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GEVREY REGULARITY ON MAXIMALLY REAL SUBMANIFOLDS

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ABSTRACT. The Fourier -Brós -Iagolnitzer (FBI) transform is the right tool to characterize microlocal analyticity, microlocal smoothness, and Gevrey regularity. In this paper, we characterize microlocal Gevrey regularity of a distribution on a maximally real submanifold of $\mathbb{C}^m$ using the FBI transform.

1. Introduction

The commonly used FBI (Fourier -Brós -Iagolnitzer) transform has the form

$$\mathcal{F}_u(x,\xi) = \int_{\mathbb{R}^m} e^{i\xi.(x-y)-|\xi||x-y|^2} u(y)dy,\ x,\xi \in \mathbb{R}^m, \tag{1.1}$$

where u is a continuous function of compact support on $\mathbb{R}^m$, or distribution of compact support in which case the integral is taken in the dual sense. This transform characterizes microlocal regularity (analyticity, smoothness, and Gevrey regularity). The regularity of solutions of linear and nonlinear partial differential equations has been studied (see [1–3, 5, 9]).

Let U be an open neighborhood of 0 in $\mathbb{R}^m$ with coordinates $x_1,\dots,x_m$, and let $Z = (Z_1,\dots,Z_m) : U \to \mathbb{C}^m$ be a C^∞ map, and $dZ_1,\dots,dZ_m$ linearly independent on U. Write $Z(x) = x + i\phi(x)$, where ϕ is a real valued, and smooth function , $\phi(0) = 0$ and $d\phi(0) = 0$. Then $\mathcal{X} = Z(U)$ is a smooth maximally real submanifold. If the Z_j's are G^s, then we obtain G^s maximally real submanifolds. For a compactly supported distribution u on a manifold $\mathcal{X}$, we define the FBI transform of u by

$$\mathcal{F}_u(z,\zeta) = \int_{\mathcal{X}} e^{i\zeta.(z-z')-\langle\zeta\rangle[z-z']^2} u(z')\Delta(z-z',\zeta)dz' \tag{1.2}$$

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in which $z \in \mathcal{X}$, $\zeta \in \mathcal{C}_1 = \{\zeta \in \mathbb{C}^m : |\Im\zeta| < |\Re\zeta|\}$. In [10], the author used the above FBI transform to characterize the smoothness of a distribution locally.

Christ ([7, Theorem 2.3]) provides characterizations of the class G^s. Additionally, in [1], Adwan and Hoepfner utilized the FBI transform, similar to the Fourier transform, to characterize Gevrey regularity. Subsequently, in [5], the authors generalized Christ's theorem to a subclass of generalized FBI transforms. This work will extend the findings from [5] to maximally real submanifolds by employing a variant of the FBI transform (1.2).

The paper's organization is as follows: In section 2, we recall some notions and definition of maximally real submanifolds. We also define the FBI transform in a maximally real submanifold of $\mathbb{C}^m$. Characterization of microlocal Gevrey regularity via the decay of the FBI transform is given in section 3.

2. FBI transform on a maximally real G^s submanifold of $\mathbb{C}^m$

We start this section by introducing the concept of a Gevrey function.

Definition 2.1. Let $s \geq 1$, and let $f \in C^\infty(\Omega)$, $\Omega \subset \mathbb{R}^m$ open. Then the function f is said to be Gevrey function of order s on Ω, if for any compact subset K of Ω there is a constant $C_K > 0$ such that

$$|\partial^\alpha f(x)| \leq C_K^{|\alpha|+1}(\alpha!)^s, \forall \alpha \in \mathbb{N}_0^n, \forall x \in K.$$

We denote the class of all Gevrey functions of order s on Ω by $G^s(\Omega)$.

Consider an involutive structure on a G^s manifold M; as usual, the tangent and cotangent bundles are denoted by $\mathcal{V}$ and T' and their fiber dimensions by m and n, respectively (for the details we refer to [4]).

Definition 2.2. Let $(M, \mathcal{V})$ be a involutive structure. A submanifold $\mathcal{X} \subset M$ is called G^s maximally real, if the pullback map

$$\pi^* : \mathbb{C}T^*M|_{\mathcal{X}} \to \mathbb{C}T^*\mathcal{X}$$

induces an isomorphism

$$T'|_{\mathcal{X}} \cong \mathbb{C}T^*\mathcal{X}.$$

Note that $\dim_{\mathbb{R}} \mathcal{X} = m$.

We refer the image of the real cotangent bundle of $\mathcal{X}$, $T'\mathcal{X}$, under the natural isomorphism $\mathbb{C}T^*\mathcal{X} \cong T'|_{\mathcal{X}}$ as the real structure bundle of $\mathcal{X}$, which is denoted by $\mathbb{R}T'_{\mathcal{X}}$. We will denote variable points in $\mathbb{C}^m$ by z or z' and dual coordinates by ζ_j $(1 \leq j \leq m)$. For any number $\kappa > 0$, we write

$$\mathcal{C}_\kappa = \{\zeta \in \mathbb{C}^m : |\Im\zeta| < \kappa|\Re\zeta|\}.$$

For any $z = (z_1, \ldots, z_m) \in \mathbb{C}^m$ put $[z]^2 = z_1^2 + \ldots + z_m^2$, and for $\zeta \in \mathcal{C}_1$ set $\langle\zeta\rangle = (\zeta.\zeta)^{\frac{1}{2}}$ (main branch of the square root).

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Let

$$M_k = \sum_{j=1}^{m} \mu_{kj} \frac{\partial}{\partial x_j}, \ 1 \le k \le m \tag{2.1}$$

be the vector fields characterized by the relations

$$M_k(z_j)|_{\mathcal{X}}) = \delta_{kj}.$$

Then the vector fields $M_1, \ldots, M_m$ form a G^s basis of $\mathbb{C}T\mathcal{X}$.

Definition 2.3. Let $\mathcal{X}$ be a maximally real submanifold of $\mathbb{C}^m$ and $z_0 \in \mathcal{X}$. Then $\mathcal{X}$ is said to be well positioned at z_0, if there is a number $\kappa, 0 < \kappa < 1$, and an open neighborhood Ω of z_0 such that the following holds:

i) $|\Im\zeta| < \kappa|\Re\zeta|; \forall z \in \Omega, \zeta \in \mathbb{R}T'_{\mathcal{X}}|_z$;
ii) $\Im\{\zeta.(z-z') + i\langle\zeta\rangle[z-z']^2\} \ge (1-\kappa)|\zeta||z-z'|^2, \ \forall z, z' \in \Omega$,
$\forall \zeta \in \left(\mathbb{R}T'_{\mathcal{X}}|_z\right) \cup \left(\mathbb{R}T'_{\mathcal{X}}|_{z'}\right)$.

Finally, $\mathcal{X}$ is very well positioned at z_0 if, given any number $\kappa, 0 \le \kappa < 1$, there exists an open neighborhood Ω of z_0 in $\mathcal{X}$ such that the above holds.

Let U be an open neighborhood of $0 \in \mathbb{R}^m$, and let $Z : U \to \mathbb{C}^m$ with $Z(x) = x + i\phi(x)$, where $\phi : U \to \mathbb{R}^m$ is a G^s map, $\phi(0) = 0$ and $d\phi(0) = 0$. Then $\mathcal{X} = \{x + i\phi(x) : x \in U\}$ is a G^s maximally real smooth submanifold of $\mathbb{C}^m$. In this case a point $(z,\zeta) \in \mathbb{R}T'_{\mathcal{X}}$, with $z \in Z(U)$, if there is $x \in U$ and $\xi \in \mathbb{R}^m$ such that

$$z = Z(x) \text{ and } \zeta = {}^tZ_x(x)^{-1}\xi.$$

It was shown in [10] (Proposition IX.2.2) that $\mathcal{X}$ defined above is very well positioned at 0, that is, given any number $\kappa, 0 \le \kappa < 1$, there is an open neighborhood Ω of 0 in $\mathcal{X}$ such that the following holds :

$$\begin{aligned} &1. \quad |\Im\zeta| < \kappa|\Re\zeta|, \\ &2. \quad \Im\left[\zeta.(z-z') + i\langle\zeta\rangle[z-z']^2\right] \ge (1-\kappa)|\zeta||z-z'|^2 \end{aligned} \tag{2.2}$$

where $z, z' \in \Omega$ and $\zeta \in (\mathbb{R}T'_{\mathcal{X}}|_z) \cup (\mathbb{R}T'_{\mathcal{X}}|_{z'})$.
For $\zeta \in \mathcal{C}_1, z \in \mathbb{C}^m$, we use the notation

$$\Delta(z,\zeta) = \det\{I + i(z \odot \zeta)/\langle\zeta\rangle\},$$

which is the Jacobian determinant of the map $\zeta \to \zeta + i\langle\zeta\rangle z$.

Definition 2.4. Let u be a compactly supported distribution in the manifold $\mathcal{X}$. For $(z,\zeta) \in \mathbb{C}^m \times \mathcal{C}_1$, the FBI transform of u defined as a duality bracket

$$\mathcal{F}_u(z,\zeta) = \int_{\mathcal{X}} e^{i\zeta.(z-z') - \langle\zeta\rangle[z-z']^2} u(z')\Delta(z-z',\zeta)dz'.$$

Note that $\mathcal{F}_u(z,\zeta) \in \mathcal{O}(\mathbb{C}^m \times \mathcal{C}^1)$.

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Definition 2.5. Define, for any $\epsilon > 0$ and $z \in \mathbb{C}^m$,

$$u^\epsilon(z) = \frac{1}{(2\pi)^m}\int_{\mathbb{R}^m} e^{-\epsilon\langle\zeta\rangle^2}\mathcal{F}_u(z,\zeta)d\zeta$$
$$= \frac{1}{(2\pi)^m}\int_{\mathbb{R}^m}\int_{\mathcal{X}} e^{i\zeta.(z-z')-\langle\zeta\rangle[z-z']^2-\epsilon\langle\zeta\rangle^2}u(z')\Delta(z-z',\zeta)dz'd\zeta$$

(Of course, since $\zeta \in \mathbb{R}^m$, we have $\langle\zeta\rangle = |\zeta|$).

Observe that for each fixed $\epsilon > 0, u^\epsilon(z) \in \mathcal{O}(\mathbb{C}^m)$.

Theorem 2.6 ([10],Theorem IX.2.2)**.** (*FBI Inversion Formula*) *Let $\mathcal{X} \subset \mathbb{C}^m$ be maximally real submanifold, $0 \in \mathcal{X}$ and $\mathcal{X}$ is well-positioned at the origin. There is a neighborhood Ω of 0 in $\mathcal{X}$ such that whatever $u \in \mathcal{E}'(\Omega)$,*

$$u(z) = \lim_{\epsilon\to 0^+} u^\epsilon(z) \text{ in } D'(\Omega).$$

To characterize the Gevrey regularity near 0 of a distribution $u \in \mathcal{E}'(\Omega)$ in terms of the "rapid decay" of its FBI transform, we need a modification of the inversion formula (see [10]). Let then K be a compact subset of 0 in Ω and define, for any $\epsilon > 0$,

$$u_K^\epsilon(z) = (2\pi^3)^{-m/2}\int\int e^{i\zeta.(z-z')-\langle\zeta\rangle[z-z']^2-\epsilon\langle\zeta\rangle^2}\mathcal{F}_u(z',\zeta)\langle\zeta\rangle^{\frac{m}{2}}dz'd\zeta, \tag{2.3}$$

where the integration with respect to (z',ζ) is carried out over $\mathbb{R}T'_{\mathcal{X}}|_K$. Property (i) of condition (2.2) ensures that $u_K^\epsilon(z)$ is an entire holomorphic function in $\mathbb{C}^m$.

3. Characterization of Gevrey micro regularity on maximally real submanifold

As before, let $U \subset \mathbb{R}^m$ be a neighborhood of 0, $Z_j(x) \in G^s(U)$ with $dZ_1,\dots,dZ_m$ linearly independent on U so that $\mathcal{X} = Z(U) \subset \mathbb{C}^m$ is a G^s maximally real submanifold and is very well positioned at the origin. Note that $u : \mathcal{X} \to \mathbb{C}$ is in $G^s(\mathcal{X})$, if and only if $u(Z(x))$ is in $G^s(U)$. We consider almost-analytic extensions of the map Z, that is, a G^s mapping $Z_\# : U + i(-1,1)^m \to \mathbb{C}^m$ such that:

i) $Z_\#(x) = Z(x)$ for every $x \in U$,
ii) there exist $C > 0$ such that $\frac{\partial Z^l_\#}{\partial \bar{z}_j}(x+iy) \le C^{N+1}(N!)^{s-1}|y|^N$, $\forall j, \forall N, 1 \le l \le m$.

Following the characterization of microregularity given in [4, Definition V.2.11 and Theorem V.3.7] and ([5, Proposition 2.1]), we define Gevery microlocal regularity as follows.

Definition 3.1. Let $u \in \mathcal{E}'(\mathcal{X})$, $\xi^0 \in \mathbb{R}^m \setminus \{0\}$. Then u is said to be micro-locally G^s regular at $(0,\xi^0)$ if there is a neighborhood V of 0, open acute cones $\Gamma^1,\dots,\Gamma^N$ in $\mathbb{R}^m\setminus\{0\}$ and C^1 functions f_j on $Z_\#(V + i\Gamma^j_\delta)$ $(\delta > 0)$ of tempered growth such that

i) $u = \sum_{j=1}^N bf_j$ near 0;
ii) $\xi^0.\Gamma^j < 0$, $\forall j$;
iii) $\left|\frac{\partial f_j}{\partial \bar{z}_k}(z)\right| \le A\exp\left(\frac{-B}{\left(dist(Z_\#(x+iy),Z(V))\right)^{\frac{1}{s-1}}}\right)$, $\forall j = 1,\dots,N$, $\forall k = 1,\dots,m$ for some $A, B > 0$.

The G^s- wave front set of u, is defined by $WF_s(u) = \{(x,\xi) : u \text{ is not } s- \text{ micro-regular at } (x,\xi)\}$.

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We now state the Gevrey regularity result as follows:

Theorem 3.2. *Let $u \in \mathcal{E}'(\mathcal{X})$, $\xi^0 \in \mathbb{R}^m \setminus \{0\}$. Then u is microlocally G^s regular at $(0, \xi^0)$, if and only if there are constants $a, b > 0$ such that*

$$|\mathcal{F}_u(z, \zeta)| \leq a e^{-b|\zeta|^{\frac{1}{s}}},$$

for (z, ζ) in a conic neighborhood of $(0, \xi^0)$ in $\mathbb{R}T'_{\mathcal{X}}$.

Proof. Suppose u is microlocally G^s regular at $(0, \xi^0)$. Then without loss of generality assume there is a neighborhood V of 0, open acute cone Γ in $\mathbb{R}^m \setminus \{0\}$ and C^1 function f on $Z_\#(V + i\Gamma_\delta)$ $(\delta > 0)$ of tempered growth such that

i) $u = bf$ on $Z(V)$;
ii) $\xi^0.\Gamma < 0$;
iii) $|\frac{\partial f}{\partial \bar{z}_j}(z)| \leq A \exp\left(\frac{-B}{\left(dist(z, Z(V))\right)^{\frac{1}{s-1}}}\right)$, for some $A, B > 0$ and $z \in Z_\#(V + i\Gamma_\delta)$.

Let $r > 0$ such that $B_{2r} = \{x : |x| < 2r\} \subset\subset V$, $\Omega = Z(B_{2r})$. Let $g \in C_0^\infty(\mathcal{X})$, $g \equiv 1$ in $Z(B_r)$ and $supp(g) \subset \Omega$.
Let $v \in \Gamma_\delta$ and $u \in \mathcal{E}'(\mathcal{X})$. Then

$$\begin{aligned}
\mathcal{F}_u(z, \zeta) &= \int_{\mathcal{X}} e^{i\zeta.(z-z') - \langle\zeta\rangle[z-z']^2} u(z') \Delta(z - z', \zeta) dz' \\
&= \lim_{t \to 0^+} \int_{B_{2r}} e^{i\zeta.(Z(x)-Z(x')) - \langle\zeta\rangle[Z(x)-Z(x')]^2} f\big(Z(x') + iZ_{x'}(x')tv)g(Z(x')\big)\Delta(Z(x) - Z(x'), \zeta) dZ(x')
\end{aligned}$$

Since $g\big(Z(x)\big) \in C^\infty$, it has almost analytic extension $\tilde{g}$ on $Z_\#(V + i\Gamma_\delta)$. Then

$$\mathcal{F}_u(Z(x), \zeta) = \lim_{t \to 0^+} \int_{B_{2r}} e^{i\zeta.(Z(x)-\tilde{Z}(x')) - \langle\zeta\rangle[Z(x)-\tilde{Z}(x')]^2} f(\tilde{Z}(x'))\tilde{g}(\tilde{Z}(x'))\Delta(Z(x) - \tilde{Z}(x'), \zeta) d\tilde{Z}(x'),$$

where $\tilde{Z}(x', v, t) = Z(x') + iZ_{x'}(x')tv$.
For $0 < \lambda < 1$, let

$$D_\lambda = \{Z(x') + iZ_{x'}(x')tv \in \mathbb{C}^m : x' \in B_{2r}, \lambda \leq t \leq 1\}.$$

Let $Q(x, x', \zeta) = i\zeta.(Z(x) - Z(x')) - \langle\zeta\rangle[Z(x) - Z(x')]^2$.
Consider the m - form

$$w(\tilde{z}) = e^{Q(x, \tilde{z}, \zeta)} \tilde{g}(\tilde{z}) f(\tilde{z}) d\tilde{z}_1 \wedge \ldots \wedge d\tilde{z}_m \; , \tilde{z} = z' + iZ_{x'}(x')y.$$

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Then by Stokes theorem we have

$$\begin{aligned}\mathcal{F}_u(Z(x),\zeta) &= \lim_{\lambda\to 0^+}\int_{B_{2r}} e^{Q_0} f(Z(x')+iZ_{x'}(x')\lambda v)\tilde{g}(Z(x')+iZ_{x'}(x')\lambda v)\Delta(Z(x)-Z(x')-iZ_{x'}(x')\lambda v,\zeta)dZ(x')\\
&= \int_{B_{2r}} e^{Q_1} f(Z(x')+iZ_{x'}(x')v)\tilde{g}(Z(x')+iZ_{x'}(x')v)\Delta(Z(x)-Z(x')-iZ_{x'}(x')v,\zeta)dZ(x')\\
&\quad+\lim_{\lambda\to 0^+}\sum_{j=1}^{m}\int_{D_\lambda} e^{Q_2}\tilde{g}(Z(x')+iZ_{x'}(x')tv)\frac{\partial f}{\partial\overline{\tilde{z}_j}}(Z(x')+iZ_{x'}(x')tv)\\
&\quad\ \Delta(Z(x)-Z(x')-iZ_{x'}(x')tv,\zeta)d\overline{\tilde{z}_j}\wedge d\tilde{z}\\
&\quad+\lim_{\lambda\to 0^+}\sum_{j=1}^{m}\int_{D_\lambda} e^{Q_2} f(Z(x')+iZ_{x'}(x')tv)\frac{\partial \tilde{g}}{\partial\overline{\tilde{z}_j}}(Z(x')+iZ_{x'}(x')tv)\\
&\quad\ \Delta(Z(x)-Z(x')-iZ_{x'}(x')tv,\zeta)d\overline{\tilde{z}_j}\wedge d\tilde{z}\\
&= I_0+I_1^\lambda+I_2^\lambda,\end{aligned}$$

where

$$Q_0(x,x',\zeta)=i\zeta.(Z(x)-Z(x')-iZ_{x'}(x')\lambda v)-\langle\zeta\rangle[Z(x)-Z(x')-iZ_{x'}(x')\lambda v]^2,$$

$$Q_1(x,x',\zeta)=i\zeta.(Z(x)-Z(x')-iZ_{x'}(x')v)-\langle\zeta\rangle[Z(x)-Z(x')-iZ_{x'}(x')v]^2,$$

and

$$Q_2(x,x',\zeta,t)=i\zeta.(Z(x)-Z(x')-iZ_{x'}(x')tv)-\langle\zeta\rangle[Z(x)-Z(x')-iZ_{x'}(x')tv]^2.$$

Let's estimate Q_2. Since $v\in\Gamma$ and $\xi^0.\Gamma<0$, there is a conic neighborhood Γ_1 of ξ^0 and a constant $c>0$ such that

$$\xi.v\le -c|v||\xi|,\ \forall\xi\in\Gamma_1.$$

Since $\mathcal{X}$ is well positioned at the origin, we have $\forall x,x'\in B_{2r}$,$\xi\in\Gamma$;

$$\Re\Big(i\zeta.(Z(x)-Z(x'))-\langle\zeta\rangle[Z(x)-Z(x')]^2\Big)\le -(1-\kappa)|\zeta||Z(x)-Z(x')|^2.$$

So we have

$$\begin{aligned}|(tZ_{x'}(x')v).\zeta| &= |(tZ_{x'}(x')v).({}^tZ_{x'}^{-1}(x')\xi)|\\
&= |tZ_x(x)Z_{x'}^{-1}(x')v.\xi|\\
&\le t(1+c_1|x-x'|)|v||\xi)| \text{ since } Z_x(x)Z_{x'}^{-1}(x')=Id+O(|x-x'|)\\
&\le tc_1'|v||\xi|\ .c_1'=1+4c_1r\end{aligned}$$

Thus after shrinking Ω, if needed, so that $|\zeta|\le 2|\xi|$, we have

$$\Re(tZ_{x'}(x')v.\zeta)\le -\frac{1}{2}c\,c_1't|v||\zeta|=-\frac{1}{2}c't|v||\zeta|.$$

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Also shrink Ω further if necessary so that $|z-z'| \leq \frac{c}{16}$ for $z, z' \in \Omega$ and $||\phi_x|| \leq \frac{1}{2}$, $\forall x \in B_{2r}$, since $|\langle\zeta\rangle| \leq |\zeta|$, we have for $\delta < \frac{c}{36}$,

$$\begin{aligned}
&\Re\bigg(- \langle\zeta\rangle\bigg(- 2itZ_{x'}(x')v.(Z(x) - Z(x')) - (tZ_{x'}(x')v).(tZ_{x'}(x')v)\bigg)\bigg) \\
&\leq |\zeta|\bigg(3t|v||Z(x) - Z(x')| + \frac{9}{4}|v|^2t^2\bigg) \\
&\leq |\zeta||v|3t(|Z(x) - Z(x')| + \frac{9}{4}|v|) \\
&\leq |\zeta||v|t\bigg(\frac{3c}{16} + \frac{9}{4}\frac{c}{36}\bigg) \text{ since } v_0 \in \Gamma_\delta, \delta < \frac{c}{36} \\
&= \frac{1}{4}tc|v||\zeta| \\
&\leq \frac{1}{4}tc_1'c|v||\zeta| = \frac{1}{4}c't|v||\zeta|.
\end{aligned}$$

Therefore, $\Re Q_2(x, x', \zeta, t) \leq -(1-\kappa)|\zeta||Z(x) - Z(x')|^2 - tc''|v||\zeta|,\ c'' = \frac{c'}{2}$.
Similarly, $\Re Q_1(x, x', \zeta) \leq -(1-\kappa)|\zeta||Z(x) - Z(x')|^2 - c''|v||\zeta|$.
Consider I_0: Choosing $|v|$ small, for $\xi \in \Gamma_1$, $\zeta = {}^tZ_{x'}^{-1}(x')\Gamma_1$, $|\zeta| \geq 1$ we have

$$\begin{aligned}
|I_0| &\leq e^{-c''|v||\zeta|} \sup_{x \in \overline{B_{2r}}} |(f\tilde{g})(Z(x') + iZ_{x'}(x')v)| \int_{B_{2r}} |\Delta| e^{-(1-\kappa)|\zeta||Z(x) - Z(x')|^2} dZ(x') \\
&\leq de^{-c''|v||\zeta|} \int_{B_{2r}} |\Delta| e^{-(1-\kappa)|\zeta||Z(x) - Z(x')|^2} dZ(x') \\
&\leq d'e^{-b|\zeta|} , b > 0 \\
&\leq d'e^{-b|\zeta|^{\frac{1}{s}}}.
\end{aligned}$$

Since $\frac{I_0}{e^{-c'|\zeta|^{\frac{1}{s}}}}$ is bounded on $\overline{B_{2r}} \times \{\zeta : |\zeta| \leq 1\}$, there exist $a_0, b_0 > 0$ such that

$$|I_0| \leq a_0 e^{-b_0|\zeta|^{\frac{1}{s}}} \text{ for } |x'| < 2r, \zeta = {}^tZ_{x'}^{-1}(x')\xi , \xi \in \Gamma_1. \tag{3.1}$$

Consider

$$\begin{aligned}
I_1^\lambda = \lim_{\lambda \to 0^+} \sum_{j=1}^m \int_{D_\lambda} & e^{Q_2} \tilde{g}(Z(x') + iZ_{x'}(x')tv) \frac{\partial f}{\partial \bar{\tilde{z}}_j}(Z(x') + iZ_{x'}(x')tv) \\
& \Delta(Z(x) - Z(x') - iZ_{x'}(x')tv, \zeta) d\bar{\tilde{z}}_j \wedge d\tilde{z} :
\end{aligned}$$

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For $\zeta \in \{{}^tZ_{x'}^{-1}(x')\Gamma_1\}, |\zeta| \geq 1$,

$$\begin{aligned}
&\left| e^{Q_2}\tilde{g}(Z(x') + iZ_{x'}(x')tv)\frac{\partial f}{\partial \bar{\tilde{z}}_j}(Z(x') + iZ_{x'}(x')tv)\right| \\
&\leq C' e^{\Re Q_2} A e^{-\frac{B}{|tv|^{\frac{1}{s-1}}}}, \quad (C' = sup_{(x',t)\in \overline{B_{2r}}\times[0,t]}|\tilde{g}(Z(x') + iZ_{x'}(x')tv)|) \\
&\leq A' e^{-(1-\kappa)|\zeta||z-z'|^2} e^{-bt|\zeta|} e^{-\frac{B'}{|t|^{\frac{1}{s-1}}}}, \text{ (taking } v \text{ small)} \\
&\leq A'\left(\frac{N}{s}\right)^{\frac{N}{s}} e^{-\frac{N}{s}}(bt|\zeta|)^{-\frac{N}{s}}\left[\frac{s-1}{s}N\right]^{\frac{(s-1)N}{s}} e^{-\frac{(s-1)}{s}N}\left(\frac{B'}{t^{\frac{1}{s-1}}}\right)^{-\frac{(s-1)}{s}N} e^{-(1-\kappa)|\zeta||z-z'|^2} \\
&\leq D^{N+1}N!|\zeta|^{-\frac{N}{s}}e^{-(1-\kappa)|\zeta||z-z'|^2}, \text{ where we used } e^{-\alpha} \leq d^d e^{-d}\alpha^{-d}, d, \alpha > 0
\end{aligned}$$

with $d = \frac{N}{s}$ for $e^{-bt|\zeta|}$ and $d = \frac{s-1}{s}N$ for $e^{-\frac{B'}{|t|^{\frac{1}{s-1}}}}$ for $N \geq 1$.
Thus

$$\begin{aligned}
&\lim_{\lambda\to 0^+}|I_1^\lambda| \\
&\leq D^{N+1}N!|\zeta|^{-\frac{N}{s}}\sum_{j=1}^m\int_0^1\int_{B_{2r}}|\Delta|e^{-(1-\kappa)|\zeta||z-z'|^2}d\bar{\tilde{z}}_j\wedge d\tilde{z} \text{ for } x' \in B_{2r}, \zeta \in \{{}^tZ_{x'}^{-1}(x')\Gamma_1\}, |\zeta| \geq 1.
\end{aligned}$$

Therefore, $\lim_{\lambda\to 0^+}|I_1^\lambda| \leq a_1 e^{-b_1|\zeta|^{\frac{1}{s-1}}}$ for $x' \in B_{2r}, \zeta \in \{{}^tZ_{x'}^{-1}(x')\Gamma_1\}, |\zeta| \geq 1$ for some $a_1, b_1 > 0$ independent of λ.
But since $\frac{|I_1^\lambda|}{e^{-b_1|\zeta|^{\frac{1}{s-1}}}}$ is bounded on $\overline{B_{2r}} \times \{\zeta : |\zeta| \leq 1\}$, there exist $A_1, B_1 > 0$ such that

$$\lim_{\lambda\to 0^+}|I_1^\lambda| \leq A_1 e^{-b_1|\zeta|^{\frac{1}{s-1}}}. \tag{3.2}$$

Consider

$$\begin{aligned}
I_2^\lambda = \lim_{\lambda\to 0^+}\sum_{j=1}^m\int_{D_\lambda} e^{Q_2} f(Z(x') + iZ_{x'}(x')tv)\frac{\partial \tilde{g}}{\partial \bar{\tilde{z}}_j}(Z(x') + iZ_{x'}(x')tv) \\
\Delta(Z(x) - Z(x') - iZ_{x'}(x')tv, \zeta)d\bar{\tilde{z}}_j \wedge d\tilde{z} :
\end{aligned}$$

Since $\frac{\partial \tilde{g}}{\partial \bar{\tilde{z}}_j} \equiv 0$ for $|x| \leq r$, the integral over $|x| \leq r$ is zero. Then for $|x'| < \frac{r}{2}$ and $|x| \geq r$ and taking $|v|$ small we have

$$\Re Q_2 \leq -(1-\kappa)|\zeta||Z(x) - Z(x')|^2 - c''|\zeta|$$

for $\zeta \in \{{}^tZ_{x'}^{-1}(x')\Gamma_1\}, |\zeta| \geq 1$. Since f is of tempered growth, there exist a constant $d > 0$ and an integer $k \geq 0$ such that

$$|f(Z(x') + iZ_{x'}(x')tv)| \leq \frac{d}{t^k|v|^k}.$$

Also since $\tilde{g}$ is almost holomorphic, there exist $c_k > 0$ such that

$$|\frac{\partial \tilde{g}}{\partial \bar{\tilde{z}}_j}| \leq c_k t^k |v|^k. \; \forall j = 1, \ldots, m.$$

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Thus we can get $A_2, B_2 > 0$ independent of λ such that

$$\lim_{\lambda\to 0^+} |I_2^\lambda| \leq A_2 e^{-B_2|\zeta|^{\frac{1}{s}}} \ \forall \zeta \in \{{}^tZ_{x'}^{-1}(x')\Gamma_1\}. x' \in B_{\frac{r}{2}} \tag{3.3}$$

Combining (3.1), (3.2) and (3.3) there are constants $a, b > 0$ such that

$$|\mathcal{F}_u(z,\zeta)| \leq ae^{-b|\zeta|^{\frac{1}{s}}},$$

for (z,ζ) in a conic neighborhood of $(0,\xi^0)$ in $\mathbb{R}T'_{\mathcal{X}}$.

For the converse let $u \in \mathcal{E}'(\Omega)$, Ω open neighborhood of 0 in $\mathcal{X}$. Let B_0 be a ball centered at zero such that $Z(B_0) = \Omega$. Suppose that there exist $a, b > 0$ such that

$$|\mathcal{F}_u(z,\zeta)| \leq ae^{-b|\zeta|^{\frac{1}{s}}},$$

for (z,ζ) in a conic neighborhood of $(0,\xi^0)$, that is, on a set of the form $\{(z,\zeta) : z \in Z(V), \zeta \in {}^tZ_x^{-1}\Gamma\}$, where V is a neighborhood of 0 ($V \subset B_0$) and Γ is a conic neighborhood of ξ^0. Applying the inversion formula (2.3) we have

$$u_{\tilde{K}}(z) = \frac{1}{(2\pi^3)^{\frac{m}{2}}} \lim_{\epsilon\to 0^+} \iint\limits_{\mathbb{R}T'_{\mathcal{X}}|_{\tilde{K}}} e^{i\zeta.(z-z')-\langle\zeta\rangle[z-z']^2-\epsilon\langle\zeta\rangle^2}\mathcal{F}_u(z',\zeta)\langle\zeta\rangle^{\frac{m}{2}}dz'd\zeta,$$

where $\tilde{K} = Z(K)$; K is a compact neighborhood of 0 in U. Extend the map Z as a map $\tilde{Z} : \mathbb{R}^m \to \mathbb{C}^m$ and let $\tilde{Z}(\mathbb{R}^m) = \tilde{\Omega}$. Then the properties of wellposedness remains valid in $\tilde{\Omega}$. Choose $d > 0$ such that $\Omega' = \{Z(x') : |Z(x')| \leq d\} \subset Z(V)$. Then we can write $u^\epsilon_{\tilde{K}}(z)$ as

$$u^\epsilon_{\tilde{K}}(z) = u^\epsilon_{\tilde{K},0}(z) + u^\epsilon_{\tilde{K},1}(z),$$

where

$$u^\epsilon_{\tilde{K},0}(z) = \frac{1}{(2\pi^3)^{\frac{m}{2}}} \int\int_{\Omega'} e^{i\zeta.(z-z')-\langle\zeta\rangle[z-z']^2-\epsilon\langle\zeta\rangle^2}\mathcal{F}_u(z',\zeta)\langle\zeta\rangle^{\frac{m}{2}}dz'd\zeta,$$

and

$$u^\epsilon_{\tilde{K},1}(z) = \frac{1}{(2\pi^3)^{\frac{m}{2}}} \int\int_{\tilde{\Omega}\setminus\Omega'} e^{i\zeta.(z-z')-\langle\zeta\rangle[z-z']^2-\epsilon\langle\zeta\rangle^2}\mathcal{F}_u(z',\zeta)\langle\zeta\rangle^{\frac{m}{2}}dz'd\zeta,$$

so that $u(z) = u_{\tilde{K},0}(z) + u_{\tilde{K},1}(z)$.

Consider $u_{\tilde{K},0}(z)$: Let $\mathcal{C}_j$, $1 \leq j \leq N$ be open, acute cones such that

$$\mathbb{R}^m = \cup_{j=1}^N \overline{\mathcal{C}_j}$$

and $\overline{\mathcal{C}_j} \cap \overline{\mathcal{C}_k}$ has measure zero when $j \neq k$. We may assume that $\xi^0 \in \mathcal{C}_1$ and $\xi^0 \notin \overline{\mathcal{C}_j}$ for $j \geq 2$. This implies that we can get acute, open cones Γ^j , $2 \leq j \leq N$ and a constant $C > 0$ such that

$$\xi^0.\Gamma^j < 0 \text{ and } y.\xi \geq C|y||\xi| \ \forall y \in \Gamma^j, \ \forall \xi \in \mathcal{C}_j.$$

For each $j = 2, \dots, N,\ z = Z_\#(x + iy) \in Z_\#(V + i\Gamma^j_\delta)$ define

$$f^\epsilon_j(z)) = \frac{1}{(2\pi^3)^{\frac{m}{2}}} \int_{\xi\in\mathcal{C}_j} \int_{\Omega'} e^{i\zeta.(z-z')-\langle\zeta\rangle[z-z']^2-\epsilon\langle\zeta\rangle^2}\mathcal{F}_u(z',\zeta)\langle\zeta\rangle^{\frac{m}{2}}dz'd\zeta.$$

Then for $2 \leq j \leq N$, each f_j^ϵ is entire and as $\epsilon \to 0^+$, the f_j^ϵ converges uniformly on compact subsets of $Z_\#(\mathbb{R}^m + i\Gamma^j)$ to

$$f_j(z) = \frac{1}{(2\pi^3)^{\frac{m}{2}}} \int_{\xi\in\mathcal{C}_j} \int_{\Omega'} e^{i\zeta.(z-z') - \langle\zeta\rangle[z-z']^2} \mathcal{F}_u(z',\zeta)\langle\zeta\rangle^{\frac{m}{2}} dz' d\zeta,$$

which is also holomorphic on $Z_\#(\mathbb{R}^m + i\Gamma^j)$ and is of tempered growth on $Z_\#(V + i\Gamma^j_\delta)$ for some $0 < \delta \leq 1$.

Then each f_j $(j \geq 2)$ has boundary value $bf_j \in D'(Z(V))$.
Let

$$g_1^\epsilon(Z(x))) = \frac{1}{(2\pi^3)^{\frac{m}{2}}} \int_{\xi\in\mathcal{C}_1} \int_{\Omega'} e^{i\zeta.(Z(x)-Z(x')) - \langle\zeta\rangle[z-z']^2 - \epsilon\langle\zeta\rangle^2} \mathcal{F}_u(z',\zeta)\langle\zeta\rangle^{\frac{m}{2}} dz' d\zeta.$$

Then by the decay of the FBI transform g_1^ϵ are smooth for all $\epsilon > 0$ and converges uniformly on $\mathbb{R}^m$ to the function

$$g_1(Z(x)) = \frac{1}{(2\pi^3)^{\frac{m}{2}}} \int_{\xi\in\mathcal{C}_1} \int_{\Omega'} e^{i\zeta.(Z(x)-Z(x')) - \langle\zeta\rangle[Z(x)-Z(x')]^2} \mathcal{F}_u(Z(x'),\zeta)\langle\zeta\rangle^{\frac{m}{2}} dZ(x') d\zeta.$$

Clearly $g_1(Z(x))$ is smooth on $\mathbb{R}^m$. We will show that $g_1(Z(x))$ is in G^s for x near the origin.

Let $\{M_1, \dots, M_m\}$ be the vector fields that satisfy

$$M_j Z_k = \delta_j^k, \ 1 \leq j, k \leq m.$$

Then $g_1(Z(x))$ will be in G^s near 0 if there exist $c > 0$ such that for all α,

$$|M^\alpha g_1(Z(x))| \leq c^{|\alpha|+1}(\alpha!)^s, \tag{3.4}$$

for x near 0. We will therefore establish estimate (3.4). Observe that since

$$\mathbb{C}^m \ni z \mapsto e^{i\zeta.(z-Z(x')) - \langle\zeta\rangle[z-Z(x')]^2}$$

is an entire function; for any α,

$$M^\alpha g_1(Z(x)) = \frac{1}{(2\pi^3)^{\frac{m}{2}}} \int_{\xi\in\mathcal{C}_1} \int_{\Omega'} \partial_z^\alpha \left\{ e^{i\zeta.(z-Z(x')) - \langle\zeta\rangle[z-Z(x')]^2} \right\}|_{z=Z(x)} \mathcal{F}_u(Z(x'),\zeta)\langle\zeta\rangle^{\frac{m}{2}} dZ(x') d\zeta.$$

We begin by estimating the term $\partial_z^\alpha \left\{ e^{i\zeta.(z-Z(x')) - \langle\zeta\rangle[z-Z(x')]^2} \right\}$ for z and x' bounded.
Clearly

$$\partial_z^\alpha \left\{ e^{i\zeta.(z-Z(x')) - \langle\zeta\rangle[z-Z(x')]^2} \right\} = \prod_{j=1}^m \partial_{z_j}^{\alpha_j} F(h_j(z_j)),$$

where $F(w) = e^w$ for $w \in \mathbb{C}$ and for each $1 \leq j \leq m$,

$$h_j(z_j) = i\zeta_j(z_j - Z_j(x')) - \langle\zeta\rangle[z_j - Z_j(x')]^2.$$

By the formula of Faá di Bruno,

$$\partial_{z_j}^{\alpha_j} F(h_j(z_j)) = \sum_{S_j} \frac{\alpha_j!}{\alpha_j^1! \dots \alpha_j^m} F^{(\alpha_j^1 + \dots + \alpha_j^m)}(h_j(z_j)) \prod_{k=1}^m \left(\frac{h_j^{(k)}(z_j)}{k!} \right)^{\alpha_j^k}$$

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where the sum is taken over the set

$$S_j = \{(\alpha_j^1, \dots, \alpha_j^m) : \sum_{k=1}^{m} k\alpha_j^k = \alpha_j\},$$

where each α_j^k is a nonnegative integer.
Note that $h_j^{(1)}(z_j) = i\zeta_j - 2\langle\zeta\rangle(z_j - Z_j(x')), h_j^{(2)}(z_j) = -2\langle\zeta\rangle$, and $h_j^{(k)}(z_j) = 0$ for $k \geq 3$. It follows that

$$\partial_{z_j}^{\alpha_j} F(h_j(z_j)) = \sum_{S_j} \frac{\alpha_j!}{\alpha_j^1!\alpha_j^2!}(-1)^{\alpha_j^2}\langle\zeta\rangle^{\alpha_j^2}\Big(i\zeta_j - 2\langle\zeta\rangle(z_j - Z_j(x'))\Big)^{\alpha_j^1} e^{h_j(z_j)},$$

and

$$S_j = \{(\alpha_j^1, \alpha_j^2) : \alpha_j^1 + 2\alpha_j^2 = \alpha_j\}.$$

Hence

$$\partial_z^\alpha\Big\{e^{i\zeta.(z-Z(x'))-\langle\zeta\rangle[z-Z(x')]^2}\Big\} = e^{h(z)}\prod_{j=1}^{m}\Bigg(\sum_{S_j}\frac{\alpha_j!}{\alpha_j^1!\alpha_j^2!}(-1)^{\alpha_j^2}\langle\zeta\rangle^{\alpha_j^2}\Big(i\zeta_j - 2\langle\zeta\rangle(z_j - Z_j(x'))\Big)^{\alpha_j^1}\Bigg), \tag{3.5}$$

where $h(z) = i\zeta.(z - Z(x')) - \langle\zeta\rangle[z - Z(x')]^2$. Since $z_j = Z_j(x)$ and $Z_j(x')$ belong to a bounded set, there exists $c > 0$ such that

$$\Big|i\zeta_j - 2\langle\zeta\rangle(z_j - Z_j(x'))\Big|^{\alpha_j^1} \leq c^{\alpha_j^1}|\zeta|^{\alpha_j^1} \leq c^{|\alpha|}|\zeta|^{\alpha_j^1}.$$

Likewise, the factor $e^{h(z)}$ is bounded for $z = Z(x)$. Therefore, the term

$$\Big|\partial_z^\alpha\Big\{e^{i\zeta.(z-Z(x'))-\langle\zeta\rangle[z-Z(x')]^2}\Big\}|_{z=Z(x)}\Big|,$$

is bounded by the sum of a finite number of terms of the form (after possibly increasing c)

$$c^{|\alpha|+1}\Big(\frac{\alpha_1!}{\alpha_1^1!\alpha_1^2!}\cdots\frac{\alpha_m!}{\alpha_m^1!\alpha_m^2!}\Big)|\zeta|^{\alpha_1^1+\dots+\alpha_m^1+\alpha_1^2+\dots+\alpha_m^2}. \tag{3.6}$$

We thus see that $|M^\alpha g_1(Z(x))|$ is bounded by a finite sum of the form

$$c^{|\alpha|+1}\Big(\frac{\alpha_1!}{\alpha_1^1!\alpha_1^2!}\cdots\frac{\alpha_m!}{\alpha_m^1!\alpha_m^2!}\Big)\int_{\mathcal{C}_1}\int_{\Omega'}|\zeta|^{\alpha_1^1+\dots+\alpha_m^1+\alpha_1^2+\dots+\alpha_m^2+\frac{m}{2}}|\mathcal{F}_u(Z(x'),\zeta)||d\zeta|dZ(x').$$

By the hypothesis on the decay of $\mathcal{F}_u(Z(x'),\zeta)$, $|M^\alpha g_1(Z(x))|$ is bounded by a finite sum of the form

$$c^{|\alpha|+1}\Big(\frac{\alpha_1!}{\alpha_1^1!\alpha_1^2!}\cdots\frac{\alpha_m!}{\alpha_m^1!\alpha_m^2!}\Big)\int_{\mathcal{C}_1}\int_{\Omega'}|\zeta|^{\alpha_1^1+\dots+\alpha_m^1+\alpha_1^2+\dots+\alpha_m^2+\frac{m}{2}}e^{-b|\zeta|^{\frac{1}{s}}}|d\zeta|dZ(x')$$

for some $c > 0$. Since $|\zeta|$ is comparable to $|\xi|$, $\xi \in \mathbb{R}^m$ and Ω' is a bounded set, the latter expression is bounded by

$$c^{|\alpha|+1}\Big(\frac{\alpha_1!}{\alpha_1^1!\alpha_1^2!}\cdots\frac{\alpha_m!}{\alpha_m^1!\alpha_m^2!}\Big)\int_{\mathbb{R}^m}|\xi|^{\alpha_1^1+\dots+\alpha_m^1+\alpha_1^2+\dots+\alpha_m^2+\frac{m}{2}}e^{-b|\xi|^{\frac{1}{s}}}d\xi. \tag{3.7}$$

We estimate the integral (3.7) in the following. Using polar coordinates

$$\int_{\mathbb{R}^m}|\xi|^{\alpha_1^1+\dots+\alpha_m^1+\alpha_1^2+\dots+\alpha_m^2+\frac{m}{2}}|e^{-b|\xi|^{\frac{1}{s}}}|d\xi = c\int_0^\infty r^{\alpha_1^1+\dots+\alpha_m^1+\alpha_1^2+\dots+\alpha_m^2+\frac{m}{2}+m-1}e^{-br^{\frac{1}{s}}}dr$$

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for some constant $c > 0$. Change variable $t = br^{\frac{1}{s}}$ and so $r = \frac{t^s}{b^s}$. Then the integral above is dominated by

$$c^{|\alpha|+1} \int_0^\infty t^{s(\alpha_1^1+\ldots+\alpha_m^1+\alpha_1^2+\ldots+\alpha_m^2+\frac{3m}{2})-1} e^{-t} dt,$$

for some $c > 0$.
Let $k = \alpha_1^1 + \ldots + \alpha_m^1 + \alpha_1^2 + \ldots + \alpha_m^2 + \frac{3m}{2}$. Then the latter integral is

$$c^{|\alpha|+1} \int_0^\infty t^{sk-1} e^{-t} dt = c^{|\alpha|+1} \Gamma(sk),$$

where $\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt$ is the Gamma function. By Sterling's formula for the Gamma function, there is a constant $c > 0$ such that for $s \geq 1$,

$$\Gamma(s) \leq cs^{s-\frac{1}{2}} e^{-s} \leq cs^s.$$

Hence

$$\Gamma(sk) \leq c^k (k^k)^s \leq (ce^s)^k (k!)^s, \tag{3.8}$$

where we have used $e^k \geq \frac{k^k}{k!}$ in the second inequality. From (3.7) and (3.8), it follows that $|M^\alpha g_1(Z(x))|$ is bounded by a finite sum of the form

$$c^{|\alpha|+1} \left(\frac{\alpha_1!}{\alpha_1^1! \alpha_1^2!} \cdots \frac{\alpha_m!}{\alpha_m^1! \alpha_m^2!} \right) \left[(\alpha_1^1 + \ldots + \alpha_m^1 + \alpha_1^2 + \ldots + \alpha_m^2 + \frac{3m}{2})! \right]^s. \tag{3.9}$$

Note that

$$\begin{aligned} &(\alpha_1^1 + \ldots + \alpha_m^1 + \alpha_1^2 + \ldots + \alpha_m^2 + \frac{3m}{2})! \\ &\qquad \leq (\alpha_1^1 + \ldots + \alpha_m^1 + \alpha_1^2 + \ldots + \alpha_m^2 + 3m)! \\ &\qquad \leq 2^{\alpha_1^1+\ldots+\alpha_m^1+\alpha_1^2+\ldots+\alpha_m^2+3m} (\alpha_1^1 + \ldots + \alpha_m^1 + \alpha_1^2 + \ldots + \alpha_m^2)!(3m)!, \end{aligned}$$

where we have used the inequality

$$(a + b)! \leq 2^{a+b} a!b! \text{ for } a, b \geq 0.$$

Therefore, after changing c, (3.9) is bounded by

$$c^{|\alpha|+1} \left(\frac{\alpha_1!}{\alpha_1^1! \alpha_1^2!} \cdots \frac{\alpha_m!}{\alpha_m^1! \alpha_m^2!} \right) \left[(\alpha_1^1 + \ldots + \alpha_m^1 + \alpha_1^2 + \ldots + \alpha_m^2)! \right]^s. \tag{3.10}$$

Observe that for each $1 \leq j \leq m$,

$$\begin{aligned} \frac{\alpha_j!}{\alpha_j^1! \alpha_j^2!} &= \frac{[(\alpha_j^1 + \alpha_j^2) + \alpha_j^2]!}{\alpha_j^1! \alpha_j^2!} \text{ since } \alpha_j = \alpha_j^1 + 2\alpha_j^2 \\ &\leq \frac{2^{\alpha_j} (\alpha_j^1 + \alpha_j^2)!}{\alpha_j^1!} \leq \frac{2^{2\alpha_j} \alpha_j^1! \alpha_j^2!}{\alpha_j^1!} = 2^{2\alpha_j} \alpha_j^2 \, !. \end{aligned}$$

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Therefore, (3.10) is bounded by

$$
\begin{aligned}
& 4^{|\alpha|}\alpha_1^2!\ldots\alpha_m^2!\Big[(\alpha_1^1+\ldots+\alpha_m^1+\alpha_1^2+\ldots+\alpha_m^2)!\Big]^s \\
& \le 4^{|\alpha|}\Big[\alpha_1^2!\ldots\alpha_m^2!(\alpha_1^1+\ldots+\alpha_m^1+\alpha_1^2+\ldots+\alpha_m^2)!\Big]^s \text{ since } s>1 \\
& \le 4^{|\alpha|}\Big[(\alpha_1^1+\ldots+\alpha_m^1+2\sum_{j=1}^m\alpha_j^2)!\Big]^s \text{ since } (a+b)!\ge a!b! \\
& = 4^{|\alpha|}(|\alpha|!)^s \\
& \le c^{|\alpha|+1}(\alpha!)^s \qquad (3.11)
\end{aligned}
$$

where in the last inequality, we used $(a+b)!\le 2^{a+b}a!b!$ repeatedly.

Finally, we need to estimate the number of terms that arise from the product in (3.5). For n a positive integer, let

$$S_n=\{(n_1,n_2): n_1+2n_2=n\}$$

where n_1,n_2 are nonnegative integers. If $(n_1,n_2)\in S_n$, then $n_2\le\frac{n}{2}$ and for each such n_2, there is at most one n_1 such that $(n_1,n_2)\in S_n$. Hence S_n has no more than $\frac{n}{2}$ elements. It follows that the product in (3.5) leads to no more than

$$
\begin{aligned}
(\frac{\alpha_1}{2})\ldots(\frac{\alpha_m}{2}) & \le|\alpha|^m \\
& \le m!e^{|\alpha|} \\
& \le c^{|\alpha|+1}, \qquad (3.12)
\end{aligned}
$$

where c is independent of α. From (3.11) and (3.12), we conclude that there is $c>0$ independent of α such that

$$\Big|M^\alpha g_1(Z(x))\Big|\le c^{|\alpha|+1}(\alpha!)^s,$$

and hence $g_1(Z(x))$ is in G^s near 0.

Thus if K' is a compact set whose interior contains 0, there is an almost analytic extension f_1 of g_1 such that $f_{1|Z(K')}=g_1$ and

$$\left|\frac{\partial f_1}{\partial\bar z_k}(z)\right|\le A\exp\left(\frac{-B}{\Big(dist(z,Z(V))\Big)^{\frac{1}{s-1}}}\right),\forall k=1,\ldots,m$$

for some $A,B>0, z\in Z_\#(V+i\Gamma_\delta^1)$.

In the sense of distributions, for each $j\ge 2$,

$$\lim_{\Gamma^j\ni y\to 0}f_j(Z_\#(x+iy))=\lim_{\epsilon\to 0^+}f_j^\epsilon(Z(x)),$$

and

$$\lim_{\Gamma^1\ni y\to 0}f_1(Z_\#(x+iy))=\lim_{\epsilon\to 0^+}g_1^\epsilon(Z(x)).$$

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Thus

$$u_{\tilde{K},0} = \lim_{\epsilon\to 0^+} \sum_{j=1}^{N} f_j^{\epsilon}(Z(x)) = \lim_{y\to 0} \sum_{j=1}^{N} f_j(Z_{\#}(x+iy)).$$

Now we prove that $u^{\epsilon}_{\tilde{K},1}(z)$ converges to a holomorphic function. However, by wellposedness, for $z' \in \tilde{\Omega} \setminus \Omega'$ and $z \in Z(V)$, (see [8]), we have

$$\Re\bigg(i\zeta.(z-z') - \langle\zeta\rangle[z-z']^2\bigg) \leq -(1-\kappa)|\zeta||z-z'|^2 \leq -c_0 < 0$$

and since we may suppose that ϵ go to zero in the integral for $u^{\epsilon}_{\tilde{K},1}(z)$, thus we obtain

$$u_{\tilde{K},1}(z) = \frac{1}{(2\pi^3)^{\frac{m}{2}}} \int\int_{\tilde{\Omega}\setminus\Omega'} e^{i\zeta.(z-z')-\langle\zeta\rangle[z-z']^2} \mathcal{F}_u(z',\zeta)\langle\zeta\rangle^{\frac{m}{2}} dz' d\zeta,$$

which defines a holomorphic function.
Therefore, u is microlocally G^s regular at $(0, \xi^0)$. □

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References

[1] Z. Adwan, G. Hoepfner, A generalization of Borel's theorem and microlocal Gevrey regularity in involutive structures, *J. Differential Equations* **245** (2008), no. 10, 2846–2870.

[2] M. S. Baouendi, C. H. Chang and F. Treves, Microlocal hypo-analyticity and extension of CR functions, *J. Differential. Geom.* **18** (1983), no. 3, 331–391.

[3] R. F. Barostichi and G. Petronilho, Gevrey micro-regularity for solutions to first order nonlinear PDE, *J. Differential Equations* **247** (2009), no. 6, 1899–1914.

[4] S. Berhanu, P. D. Cordaro, and J. Hounie, An introduction to involutive structures, Cambridge University Press, Cambridge, 2008.

[5] S. Berhanu and Abraham Hailu, Characterization of Gevrey regularity by a class of FBI transforms, Springer -Verlag (Birkhauser), Novel Methods in Harmonic Analysis, **2** (2017) 451–482.

[6] S. Berhanu and J. Hounie, A class of FBI transforms, *Comm. Partial Differential Equations* **37** (2012), no. 1, 38–57.

[7] M. Christ, Intermediate optimal Gevrey exponents occur, *Comm. Partial Differential Equations,* **22** (1997), no. 3-4, 359–379.

[8] P. D. Cordaro and F. Treves,Hyperfunctions on hypo-analytic manifolds, Annals of Mathematics Studies, 136, Princeton University Press, Princeton, 1994.

[9] H. Komatsu, Microlocal analysis in Gevrey classes and complex domains, Microlocal Analysis and Applications, Lect. Notes in Math, 1495, Springer, Berlin, 1989.

[10] F. Treves, Hypo-analytic structures, Local theory, Princeton University Press, Princeton, 1992.

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