right) \\ &= g^{-1}\left(\frac{\frac{g(f_1(t))}{g(f_2(t))} - \frac{g(f_1(qt+(1-q)\alpha))}{g(f_2(qt+(1-q)\alpha))}}{(1-q)(t-\alpha)}\right) \\ &= g^{-1}\left(\frac{g\left(f_1(t)\right) g\left(f_2\left(qt+(1-q)\alpha\right)\right) - g\left(f_2(t)\right) g\left(f_1\left(qt+(1-q)\alpha\right)\right)}{g\left(f_2(t)\right) g\left(f_2\left(qt+(1-q)\alpha\right)\right)(1-q)(t-\alpha)}\right) \\ &= g^{-1}\left(\frac{g\left(f_2(t)\right)\left(\frac{g(f_1(t)) - g(f_1(qt+(1-q)\alpha))}{(1-q)(t-\alpha)}\right) - g\left(f_1(t)\right)\left(\frac{g(f_2(t)) - g(f_2(qt+(1-q)\alpha))}{(1-q)(t-\alpha)}\right)}{g\left(f_2(t)\right) g\left(f_2\left(qt+(1-q)\alpha\right)\right)}\right) \\ &= f_2(t) \odot {}_{\alpha}D_q^{\oplus} f_1(t) \ominus f_1(t) \odot {}_{\alpha}D_q^{\oplus} f_2(t) \oslash f_2(t) \odot f_2\left(qt + (1-q)\alpha\right). \end{aligned} $$

□

Remark 3.9. *The proof of Theorem 3.8 for ${}^{\beta}D_q^{\oplus}$ follows a similar approach and is thus omitted.*

Definition 3.10. *Let $0 < q < 1$, $0 < \alpha < \beta$, $0 < \theta < \delta$ and $k \in \mathbb{Z}^+$and let $g : [\alpha, \beta] \to [0, \infty]$ be a monotone and continuous function . The g-integral by using q-calculus for a measurable function $\varrho : [\alpha, \beta] \to [\theta, \delta]$ is :*

$$ \int_{[\alpha,\beta]}^{\oplus} \varrho(t) \odot d_q t = g^{-1}\left(\int_{\alpha}^{\beta} g(\varrho(t)) d_q t\right) = g^{-1}\left((1-q)\sum_{j=0}^{k-1} \gamma_j g(\varrho(\gamma_j))\right). $$

We will use the following notations: $\gamma_j = \beta q^j$, for $j \in \{0, 1, \cdots, k\}$, $\alpha = \gamma_k = \beta q^k$.

Definition 3.11. *Let $0 < q < 1$, $0 < \alpha < \beta$, $0 < \theta < \delta$ and $k \in \mathbb{Z}^+$and let $g : [\alpha, \beta] \to [0, \infty]$ be a monotone and continuous function . The pseudo-q-integral by using q_α-calculus for a measurable continuous function $\varrho : [\alpha, \beta] \to [\theta, \delta]$ for $\xi \in [\alpha, \beta]$ is:*

$$ \int_{[\alpha,\xi]}^{\oplus} \varrho(t) \odot {}_{\alpha}d_q t = g^{-1}\left(\int_{\xi}^{\alpha} g\big(\varrho(t)\big)_{\alpha} d_q t\right) = g^{-1}\left((1-q)(\xi-\alpha)\sum_{j=0}^{\infty} q^j g\big(\varrho(q^j\xi + (1-q^j)\alpha)\big)\right). \tag{3.6} $$

Similarly, we define the g-integral by using q^β integral of ϱ on $[\alpha, \beta]$:

$$ \int_{[\xi,\beta]}^{\oplus} \varrho(t) \odot {}^{\beta}d_q t = g^{-1}\left(\int_{\xi}^{\beta} g\big(\varrho(t)\big)^{\beta} d_q t\right) = g^{-1}\left((1-q)(\beta-\xi)\sum_{j=0}^{\infty} q^j g\big(\varrho(q^j\xi + (1-q^j)\beta)\big)\right). \tag{3.7} $$

Remark 3.12. *If we take $\alpha = 0$ in (3.6) or $\beta = 1$ in (3.7), we obtain the g integral by using q-calculus (Definition 3.10):*

$$ \int_{[0,1]}^{\oplus} \varrho(t) \odot d_q t = g^{-1}\left(\int_{0}^{1} g(\varrho(t)) d_q t\right) = g^{-1}\left((1-q)\sum_{j=0}^{\infty} q^j g(\varrho(q^j))\right) $$

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Analogously, if $\beta = 1$ and $\xi = \alpha = 0$ in (3.7) then:

$$\int_{[0,1]}^{\oplus} \varrho(t) \odot d_q t = g^{-1}\Big(\int_0^1 g(\varrho(t)) d_q t \Big) = g^{-1}\big((1-q)\sum_{j=0}^{\infty} q^j g(\varrho(1-q^k))\big).$$

Example 3.13. *Let $\alpha, \beta, m, d \in \mathbb{R}$ and $q \in (0,1)$ and consider a function $g : \mathbb{R} \to \mathbb{R}$ defined by the rule $\varrho(t) = mt + d$ for all real numbers t. Then, provided that $\alpha < \beta$, we have*

$$\begin{aligned}
\int_{[\alpha,\xi]}^{\oplus} \varrho(t) \odot \ {}_a\mathrm{d}_q t &= g^{-1}\left(\int_{\alpha}^{\xi} g(mt+d) \ {}_\alpha \mathrm{d}_q t \right) \\
&= g^{-1}\left((1-q)(\xi-\alpha) \sum_{j=0}^{\infty} q^j g(mt+d) \right) \\
&= g^{-1}\left((1-q)(\xi-\alpha) \sum_{j=0}^{\infty} q^j \Big(m\big(\beta q^j + (1-q^j)\alpha\big) + d \Big) \right) \\
&= g^{-1}\left((1-q)(\beta-\alpha) \sum_{j=0}^{\infty} q^j \big(m(\beta-\alpha) q^j + (m\alpha + d) \big) \right) \\
&= g^{-1}\left((1-q)(\beta-\alpha) \left(\frac{m(\beta-\alpha)}{1-q^2} + \frac{m\alpha+d}{1-q} \right) \right) \\
&= g^{-1}\left((\beta-\alpha) \left(\frac{m(\beta+\alpha q)}{1+q} + d \right) \right) \\
&= \frac{(\beta-\alpha)\left(\frac{m(\beta+\alpha q)}{1+q} + d\right) - d}{m}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\int_{[\xi,\beta]}^{\oplus} \varrho(t) \odot \ {}^{\beta}\mathrm{d}_q t &= g^{-1}\left(\int_{t}^{\beta} g(mt+d) \ {}^{\beta} \mathrm{d}_q t \right) \\
&= g^{-1}\left((1-q)(\beta-\xi) \sum_{j=0}^{\infty} q^j g\Big(m\big[\alpha q^j + (1-q)^j \beta + d\big] \Big) \right) \\
&= g^{-1}\left((\beta-\alpha) \left(\frac{m(\alpha+\beta q)}{1+q} + d \right) \right) \\
&= \frac{(\beta-\alpha)\left(\frac{m(\alpha+\beta q)}{1+q} + d\right) - d}{m}.
\end{aligned}$$

Theorem 3.14. *Some properties of the pseudo-q-integral are mentioned below:*

(i) $\int_{[\alpha,\beta]}^{\oplus} \big(f_1(t) \oplus f_2(t)\big) \odot \mathrm{d_q}t = \int_{[\alpha,\beta]}^{\oplus} f_1(t) \odot \mathrm{d_q}t \oplus \int_{[\alpha,\beta]}^{\oplus} f_2(t) \odot \mathrm{d_q}t$;

(ii) $\int_{[\alpha,\beta]}^{\oplus} \big(c \odot f_1(t)\big) \odot \mathrm{d_q}t = c \odot \int_{[\alpha,\beta]}^{\oplus} f_1(t) \odot \mathrm{d_q}t$;

(iii) *If* $f_1 \leq_g f_2$, *then* $\int_{[\alpha,\beta]}^{\oplus} f_1(t) \odot \mathrm{d_q}t \leq_g \int_{[\alpha,\beta]}^{\oplus} f_2(t) \odot \mathrm{d_q}t$;

(v) *If* $0 < j < k$, *then* $\int_{[\alpha,\beta]}^{\oplus} f_1(t) \odot \mathrm{d_q}t = \int_{[\alpha,\gamma_j]}^{\oplus} f_1(t) \odot \mathrm{d_q}t \oplus \int_{[\gamma_j,\beta]}^{\oplus} f_1(t) \odot \mathrm{d_q}t$.

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Theorem 3.15. *The rule of pseudo-q-integration by parts is*

$$\int_{[0,\alpha]}^{\oplus} f_1(t) \odot D_q^{\oplus} f_2(t) \odot \mathrm{d}_q t = (f_1 \odot f_2)(\alpha) - \lim_{k\to\infty} (f \odot g)(\alpha q^k) - \int_{[0,\alpha]}^{\oplus} D_q^{\oplus} f_2(t) \odot f_1(qt) \odot \mathrm{d}_q t. \tag{3.8}$$

If f_1 and f_2 are g-q-regular at zero, then the limit on the right side of (3.8) can be replaced by $(f_1 \odot f_2)(0)$.

The following theorem is analogous to the fundamental theorem of usual calculus. Its proof is straightforward and, thus, is omitted.

Theorem 3.16. *Let f be a g-q-regular function at zero on a q-geometric set Ω containing zero. Define*

$$F(\omega) := \int_{[\gamma,\zeta]}^{\oplus} f(t) \odot \mathrm{d}_q t \qquad (\omega \in \Omega),$$

where γ is a fixed point in Ω. Then F is g-q-regular at zero. Furthermore, $D_q F(\omega)$ exists for each $\omega \in \Omega$ and we have

$$D_q^{\oplus} F(\omega) = f(\omega)$$

for all $\omega \in \Omega$.

Conversely, if α and β are two points in Ω, then

$$\int_{[\alpha,\beta]}^{\oplus} D_q^{\oplus} f(t) \odot \mathrm{d}_q t = f(\beta) \ominus f(\alpha).$$

Theorem 3.17. *Suppose that ϱ is continuous on $[\gamma, \delta]$. Then we have*

$$D_q^{\oplus}\Big(\int_{[\gamma,\delta]}^{\oplus} \varrho(t) \odot \mathrm{d}_q t\Big) = \varrho(t)$$

for all $t \in (\gamma, \delta)$.

Proof. We have

$$\begin{aligned} D_q^{\oplus}\Big(\int_{[\gamma,\delta]}^{\oplus} \varrho(t) \odot \mathrm{d}_q t\Big) &= D_q^{\oplus}\bigg(g^{-1}\Big(\int_{\gamma}^{\delta} g\big(\varrho(t)\big) \mathrm{d}_q t\Big)\bigg) = g^{-1}\Bigg(D_q g\bigg(g^{-1}\Big(\int_{\gamma}^{\delta} g\big(\varrho(t)\big) \mathrm{d}_q t\Big)\bigg)\Bigg) \\ &= g^{-1}\Big(D_q \int_{\gamma}^{\delta} g\big(\varrho(t)\big) \mathrm{d}_q t\Big) = \varrho(t), \end{aligned}$$

where we used the fundamental theorem of classical calculus. □

The next three theorems can be proved by utilizing Definition 3.11 and performing some elementary calculations.

Theorem 3.18. *For any $q \in (0, 1)$ and any arbitrary continuous function $\varrho : [\alpha, \beta] \to \mathbb{R}$, which is g-$q_\alpha$-differentiable, we have*

(i) $\int_{[\alpha,\xi]}^{\oplus} {}_{\alpha}D_q^{\oplus} \varrho(t) \odot {}_{\alpha}d_q t = \varrho(\xi) \ominus \varrho(\alpha)$;
(ii) $\int_{[\xi,\beta]}^{\oplus} {}^{\beta}D_q^{\oplus} \varrho(t) \odot {}^{\beta}d_q(t) = \varrho(\beta) \ominus \varrho(\xi)$.

Theorem 3.19. *Let $f_1, f_2 : [\alpha, \beta] \to \mathbb{R}$. Integration by parts says*

$$(3.9)\quad \int_{[\alpha,\beta]}^{\oplus} f_1\left(qt+(1-q)\alpha\right) \odot {}_{\alpha}D_q^{\oplus} f_2(t) \odot {}_{\alpha}d_q t = [f_1(t) \odot f_2(t)]_{\alpha}^{\beta} - \int_{[\alpha,\beta]}^{\oplus} f_2(t) \odot {}_{\alpha}D_q f_1(t) \odot {}_{\alpha}d_q t.$$

and

$$(3.10)\quad \int_{[\alpha,\beta]}^{\oplus} f_1\left(qt+(1-q)\beta\right) \odot {}^{\beta}D_q^{\oplus} f_2(t) \odot {}^{\beta}d_q t = [f_1(t) \odot f_2(t)]_{\alpha}^{\beta} - \int_{[\alpha,\beta]}^{\oplus} f_2(t) \odot {}^{\beta}D_q f_1(t) \odot {}^{\beta}d_q t.$$

Theorem 3.20. *Let $\varrho : [\alpha, \beta] \to \mathbb{R}$ be a continuous function. Then we have*

(i) ${}_{\alpha}D_q^{\oplus} \int_{[\alpha,\xi]}^{\oplus} \varrho(t) \odot {}_{\alpha}d_q t = \varrho(\xi).$*;*
(ii) $\int_{[\gamma,\xi]}^{\oplus} {}_{\alpha}D_q^{\oplus} \varrho(t) \odot {}_{\alpha}d_q t = \varrho(\xi) \ominus \varrho(\gamma)$ *for all* $\gamma \in (\alpha, \xi)$*;*
(iii) ${}^{\beta}D_q^{\oplus} \int_{[\xi,\beta]}^{\oplus} \varrho(t) \odot {}^{\beta}d_q t = \varrho(\xi)$*;*
(v) $\int_{[\gamma,\xi]}^{\oplus} {}^{\beta}D_q^{\oplus} \varrho(t) \odot {}^{\beta}d_q t = \varrho(\xi) \ominus \varrho(\gamma)$ *for all* $\gamma \in (\xi, \beta)$.

4. Chebyshev's Inequality by Pseudo-q-Integral

In 2002, Strboja proved Chebyshev's inequality using a pseudo integral [37]. In 2004, Chebyshev's inequality has been proven in q-calculus [17]. We aim to establish Chebyshev's inequality using the q-pseudo integral type.

Lemma 4.1. *Let f_1 and f_2 be both q-decreasing or both q-increasing ($f_1, f_2 : [0,1] \to [\alpha, \beta]$). If an additive generator $g : [\alpha, \beta] \to [0, \infty]$ is a decreasing function, then the compositions $g \circ f_1$ and $g \circ f_2$ are monotonic q-increasing or monotonic q-decreasing functions and nonegative.*

Theorem 4.2. *Let f_1 and f_2 be both q-decreasing or both q-increasing ($f_1, f_2 : [0,1] \to [\alpha, \beta]$). If an additive generator $g : [\alpha, \beta] \to [0, \infty)$ of the pseudo-addition $\oplus$ and pseudo-multiplication $\odot$ is a q-increasing function then:*

$$(4.1)\qquad \int_{[0,1]}^{\oplus} (f_1(t) \odot f_2(t)) \odot d_q t \geq \int_{[0,1]}^{\oplus} f_1(t) \odot d_q t \odot \int_{[0,1]}^{\oplus} f_2(t) \odot d_q t$$

Proof. According to Definition 3.10

$$\int_{[0,1]}^{\oplus} (f_1(t) \odot f_2(t)) \odot d_q t = g^{-1}\left((1-q)\sum_{j=0}^{k-1} \gamma_j g(f_1(\gamma_j) \odot g(f_2(\gamma_j)))\right).$$

Using the pseudo multiply property:

$$= g^{-1}\left((1-q)\sum_{j=0}^{k-1} \gamma_j g(g^{-1}(g f_1(\gamma_j)) g(f_2(\gamma_j)))\right) = g^{-1}\left((1-q)\sum_{j=0}^{k-1} \gamma_j g(f_1(\gamma_j) g(f_2(\gamma_j))\right).$$

According to Lemma 4.1

$$\geq g^{-1}\left((1-q)(\sum_{j=0}^{k-1} \gamma_j g(f_1(\gamma_j))(1-q)(\sum_{j=0}^{k-1} \gamma_j g(f_2(\gamma_j)))\right)$$

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or, equivalently,

$$= g^{-1}\left[g(g^{-1}((1-q)\sum_{j=0}^{k-1}\gamma_j g(f_1(\gamma_j))) \times g(g^{-1}((1-q)\sum_{j=0}^{k-1}\gamma_j g(f_2(\gamma_j)))\right].$$

Using the pseudo integral property:

$$g^{-1}\left((g\int_{[0,1]}^{\oplus} f_1(t)\odot d_qt) \times (g\int_{[0,1]}^{\oplus} f_2(t)\odot d_qt)\right) = g^{-1}\left((g\int_{[0,1]}^{\oplus} f_1(t)\odot d_qt) \odot (g\int_{[0,1]}^{\oplus} f_2(t)\odot d_qt)\right).$$

Using the pseudo multiply property:

$$= \int_{[0,1]}^{\oplus} f_1(t)\odot d_qt \times \int_{[0,1]}^{\oplus} f_2(t)\odot d_qt.$$

□

The next example is a small application for Theorem 4.2.

Example 4.3. *Let $f_1(x) = f_2(x) = x$ and $g(x) = x^2$ with $D_g = [0, +\infty)$. Then obviously we have $g^{-1}(x) = \sqrt{x}$. We first go on the left-hand side of (4.1). Note that*

$$f_1(t)\odot f_2(t) = g^{-1}\left(g\big(f_1(t)\big)g\big(f_2(t)\big)\right) = g^{-1}(t^4) = t^2.$$

So taking the interval $[0,1]$ as the integration range, we obtain

$$\int_{[0,1]}^{\oplus} \big(f_1(t)\odot f_2(t)\big)\odot d_qt = \int_{[0,1]}^{\oplus} t^2 d_qt = g^{-1}\left((1-q)\sum_{j=0}^{k-1}\gamma_j g(\gamma_j^2)\right). \tag{4.2}$$

Setting $\gamma_j = q^j$ we get

$$\sum_{j=0}^{k-1}\gamma_j g(\gamma_j^2) = \sum_{j=0}^{k-1} q^j g(q^{2j}) = \sum_{j=0}^{k-1} q^j q^{4j} = \sum_{j=0}^{k-1} q^{5j} = \frac{1-q^{5k}}{1-q^5}$$

and thus,

$$(1-q)\sum_{j=0}^{k-1}\gamma_j g(\gamma_j^2) = (1-q)\times\frac{1-q^{5k}}{1-q^5} = \frac{1-q-q^{5k}+q^{5k+1}}{1-q^5}. \tag{4.3}$$

Therefore, by replacing (4.3) in (4.2), the left-hand-side of (4.1) is

$$\int_{[0,1]}^{\oplus} \big(f_1(t)\odot f_2(t)\big)\odot d_qt = g^{-1}\left((1-q)\sum_{j=0}^{k-1}\gamma_j g(\gamma_j^2)\right) = \sqrt{\frac{1-q-q^{5k}+q^{5k+1}}{1-q^5}}.$$

On the other hand, for the the right-hand-side of (4.1) note that

$$\int_{[0,1]}^{\oplus} f_1(t)\odot d_qt = \int_{[0,1]}^{\oplus} f_2(t)\odot d_qt = g^{-1}\left((1-q)\sum_{j=0}^{k-1}\gamma_j g(\gamma_j)\right). \tag{4.4}$$

Setting $\gamma_j = q^j$ again we get

$$\sum_{j=0}^{k-1}\gamma_j g(\gamma_j) = \sum_{j=0}^{k-1} q^j (q^j)^2 = \sum_{j=0}^{k-1} q^j q^{2j} = \sum_{j=0}^{k-1} q^{3j} = \frac{1-q^{3k}}{1-q^3}$$

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and thus,

$$(1-q)\sum_{j=0}^{k-1}\gamma_j g(\gamma_j) = (1-q)\times\frac{1-q^{3k}}{1-q^3} = \frac{1-q-q^{3k}+q^{3k+1}}{1-q^3}. \tag{4.5}$$

Therefore, by replacing (4.5) in (4.4), the right-hand-side of (4.1) is

$$\begin{aligned}\int_{[0,1]}^{\oplus} f_1(t)\odot d_qt \odot \int_{[0,1]}^{\oplus} f_2(t)\odot d_qt &= g^{-1}\left((1-q)\sum_{j=0}^{k-1}\gamma_j g(\gamma_j)\right)\odot g^{-1}\left((1-q)\sum_{j=0}^{k-1}\gamma_j g(\gamma_j)\right)\\ &= \sqrt{\frac{1-q-q^{3k}+q^{3k+1}}{1-q^3}}\odot\sqrt{\frac{1-q-q^{3k}+q^{3k+1}}{1-q^3}}\\ &= \frac{1-q-q^{3k}+q^{3k+1}}{1-q^3}.\end{aligned}$$

Consequently, from Theorem 4.2 we find

$$\sqrt{\frac{1-q-q^{5k}+q^{5k+1}}{1-q^5}} \geq \frac{1-q-q^{3k}+q^{3k+1}}{1-q^3},$$

or equivalently,

$$\frac{1-q-q^{5k}+q^{5k+1}}{1-q^5} \geq \left(\frac{1-q-q^{3k}+q^{3k+1}}{1-q^3}\right)^2. \tag{4.6}$$

Finally, it is worth mentioning that since $q\in(0,1)$, it follows for sufficiently large k that

$$-q^{5k}+q^{5k+1}\simeq -q^{3k}+q^{3k+1}\simeq 0.$$

Hence (4.6) is reduced to

$$\frac{1}{1-q^5}\geq\frac{1-q}{(1-q^3)^2}.$$

Remark 4.4. *In the general case, with normal addition and multiplication, Theorem 4.2 becomes the same theorem in [17].*

Remark 4.5. *Inequality (4.1) is valid when ${}_\alpha d_qt$ and ${}^\beta d_qt$ are substituted for d_qt. In this case, the proof is similar to the proof of Theorem 4.2.*

5. Hermite-Hadamard Inequality by Pseudo-q-Integral

In this section, we shall consider the special case of the integral $\int_{[\alpha,\beta]}^{g}\varrho(t)\otimes dt$, i.e., $\int_{[\alpha,\beta]}^{\oplus}\varrho(t)\otimes dt$ given by Definition 2.2. The aim of this part is a g-analogue of this inequality.

Definition 5.1 ([21])**.** *A function $\varrho:[\alpha,\beta]\to\mathbb{R}_g$ is said to be pseudo-convex on $[\alpha,\beta]$, if for all $t,s\in[\alpha,\beta]$ and all $0\leq\eta\leq1$*

$$\varrho\big(\eta t+(1-\eta)s\big)\leq_g \eta\odot\varrho(t)\oplus(1-\eta)\odot\varrho(s).$$

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Theorem 5.2 ([21]). *Let $\varrho : [\alpha, \beta] \to \mathbb{R}_g$ be a pseudo-convex function. Then the following Hermite-Hadamard inequality holds:*

$$\varrho\left(\frac{\alpha+\beta}{2}\right) \leq_g \left(\frac{1}{\beta-\alpha}\right) \odot \int_{[\alpha,\beta]}^{\oplus} \varrho(t) \odot \mathrm{d}t \leq_g \frac{1}{2} \odot \big(\varrho(\alpha) \oplus \varrho(\beta)\big).$$

Theorem 5.3. [**q-Hermite-Hadamard Inequality**] [23] *Let f be q-convex on $[\alpha, \beta]$. Then*

$$\int_{\alpha}^{\beta} \varrho(t)\mathrm{d}qt \leq \frac{1}{1+q}\big[(\beta q-\alpha)\varrho(\alpha) + (\beta - q\alpha)\varrho(\beta)\big], \tag{5.1}$$

$$\int_{\alpha}^{\beta} \varrho(qt)\mathrm{d}qt \leq \frac{1}{1+q}\big[(\beta - q\alpha)\varrho(\alpha) + (q\beta-\alpha)\varrho(\beta)\big], \tag{5.2}$$

$$\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta} \frac{\varrho(t)+\varrho(qt)}{2}\mathrm{d}qt \leq \frac{\varrho(\alpha)+\varrho(\beta)}{2}. \tag{5.3}$$

We want to prove the q-Hermite-Hadamard inequality for pseudo integral using q-calculus by Theorems 5.2 and 5.3.

Theorem 5.4. *Let $\varrho : [\alpha, \beta] \to \mathbb{R}_g$ be a pseudo-convex and a q-convex function. Then:*

(1)

$$g^{-1}\left(\frac{1}{\beta-\alpha}\right)\int_{[\alpha,\beta]}^{\oplus} \frac{1}{2} \odot (\varrho(t) \oplus \varrho(qt)) \odot \mathrm{d}_q t \leq \frac{1}{2} \odot (\varrho(\alpha) \oplus \varrho(\beta)). \tag{5.4}$$

(2)

$$\varrho\left(\frac{\alpha+\beta}{2}\right) \leq g^{-1}\left(\frac{1}{\beta-\alpha}\right)\int_{[\alpha,\beta]}^{\oplus} \varrho(t)\mathrm{d}_q t. \tag{5.5}$$

Proof. (1) According to the definitions of $\oplus$ and $\odot$ we have

$$\frac{1}{2} \odot \big(\varrho(\alpha) \oplus \varrho(\beta)\big) = g^{-1}\Big(\frac{1}{2}g\big(\varrho(\alpha) \oplus \varrho(\beta)\big)\Big) = g^{-1}\left(\frac{1}{2}g\Big(g^{-1}\Big(g\big(\varrho(\alpha)\big) + g\big(\varrho(\beta)\big)\Big)\Big)\right).$$

According to (5.3), we have

$$g^{-1}\left(\frac{1}{2}g\Big(g^{-1}\Big(\frac{2}{\beta-\alpha}\int_{\alpha}^{\beta}\frac{g(\varrho(t))+g(\varrho(qt))}{2}\mathrm{d}_q t\Big)\Big)\right)$$
$$= g^{-1}\left(\frac{1}{\beta-\alpha}g\Big(g^{-1}\Big(\int_{\alpha}^{\beta}\frac{g(\varrho(t))+g(\varrho(qt))}{2}\mathrm{d}_q t\Big)\Big)\right).$$

According to the Definition 2.6,

$$= g^{-1}\Big(\frac{1}{\beta-\alpha}g\Big(\int_{[\alpha,\beta]}^{\oplus}\frac{1}{2} \odot (\varrho(t) \oplus \varrho(qt)) \odot \mathrm{d}_q t\Big)\Big).$$

According to the Definition 2.3,

$$= g^{-1}\left(\frac{1}{\beta-\alpha}\right)\int_{[\alpha,\beta]}^{\oplus}\frac{1}{2} \odot (\varrho(t) \oplus \varrho(qt)) \odot \mathrm{d}_q t.$$

Thenwe have:

$$g^{-1}\Big(\frac{1}{\beta-\alpha}\Big)\int_{[\alpha,\beta]}^{\oplus}\frac{1}{2} \odot (\varrho(t) \oplus \varrho(qt)) \odot \mathrm{d}_q t \leq \frac{1}{2} \odot (\varrho(\alpha) \oplus \varrho(\beta)). \tag{5.6}$$

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(2) Since ϱ is a convex function on $[\alpha,\beta]$, then we have the following inequality $(0\leq\eta\leq 1)$

$$\varrho\left(\frac{\alpha+\beta}{2}\right)=\varrho\left[\frac{(1-\eta)\alpha+\eta\beta+\eta\alpha+(1-\eta)\beta}{2}\right]\leq\frac{1}{2}\Big\{\varrho\big[(1-\eta)\alpha+\eta\beta\big]+\varrho\big[\eta\alpha+(1-\eta)\beta\big]\Big\}$$

Taking q-integration over $\eta\in[0,1]$, we have

$$\int_0^1\varrho\left(\frac{\alpha+\beta}{2}\right)\mathrm{d}_q\eta\leq\frac{1}{2}\Bigg\{\int_0^1\varrho\big[(1-\eta)\alpha+\eta\beta\big]\,\mathrm{d}_q\eta+\int_0^1\varrho\big[\eta\alpha+(1-\eta)\beta\big]\,\mathrm{d}_q\eta\Bigg\}$$

Since g is increasing , then g^{-1} is also increasing function. So we can write

$$\begin{aligned}g^{-1}\left\{\int_0^1 g\left(\varrho\left(\frac{\alpha+\beta}{2}\right)\right)\mathrm{d}_q\eta\right\}&=g^{-1}\left\{g\left(\varrho\left(\frac{\alpha+\beta}{2}\right)\right)\int_0^1\mathrm{d}_q\eta\right\}\\&\leq g^{-1}\Bigg\{\frac{1}{2}\left\{\int_0^1 g\big[\varrho\big((1-\eta)\alpha+\eta\beta\big)\big]\,\mathrm{d}_q\eta\right.\\&\left.+\int_0^1 g[\varrho(\eta\alpha+(1-\eta)\beta]\mathrm{d}_q\eta\right\}.\end{aligned}\tag{5.7}$$

Since $\int_0^1\mathrm{d}_q\eta=1$, the Eqn. (5.7) can be rewritten as follows:

$$\begin{aligned}&g^{-1}\Bigg\{g\left(\varrho\left(\frac{\alpha+\beta}{2}\right)\right)\Bigg\}\leq g^{-1}\Bigg\{g\left(g^{-1}\left(\frac{1}{2}\right)\right)g\left(g^{-1}\left(\int_0^1 g\big[f\big((1-\eta)\alpha+\eta\beta\big)\big]\,\mathrm{d}_q\eta\right.\right.\\&\left.\left.+\int_0^1 g\big[\varrho\big(\eta\alpha+(1-\eta)\beta\big)\big]\,\mathrm{d}_q\eta\right)\right)\Bigg\}g^{-1}\Bigg\{g\left(\varrho\left(\frac{\alpha+\beta}{2}\right)\right)\Bigg\}\\&\leq g^{-1}\left(\frac{1}{2}\right)\odot g^{-1}\left(\int_0^1 g\big[\varrho\big((1-\eta)\alpha+\eta\beta\big)\big]\,\mathrm{d}_q\eta\right.\\&\left.+\int_0^1 g\big[\varrho\big(\eta\alpha+(1-\eta)\beta\big)\big]\,\mathrm{d}_q\eta\right).\end{aligned}$$

Hence

$$\begin{aligned}\varrho\left(\frac{\alpha+\beta}{2}\right)&\leq g^{-1}\left(\frac{1}{2}\right)\odot g^{-1}\Bigg\{g\left(g^{-1}\left(\int_0^1 g\big[\varrho\big((1-\eta)\alpha+\eta\beta\big)\big]\,\mathrm{d}_q\eta\right)\right)\\&+g\left(g^{-1}\left(\int_0^1 g\big[\varrho\big(\eta\alpha+(1-\eta)\beta\big)\big]\,\mathrm{d}_q\eta\right)\right)\Bigg\}\\&=g^{-1}\left(\frac{1}{2}\right)\odot\Bigg\{g^{-1}\left(\int_0^1 g\big[\varrho\big((1-\eta)\alpha+\eta\beta\big)\big]\,\mathrm{d}_q\eta\right)\\&\oplus g^{-1}\left(\int_0^1 g\big[\varrho\big(\eta\alpha+(1-\eta)\beta\big)\big]\,\mathrm{d}_q\eta\right)\Bigg\}\end{aligned}$$

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Making the changes of variable $\mu = [(1-\eta)\alpha + \eta\beta]$ and $\nu = [\eta\alpha + (1-\eta)\beta]$, we have

$$\begin{aligned}\varrho\left(\frac{\alpha+\beta}{2}\right) &\leq g^{-1}\left(\frac{1}{2}\right)\odot\left\{g^{-1}\left(\int_\alpha^\beta g\big(\varrho(\mu)\big)\frac{1}{\beta-\alpha}\,\mathrm{d}_q\mu\right)\oplus g^{-1}\left(\int_\beta^\alpha g\big(\varrho(\nu)\big)\frac{1}{\alpha-\beta}\,\mathrm{d}_q\nu\right)\right\}\\ &\leq g^{-1}\left(\frac{1}{2}\right)\odot\left\{g^{-1}\left(\int_\alpha^\beta g\big(\varrho(\mu)\big)\frac{1}{\beta-\alpha}\,\mathrm{d}_q\mu\right)\oplus g^{-1}\left(\int_\alpha^\beta g\big(\varrho(\nu)\big)\frac{1}{\beta-\alpha}\,\mathrm{d}_q\nu\right)\right\}\end{aligned}$$

Admitting $x = \mu = \nu$, we can write

$$\begin{aligned}\varrho\left(\frac{\alpha+\beta}{2}\right) &\leq g^{-1}\left(\frac{1}{2}\right)\odot\left\{g^{-1}\left(\frac{1}{\beta-\alpha}\int_\alpha^\beta g\big(\varrho(t)\big)\,\mathrm{d}_qt\right)\oplus g^{-1}\left(\frac{1}{\beta-\alpha}\int_\alpha^\beta g\big(\varrho(t)\big)\,\mathrm{d}_qt\right)\right\}\\ &= g^{-1}\left(\frac{1}{2}\right)\odot\left\{g^{-1}\left\{g\left(g^{-1}\left(\frac{1}{\beta-\alpha}\int_\alpha^\beta g\big(\varrho(t)\big)\,\mathrm{d}_qt\right)\right)\right.\right.\\ &\left.\left.+\, g\left(g^{-1}\left(\frac{1}{\beta-\alpha}\int_\alpha^\beta g\big(\varrho(t)\big)\,\mathrm{d}_qt\right)\right)\right\}\right\}\\ &= g^{-1}\left(\frac{1}{2}\right)\odot\left\{g^{-1}\left\{\frac{2}{\beta-\alpha}\int_\alpha^\beta g\big(\varrho(t)\big)\,\mathrm{d}_qt\right\}\right\}\\ &= g^{-1}\left(\frac{1}{2}\right)\odot\left\{g^{-1}\left\{g\left(g^{-1}\left(\frac{2}{\beta-\alpha}\right)\right)g\left(g^{-1}\left(\int_\alpha^\beta g\big(\varrho(t)\big)\,\mathrm{d}_qt\right)\right)\right\}\right\}\\ &= \left\{g^{-1}\left(\frac{1}{2}\right)\odot g^{-1}\left(\frac{2}{\beta-\alpha}\right)\right\}\odot\left\{g^{-1}\left(\int_\alpha^\beta g\big(\varrho(t)\big)\,\mathrm{d}_qt\right)\right\}\\ &= \left\{g^{-1}\left\{g\left(g^{-1}\left(\frac{1}{2}\right)\right)g\left(g^{-1}\left(\frac{2}{\beta-\alpha}\right)\right)\right\}\right\}\odot\left\{g^{-1}\left(\int_\alpha^\beta g\big(\varrho(t)\big)\,\mathrm{d}_qt\right)\right\}\\ &= \left\{g^{-1}\left(\frac{1}{2}\,\frac{2}{\beta-\alpha}\right)\right\}\odot\left\{g^{-1}\left(\int_\alpha^\beta g\big(\varrho(t)\big)\,\mathrm{d}_qt\right)\right\}.\end{aligned}$$

Finally, we obtain

$$\varrho\left(\frac{\alpha+\beta}{2}\right)\leq g^{-1}\left(\frac{1}{\beta-\alpha}\right)\odot\int_{[\alpha,\beta]}^{\oplus}\varrho(t)\,\mathrm{d}_qt. \tag{5.8}$$

Then we have:

$$g^{-1}\left(\frac{1}{\beta-\alpha}\right)\odot\int_{[\alpha,\beta]}^{\oplus}\varrho(t)\,\mathrm{d}_qt\leq\frac{1}{2}\odot\big[\varrho(\alpha)\oplus\varrho(\beta)\big]. \tag{5.9}$$

□

The next example is a small application for Theorem 5.4.

Example 5.5. *Let $\alpha = 0$, $\beta = 1$, $\varrho(x) = x$ and $g(x) = x^2$ with $D_g = [0, +\infty)$. Then obviously we have $g(x) = \sqrt{x}$. We first go on the left-hand side of (5.4). Note that*

$$\varrho(t)\oplus\varrho(qt) = g^{-1}\left(g\big(\varrho(t)\big)+g\big(\varrho(qt)\big)\right) = g^{-1}(t^2+q^2t^2) = \sqrt{t^2+q^2t^2} = t\sqrt{1+q^2}.$$

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So taking the interval $[\alpha,\beta]=[0,1]$ *as the integration range, we obtain*

$$g^{-1}\left(\frac{1}{\beta-\alpha}\right)\int_{[\alpha,\beta]}^{\oplus}\frac{1}{2}\odot\big(\varrho(t)\oplus\varrho(qt)\big)\mathrm{d}_qt=\int_{[0,1]}^{\oplus}\frac{1}{2}\odot(t\sqrt{1+q^2})\mathrm{d}_qt$$
$$=\int_{[0,1]}^{\oplus}g^{-1}\left(\frac{1}{2}(t\sqrt{1+q^2})^2\right)\mathrm{d}_qt$$
$$=\int_{[0,1]}^{\oplus}g^{-1}\left(\frac{t^2(1+q^2)}{2}\right)\mathrm{d}_qt$$
$$=\int_{[0,1]}^{\oplus}t\sqrt{\frac{1+q^2}{2}}\mathrm{d}_qt$$
$$=g^{-1}\left((1-q)\sum_{j=0}^{k-1}\gamma_j g\left(\frac{\gamma_j\sqrt{1+q^2}}{2}\right)\right). \tag{5.10}$$

Setting $\gamma_j=q^j$ *we get*

$$\sum_{j=0}^{k-1}\gamma_j g\left(\frac{\gamma_j\sqrt{1+q^2}}{2}\right)=\sum_{j=0}^{k-1}q^j\times\frac{q^{2j}(1+q^2)}{4}=\frac{1+q^2}{4}\sum_{j=0}^{k-1}q^{3j}=\frac{(1+q^2)(1-q^{3k})}{4(1-q^3)}$$

and thus,

$$(1-q)\sum_{j=0}^{k-1}\gamma_j g\left(\frac{\gamma_j\sqrt{1+q^2}}{2}\right)=(1-q)\times\frac{1+q^2}{4}\times\frac{1-q^{3k}}{1-q^3}=\frac{(1+q^2)(1-q^{3k})}{4(1+q+q^2)}. \tag{5.11}$$

Therefore, by replacing (5.11) *in* (5.10)*, the left-hand side of* (5.4) *is*

$$g^{-1}\left(\frac{1}{\beta-\alpha}\right)\int_{[\alpha,\beta]}^{\oplus}\frac{1}{2}\odot\big(\varrho(t)\oplus\varrho(qt)\big)\mathrm{d}_qt=g^{-1}\left((1-q)\sum_{j=0}^{k-1}\gamma_j g\left(\frac{\gamma_j\sqrt{1+q^2}}{2}\right)\right)$$
$$=\frac{1}{2}\sqrt{\frac{(1+q^2)(1-q^{3k})}{1+q+q^2}}.$$

On the other hand, the right-hand side of (5.4) *is*

$$\frac{1}{2}\odot\big(\varrho(\alpha)\oplus\varrho(\beta)\big)=\frac{1}{2}\odot\big(\varrho(0)\oplus\varrho(1)\big)=\frac{1}{2}\odot(0\oplus 1)=\frac{1}{2}\odot g^{-1}\big(g(0)+g(1)\big)$$
$$=\frac{1}{2}\odot g^{-1}(1)=\frac{1}{2}\odot 1=g^{-1}\left(g\left(\frac{1}{2}\right)g(1)\right)=g^{-1}\left(\frac{1}{4}\right)=\frac{1}{2}.$$

Consequently, from Theorem 5.4(1) *we find*

$$\frac{1}{2}\sqrt{\frac{(1+q^2)(1-q^{3k})}{1+q+q^2}}\leq\frac{1}{2},$$

or equivalently,

$$(1+q^2)(1-q^{3k})\leq 1+q+q^2. \tag{5.12}$$

Finally, it is worth mentioning that since $q\in(0,1)$*, it follows for sufficiently large* k *that* $q^{3k}\simeq 0$*. Hence* (5.12) *is reduced to*

$$1+q^2\leq 1+q+q^2.$$

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Now, for (5.5), note that

$$g^{-1}\left(\frac{1}{\beta-\alpha}\right)\int_{[\alpha,\beta]}^{\oplus}\varrho(t)\mathrm{d}_q t=\int_{[0,1]}^{\oplus}t\mathrm{d}_q t=g^{-1}\left((1-q)\sum_{j=0}^{k-1}\gamma_j g(\gamma_j)\right). \tag{5.13}$$

Setting $\gamma_j=q^j$ *we get*

$$\sum_{j=0}^{k-1}\gamma_j g(\gamma_j)=\sum_{j=0}^{k-1}q^j g(q^j)=\sum_{j=0}^{k-1}q^j q^{2j}=\sum_{j=0}^{k-1}q^{3j}=\frac{1-q^{3k}}{1-q^3}$$

and thus,

$$(1-q)\sum_{j=0}^{k-1}\gamma_j g(\gamma_j)=(1-q)\times\frac{1-q^{3k}}{1-q^3}=\frac{1-q^{3k}}{1+q+q^2}. \tag{5.14}$$

Therefore, by replacing (5.14) in (5.13), the right-hand side of (5.5) is

$$g^{-1}\left(\frac{1}{\beta-\alpha}\right)\int_{[\alpha,\beta]}^{\oplus}\varrho(t)\mathrm{d}_q t=g^{-1}\left(\frac{1-q^{3k}}{1+q+q^2}\right)=\sqrt{\frac{1-q^{3k}}{1+q+q^2}}.$$

Consequently, from Theorem 5.4(2) we find

$$\frac{1}{2}\leq\sqrt{\frac{1-q^{3k}}{1+q+q^2}},$$

or equivalently,

$$1+q+q^2\leq 4(1-q^{3k}). \tag{5.15}$$

Finally, it is worth mentioning that since $q\in(0,1)$*, it follows for sufficiently large* k *that* $q^{3k}\simeq 0$. *Hence (5.15) reduces to*

$$q+q^2\leq 3.$$

Remark 5.6. *In the general case, using standard addition and multiplication, Theorem 5.4 becomes the same theoremas presented in [17].*

6. Hermite-Hadamard Inequality by Pseudo-q_a, ^bq-Integral

Definition 6.1. *For a function* $f_2:[0,1]\to\mathbb{R}^{\oplus}$, *a function* $f_1:[\alpha,\beta]\to\mathbb{R}_g$ *is said to be Pseudo-*f_2*-convex on* $[\alpha,\beta]$, *if for all* t *and* $s\in[\alpha,\beta]$ *and all* $0\leq\eta\leq 1$ *:*

$$f_1(\eta t+(1-\eta)s)\leq_g f_2(\eta)\odot f_1(t)\oplus(1-f_2(\eta))\odot f_1(s). \tag{6.1}$$

If $f_2(t)=t$, *then* f_1 *is a pseudo-convex function.*

Theorem 6.2. *Let* $f_2:[0,1]\to\mathbb{R}^{\oplus}$ *be a function with* $f_2(\frac{1}{2})\neq 0$, *and let* $f_1:[\alpha,\beta]\to\mathbb{R}_g$ *be a pseudo-*f_2*-convex function and* $q\in(0,1)$. *Then we have*

$$\begin{aligned}\frac{1}{f_2\left(\frac{1}{2}\right)}f_1\left(\frac{\alpha+\beta}{2}\right)&\leq g^{-1}\left(\frac{1}{\beta-\alpha}\right)\Big[\int_{[\alpha,\beta]}^{\oplus}f_1(t)\odot\ {}_{\alpha}\mathrm{d}_q t\oplus\int_{[\alpha,\beta]}^{\oplus}f_1(t)\odot\ {}^{\beta}\mathrm{d}_q t\Big]\\&\leq\Big[\int_{[0,1]}^{\oplus}f_2(t)\odot\mathrm{d}_q t\oplus\int_{[0,1]}^{\oplus}f_2(1-t)\odot\mathrm{d}_q t\Big]\odot\big[f_1(\alpha)\oplus f_1(\beta)\big].\end{aligned} \tag{6.2}$$

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Proof. From the f_2-convexity of f_1, we have

$$\frac{1}{f_2\left(\frac{1}{2}\right)} f_1\left(\frac{t+s}{2}\right) \leq f_1(t) + f_1(s).$$

By taking q_α-integration over $[0,t]$ for $t \in [0,1]$ we obtain

$$\int_0^t \frac{1}{f_2\left(\frac{1}{2}\right)} f_1\left(\frac{t+s}{2}\right)\ {}_\alpha \mathrm{d_q} t \leq \int_0^t \left(f_1(t) + f_1(s)\right)\ {}_\alpha \mathrm{d}_q t$$

and putting $t = \eta\beta + (1-\eta)\alpha$ and $s = \eta\alpha + (1-\eta)\beta$ we find

$$\int_0^t \frac{1}{f_2\left(\frac{1}{2}\right)} f_1\left(\frac{\alpha+\beta}{2}\right)\ {}_\alpha \mathrm{d}_q t \leq \int_0^t \Big(f\big(\eta\beta + (1-\eta)\alpha\big) + f_1\big(\eta\alpha + (1-\eta)\beta\big)\Big)\ {}_\alpha \mathrm{d}_q t.$$

Since g is increasing, it follows that g^{-1} is also increasing. Thus, we can write:

$$\begin{aligned}
g^{-1}\Bigg[\int_0^t g\Big(\frac{1}{f_2\left(\frac{1}{2}\right)} f_1\left(\frac{\alpha+\beta}{2}\right)\Big)\ {}_\alpha \mathrm{d}_q t\Bigg] &\leq g^{-1}\Bigg[\int_0^t g\Big(f_1\big(\eta\beta + (1-\eta)\alpha\big)\ {}_\alpha \mathrm{d}_q t\Big) \\
&+ \int_0^t g\Big(f_1\big(\eta\alpha + (1-\eta)\beta\big)\ {}_\alpha \mathrm{d}_q t\Big)\Bigg] \\
&\leq g^{-1}\Bigg[g\Bigg[g^{-1}\Bigg(\int_0^t g\Big(f_1\big(\eta\beta + (1-\eta)\alpha\big)\Big)\ {}_\alpha \mathrm{d}_q t \\
&+ \int_0^t g\Big(f_1\big(\eta\alpha + (1-\eta)\beta\big)\Big)\ {}_\alpha \mathrm{d}_q t\Big)\Bigg)\Bigg]\Bigg] \\
&\leq g^{-1}\Bigg(\int_0^t g\Big(f_1\big(\eta\beta + (1-\eta)\alpha\big)\Big)\ {}_\alpha \mathrm{d}_q t\Bigg) \\
&\oplus g^{-1}\Bigg(\int_0^t g\Big(f_1\big(\eta\alpha + (1-\eta)\beta\big)\Big)\ {}_\alpha \mathrm{d}_q t\Bigg). \qquad (6.3)
\end{aligned}$$

Now, by taking q^β-integration over $[t,1]$ for $t \in [0,1]$ we obtain

$$\int_t^1 \frac{1}{f_2\left(\frac{1}{2}\right)} f_1\left(\frac{t+s}{2}\right)\ {}^\beta \mathrm{d}_q t \leq \int_t^1 \left(f_1(t) + f_1(s)\right)\ {}^\beta \mathrm{d}_q t$$

and putting $t = \eta\beta + (1-\eta)\alpha$ and $s = \eta\alpha + (1-\eta)\beta$ we find

$$\int_t^1 \frac{1}{f_2\left(\frac{1}{2}\right)} f_1\left(\frac{\alpha+\beta}{2}\right)\ {}^\beta \mathrm{d}_q t \leq \int_t^1 \Big(f_1\big(\eta\beta + (1-\eta)\alpha\big) + f_1\big(\eta\alpha + (1-\eta)\beta\big)\Big)\ {}^\beta \mathrm{d}_q t.$$

Since g and g^{-1} are both increasing functions, we can similarly write

$$\begin{aligned}
g^{-1}\Bigg[\int_t^1 g\Big(\frac{1}{f_2\left(\frac{1}{2}\right)} f_1\left(\frac{\alpha+\beta}{2}\right)\Big)\ {}^\beta \mathrm{d}_q t\Bigg] &\leq g^{-1}\Bigg(\int_t^1 g\Big(f_1\big(\eta\beta + (1-\eta)\alpha\big)\Big)\ {}^\beta \mathrm{d}_q t\Bigg) \\
&\oplus g^{-1}\Bigg(\int_t^1 g\Big(f_1\big(\eta\alpha + (1-\eta)\beta\big)\Big)\ {}^\beta \mathrm{d}_q t\Bigg). \qquad (6.4)
\end{aligned}$$

By summing the inequalities (6.3) and (6.4) and setting $\mu = (1-t)\alpha + t\beta$ and $\nu = t\alpha + (1-t)\beta$, we get

$$\frac{1}{f_2\left(\frac{1}{2}\right)} f_1\left(\frac{\alpha+\beta}{2}\right) \leq g^{-1}\left(\frac{1}{\beta-\alpha}\right)\Big[\int_\alpha^\beta f_1(t) \odot\ {}_\alpha \mathrm{d}_q t \oplus \int_\alpha^\beta f_1(t) \odot\ {}^\beta \mathrm{d}_q t\Big],$$

which proves the first inequality.

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To prove the second inequality, note that from the definition of pseudo-q_a-integral we have

$$\begin{aligned}\frac{1}{\beta-\alpha}\int_{[\alpha,\beta]}^{\oplus} f_1(t)\odot\ {}_{\alpha}\mathrm{d}_q t &= g^{-1}\Big(\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta} g\big(f_1(t)\big)\ {}_{\alpha}\mathrm{d}_q t\Big)\\ &= g^{-1}\Big((1-q)\sum_{j=0}^{\infty} q^j g\Big(f_1\big(\alpha+q^j(\beta-\alpha)\big)\Big)\Big)\\ &= \int_{[0,1]}^{\oplus} f_1\big(\alpha+\eta(\beta-\alpha)\big)\odot \mathrm{d}_q\eta.\end{aligned}$$

By using the pseudo-f_2-convexity of f_1, we may estimate the last expression as

$$\int_{[0,1]}^{\oplus} f_1\big(\alpha+\eta(\beta-\alpha)\big)\odot \mathrm{d}_q\eta \le f_1(\beta)\odot\int_{[0,1]}^{\oplus} f_2(t)\odot \mathrm{d}_q\eta \oplus f_1(\alpha)\odot\int_{[0,1]}^{\oplus} f_2(1-\eta)\odot \mathrm{d}_q\eta$$

and similarly,

$$\begin{aligned}\frac{1}{\beta-\alpha}\int_{[\alpha,\beta]}^{\oplus} f_1(t)\odot\ {}^{\beta}\mathrm{d}_q t &= g^{-1}\Big(\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta} g\big(f_1(t)\big)\ {}^{b}\mathrm{d}_q t\Big)\\ &= g^{-1}\Big((1-q)\sum_{j=0}^{\infty} q^j g\Big(f_1\big(\beta-q^j(\beta-\alpha)\big)\Big)\Big)\\ &= \int_{[0,1]}^{\oplus} f_1\big(\beta-\eta(\beta-\alpha)\big)\odot \mathrm{d}_q\eta.\end{aligned}$$

Using the pseudo-f_2-convexity of f_1 once more yields

$$\int_{[0,1]}^{\oplus} f_1\big(\beta-\eta(\beta-\alpha)\big)\odot \mathrm{d}_q\eta \le f_1(\alpha)\odot\int_{[0,1]}^{\oplus} f_2(\eta)\odot \mathrm{d}_q\eta \oplus f_1(\beta)\odot\int_{[0,1]}^{\oplus} f_2(1-\eta)\odot \mathrm{d}_q\eta.$$

Finally, a summation completes the proof. □

The following example is a small application for Theorem 6.2.

Example 6.3. *Let $\alpha=0$, $\beta=1$, $f_1(x)=x$, $f_2(x)=1$ and $g(x)=x^2$ with $D_g=[0,+\infty)$. Note that $f_2(\frac{1}{2})=1\neq 0$ and f_1 is a pseudo-f_2-convex function. Then obviously we have $g(x)=\sqrt{x}$. The left hand-side of (6.2) is very simple:*

$$\frac{1}{f_2\left(\frac{1}{2}\right)} f_1\left(\frac{\alpha+\beta}{2}\right) = 1\times\frac{1}{2}=\frac{1}{2}.$$

We will now discuss the middle term of (6.2). Note that

$$\begin{aligned}\int_{[\alpha,\beta]}^{\oplus} f_1(t)\odot\ {}_{\alpha}\mathrm{d}_q t &= \int_{[0,1]}^{\oplus} t\odot\ {}_{\alpha}\mathrm{d}_q t\\ &= g^{-1}\left((1-q)(1-0)\sum_{j=0}^{\infty}\gamma_j g\big(\gamma_j\times 0+(1-\gamma_j)\times 1\big)\right)\\ &= g^{-1}\left((1-q)\sum_{j=0}^{\infty}\gamma_j g(1-\gamma_j)\right).\end{aligned} \tag{6.5}$$

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Setting $\gamma_j = q^j$ *we get*

$$\sum_{j=0}^{\infty} \gamma_j g(1-\gamma_j) = \sum_{j=0}^{\infty} q_j (1-q_j)^2 = \sum_{j=0}^{\infty} (q_j - 2q_j^2 + q_j^3)$$
$$= \frac{1}{1-q} - \frac{2}{1-q^2} + \frac{1}{1-q^3} = \frac{q+q^3}{(1-q)(1+q)(1+q+q^2)}$$

and thus,

$$(1-q)\sum_{j=0}^{\infty} \gamma_j g(1-\gamma_j) = (1-q) \times \frac{q+q^3}{(1-q)(1+q)(1+q+q^2)} = \frac{q+q^3}{(1+q)(1+q+q^2)}. \tag{6.6}$$

Therefore, by replacing (6.6) *in* (6.5) *we obtain*

$$\int_{[\alpha,\beta]}^{\oplus} f_1(t) \odot \ {}_{\alpha}\mathrm{d}_q t = g^{-1}\left((1-q)\sum_{j=0}^{\infty} \gamma_j g(1-\gamma_j)\right) = \sqrt{\frac{q+q^3}{(1+q)(1+q+q^2)}}.$$

Similarly,

$$\int_{[\alpha,\beta]}^{\oplus} f_1(t) \odot \ {}^{\beta}\mathrm{d}_q t = \int_{[0,1]}^{\oplus} t \odot \ {}^{\beta}\mathrm{d}_q t$$
$$= g^{-1}\left((1-q)(1-0)\sum_{j=0}^{\infty} \gamma_j g\big(\gamma_j \times 1 + (1-\gamma_j) \times 0\big)\right)$$
$$= g^{-1}\left((1-q)\sum_{j=0}^{\infty} \gamma_j g(\gamma_j)\right). \tag{6.7}$$

Setting $\gamma_j = q^j$ *again we get*

$$\sum_{j=0}^{\infty} \gamma_j g(\gamma_j) = \sum_{j=0}^{\infty} q^j q^{2j} = \sum_{j=0}^{\infty} q^{3j} = \frac{1}{1-q^3}$$

and thus,

$$(1-q)\sum_{j=0}^{\infty} \gamma_j g(\gamma_j) = (1-q) \times \frac{1}{1-q^3} = \frac{1}{1+q+q^2}. \tag{6.8}$$

Therefore, by replacing (6.8) *in* (6.7) *we obtain*

$$\int_{[\alpha,\beta]}^{\oplus} f_1(t) \odot \ {}^{\beta}\mathrm{d}_q t = g^{-1}\left((1-q)\sum_{j=0}^{\infty} \gamma_j g(\gamma_j)\right) = \sqrt{\frac{1}{1+q+q^2}}.$$

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Hence the middle term of (6.2) *is*

$$\begin{aligned} g^{-1}\left(\frac{1}{\beta-\alpha}\right)\Big[\int_{[\alpha,\beta]}^{\oplus} f_1(t)\odot\ {}_{\alpha}\mathrm{d}_q t \oplus \int_{[\alpha,\beta]}^{\oplus} f_1(t)\odot\ {}^{\beta}\mathrm{d}_q t\Big] \\ = g^{-1}\left(g\left(\int_{[\alpha,\beta]}^{\oplus} f_1(t)\odot\ {}_{\alpha}\mathrm{d}_q t\right)+g\left(\int_{[\alpha,\beta]}^{\oplus} f_1(t)\odot\ {}^{\beta}\mathrm{d}_q t\right)\right) \\ = g^{-1}\left(\frac{q+q^3}{(1+q)(1+q+q^2)}+\frac{1}{1+q+q^2}\right) \\ = g^{-1}\left(\frac{1+2q+q^3}{(1+q)(1+q+q^2)}\right)=\sqrt{\frac{1+2q+q^3}{(1+q)(1+q+q^2)}}. \end{aligned}$$

On the other hand, for the right-hand side of (6.2) *note that*

$$\int_{[0,1]}^{\oplus} f_2(t)\odot d_q t=\int_{[0,1]}^{\oplus} f_2(1-t)\odot d_q t=\int_{[0,1]}^{\oplus} 1\odot d_q t=g^{-1}\left((1-q)\sum_{j=0}^{\infty}\gamma_j g(1)\right). \tag{6.9}$$

Setting $\gamma_j=q^j$ *once more we get*

$$\sum_{j=0}^{\infty}\gamma_j g(1)=\sum_{j=0}^{\infty} q^j\times 1^2=\sum_{j=0}^{\infty} q^j=\frac{1}{1-q}$$

and thus,

$$(1-q)\sum_{j=0}^{\infty}\gamma_j g(1)=(1-q)\times\frac{1}{1-q}=1. \tag{6.10}$$

Therefore, by replacing (6.10) *in* (6.9) *we obtain*

$$\int_{[0,1]}^{\oplus} f_2(t)\odot d_q t\oplus\int_{[0,1]}^{\oplus} f_2(1-t)\odot d_q t=g^{-1}(1)\oplus g^{-1}(1)=1\oplus 1=g^{-1}\big(g(1)+g(1)\big)=g^{-1}(2)=\sqrt{2}.$$

In addition,

$$f_1(\alpha)\oplus f_1(\beta)=f_1(0)\oplus f_1(1)=0\oplus 1=g^{-1}\big(g(0)+g(1)\big)=g^{-1}(1)=1.$$

Therefore, the right-hand side of (6.2) *is*

$$\begin{aligned} \Big[\int_{[0,1]}^{\oplus} f_2(t)\odot \mathrm{d}_q t\oplus\int_{[0,1]}^{\oplus} f_2(1-t)\odot \mathrm{d}_q t\Big]\odot\big[f_1(\alpha)\oplus f_1(\beta)\big] \\ =\sqrt{2}\odot 1=g^{-1}\big(g(\sqrt{2})g(1)\big)=g^{-1}(2)=\sqrt{2}. \end{aligned}$$

Consequently, from Theorem 6.2 *we find*

$$\frac{1}{2}\le\sqrt{\frac{1+2q+q^3}{(1+q)(1+q+q^2)}}\le\sqrt{2},$$

or equivalently,

$$\frac{1}{4}\le\frac{1+2q+q^3}{(1+q)(1+q+q^2)}\le 2.$$

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7. Conclusion

In this paper, we formulated and proved Chebyshev's inequality using pseudo-q-integral as well as Hermite-Hadamard inequality using pseudo-integral and Pseudo-q_a, ^bq-Integral. In our future research, we will explore some other well-known inequalities such as Minkowski's, Young's and Strakowski's inequality and we will try to formulate and prove them using pseudo-q-integral and Pseudo-q_a, ^bq-Integral.

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