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SOME RESULTS ON ROUGH $\mathcal{I}_2$-LACUNARY STATISTICAL CONVERGENCE OF COMPLEX UNCERTAIN SEQUENCES

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ABSTRACT. This paper introduces rough $\mathcal{I}_2$-lacunary statistical convergence for complex uncertain double sequences (CUDS), extending concepts of rough convergence, rough lacunary statistical convergence, and rough $\mathcal{I}_2$-convergence. We explore this notion across four aspects of uncertainty: almost surely, measure, mean, and distribution. Additionally, we investigate rough $\mathcal{I}_2$-lacunary statistical convergence in p-distance and metric spaces for CUDS. The study illustrates the interconnectedness of these convergence concepts and offers observations on their relationships.

1. Introduction

Statistical convergence, first studied by Fast [18], was later extended to double sequences by Mursaleen and Edely [29] and Tripathy [41], to improve comprehension in the context of summability theory.

The notion of $\mathcal{I}$-convergence was initially improved by Kostyrko et al. [24] as an extension of statistical convergence. Das et al. [7] proposed the concept of $\mathcal{I}$-convergence for double sequences within a metric space, elucidating several characteristics of this convergence. Subsequently, Savaş and Das [36] expanded upon this concept to introduce $\mathcal{I}$-statistical convergence. Further exploration in this area can be observed in the research conducted by [19–21,30,37–39]. These subsequent studies delve deeper into the properties and applications of $\mathcal{I}$-convergence and $\mathcal{I}$-statistical convergence, contributing to the ongoing development of convergence theory.

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The concept of rough convergence originated with Phu [33] in the context of finite-dimensional normed spaces. Aytar [4] extended rough convergence to rough statistical convergence. Additional literature can be found in [26, 27]. Pal et al. [31] and Dündar and Çakan [15] further expanded the concept to rough $\mathcal{I}$-convergence, employing ideals of $\mathbb{N}$. Further exploration and application are documented in [1–3, 14, 16, 17, 22, 28, 40]. These advancements mark significant contributions to the understanding of convergence theory, particularly in the realm of rough and $\mathcal{I}$-convergence.

In 2007, Liu [25] introduced uncertainty theory, a framework encompassing various forms of convergence for uncertain sequences, including convergence in measure, distribution, mean, and almost sure convergence. Peng [32] extended this theory to complex uncertain variables, followed by Chen et al. [5], who studied the convergence of complex uncertain sequences using these variables. Tripathy and Nath [42] proposed the statistical convergence of complex uncertain sequences within uncertainty theory in 2017. Subsequently, Debnath and Das [11, 12] introduced rough convergence and rough statistical convergence for complex uncertain sequences, leading to significant developments in the field. For further insights and developments, refer to [8–10, 13, 23, 34, 35, 43].

Building on the foundations laid by prior research, this paper introduces the concept of rough $\mathcal{I}_2$-lacunary statistical convergence for complex uncertain sequences across four dimensions of uncertainty: almost surely, measure, mean, and distribution. Additionally, we investigate the implications of rough $\mathcal{I}_2$-lacunary statistical convergence in p-distance and within the metric framework for complex uncertain sequences. Moreover, we aim to elucidate the interconnections between various forms of rough $\mathcal{I}_2$-lacunary statistical convergence for complex uncertain sequences through a diagrammatic representation. Through these explorations, we contribute to the evolving discourse on convergence theory in uncertain contexts, offering insights into the behavior and relationships of complex uncertain sequences under different convergence criteria.

2. Preliminaries

In this section, we give significant existing conceptions and results which are crucial for our results.

During the paper, r is a nonnegative real number and $\mathbb{R}^n$ denotes the real n-dimensional space with the norm $\|.\|$. new line Consider a sequence $x = (x_i) \subset \mathbb{R}^n$.

The sequence $x = (x_i)$ is said to be r-convergent to x_*, denoted by $x_i \xrightarrow{r} x_*$ provided that

$$\forall \varepsilon > 0 \ \exists i_\varepsilon \in \mathbb{N} : \ \ i \geq i_\varepsilon \Rightarrow \|x_i - x_*\| < r + \varepsilon.$$

The set

$$\mathrm{LIM}^r x := \{x_* \in \mathbb{R}^n : x_i \xrightarrow{r} x_*\}$$

is called the r-limit set of the sequence $x = (x_i)$. A sequence $x = (x_i)$ is said to be r-convergent if $\mathtt{LIM}^r x \neq \emptyset$. In this case, r is called the convergence degree of the sequence $x = (x_i)$. For $r = 0$, we get the ordinary convergence. There are several reasons for this interest (see [33]).

On applying the notion of ideals, Kostyrko et al. [24] determined the concept of $\mathcal{I}$ and $\mathcal{I}^*$-convergence.

Let $Y \neq \emptyset$. $\mathcal{I} \subset 2^Y$ is called an ideal on Y provided that

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(a) for all $S, T \in \mathcal{I}$ implies $S \cup T \in \mathcal{I}$;

(b) for all $S \in \mathcal{I}$ and $T \subset S$ implies $T \in \mathcal{I}$.

Let $Y \neq \emptyset$. $\mathcal{F} \subset 2^Y$ is named a filter on Y provided that

(a) for all $S, T \in \mathcal{F}$ implies $S \cap T \in \mathcal{F}$;

(b) for all $S \in \mathcal{F}$ and $T \supset S$ implies $T \in \mathcal{F}$.

An ideal $\mathcal{I}$ is known as non-trivial provided that $Y \notin \mathcal{I}$ and $\mathcal{I} \neq \emptyset$. A non-trivial ideal $\mathcal{I} \subset P(Y)$ is known as an admissible ideal in Y iff $\mathcal{I} \supset \{\{w\} : w \in Y\}$. $\mathcal{I}_d$ is defined as the set of all subsets of $\mathbb{N}$ whose natural density is zero forms a non-trivial admissible ideal. Then, the filter $\mathcal{F} = \mathcal{F}(\mathcal{I}) = \{Y - S : S \in \mathcal{I}\}$ is called the filter connected with the ideal.

A nontrivial ideal $\mathcal{I}_2$ of $\mathbb{N} \times \mathbb{N} = \mathbb{N}^2$ is called strongly admissible (also admissible ideal) when $\{i\} \times \mathbb{N}$ and $\mathbb{N} \times \{i\}$ belong to $\mathcal{I}_2$ for each $i \in \mathbb{N}$.

Let

$$\mathcal{I}_2^0 = \left\{K \subseteq \mathbb{N}^2 : (\exists m\,(K) \in \mathbb{N}) \ \ (\exists i, j \geq m\,(K) \Rightarrow (i,j) \notin K)\right\}.$$

Then, $\mathcal{I}_2^0$ is a nontrivial strongly admissible ideal and obviously an ideal $\mathcal{I}_2$ is strongly admissible iff $\mathcal{I}_2^0 \subseteq \mathcal{I}_2$.

Let $x = (x_{mn})$ be a double sequence in a normed space $(X, \|.\|)$ and r be a non negative real number. x is said to be r-statistically convergent to ξ, denoted by $x \xrightarrow{r-st_2} \xi$, if for $\varepsilon > 0$ we have $d(A(\varepsilon)) = 0$, where $A(\varepsilon) = \{(m,n) \in \mathbb{N} \times \mathbb{N} : \|x_{mn} - \xi\| \geq r + \varepsilon\}$.

Let (X, ρ) be a metric space. A double sequence $x = (x_{mn})$ in X is said to be $\mathcal{I}_2$-convergent to $L \in X$, if for any $\varepsilon > 0$ we have $A(\varepsilon) = \{(m,n) \in \mathbb{N} \times \mathbb{N} : \rho(x_{mn}, L) \geq \varepsilon\} \in \mathcal{I}_2$. In this case, we say that x is $\mathcal{I}_2$-convergent and we write

$$\mathcal{I}_2 - \lim_{m,n\to\infty} x_{mn} = L.$$

A double sequence $x = (x_{mn})$ is said to be rough convergent (r-convergent) to x_* with the roughness degree r, denoted by $x_{mn} \xrightarrow{r} x_*$ provided that

$$\forall \varepsilon > 0 \ \ \exists k_\varepsilon \in \mathbb{N} : \ \ m, n \geq k_\varepsilon \Rightarrow \|x_{mn} - x_*\| < r + \varepsilon,$$

or equivalently, if

$$\limsup \|x_{mn} - x_*\| \leq r.$$

A double sequence $x = (x_{mn})$ is said to be r-$\mathcal{I}_2$-convergent to x_* with the roughness degree r, denoted by $x_{mn} \xrightarrow{r-\mathcal{I}_2} x_*$ provided that

$$\{(m,n) \in \mathbb{N} \times \mathbb{N} : \|x_{mn} - x_*\| \geq r + \varepsilon\} \in \mathcal{I}_2,$$

for every $\varepsilon > 0$; or equivalently, if the condition

$$\mathcal{I}_2 - \limsup \|x_{mn} - x_*\| \leq r$$

is satisfied.

Now, we give the definition of $\mathcal{I}_2$-asymptotic density of $\mathbb{N}^2$.

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A subset $K \subset \mathbb{N} \times \mathbb{N}$ is said to have $\mathcal{I}_2$-asymptotic density, $d_{\mathcal{I}_2}(K)$, if

$$d_{\mathcal{I}_2}(K) = \mathcal{I}_2 - \lim_{m,n\to\infty} \frac{|K(m,n)|}{m.n},$$

where $K(m,n) = \{(j,k) \in \mathbb{N} \times \mathbb{N} : j \le m, k \le n; (j,k) \in K\}$ and $|K(m,n)|$ denotes the number of elements of the set $K(m,n)$.

A double sequence $x = \{x_{jk}\}$ of real numbers is $\mathcal{I}_2$-statistically convergent to ξ, and we write $x \overset{\mathcal{I}_2-st}{\rightarrow} \xi$, if for any $\varepsilon > 0$ and $\delta > 0$

$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N}: \frac{1}{mn} |\{(j,k) : \|x_{jk} - \xi\| \ge \varepsilon,\ j \le m, k \le n\}| \ge \delta\right\} \in \mathcal{I}_2.$$

Let $x = \{x_{jk}\}$ be a double sequence in a normed linear space $(X, \|.\|)$ and r be a non negative real number. Then, x is said to be rough $\mathcal{I}_2$-statistical convergent to ξ or r-$\mathcal{I}_2$-statistical convergent to ξ, if for any $\varepsilon > 0$ and $\delta > 0$

$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N}: \frac{1}{mn} |\{(j,k),\ j \le m, k \le n : \|x_{jk} - \xi\| \ge r + \varepsilon\}| \ge \delta\right\} \in \mathcal{I}_2.$$

In this case, ξ is called the rough $\mathcal{I}_2$-statistical limit of $x = \{x_{jk}\}$ and we denote it by $x \overset{r-\mathcal{I}_2-st}{\longrightarrow} \xi$.

The double sequence $\theta = \{\theta_{mn}\} = (k_{mn})$ is called a double lacunary sequence if $k_{00} = 0$, $h_{m,n} = k_{m,n} - k_{m-1,n-1} \to \infty$, as m, n tend to ∞, independent of each other.

A double sequence $x = \{x_{jk}\}$ of numbers is said to be $\mathcal{I}_2$-lacunary statistical convergent or $S_{\theta_2}(\mathcal{I}_2)$-convergent to x_0, if for each $\rho > 0$ and $\sigma > 0$,

$$\left\{(m,n) \in \mathbb{N}^2 : \frac{1}{h_{mn}} |\{(j,k) \in I_{mn} : |x_{jk} - x_0| \ge \rho\}| \ge \sigma\right\} \in \mathcal{I}_2,$$

where $I_{mn} = (k_{m-1,n-1}, k_{m,n}]$. In this case, we write $x_{jk} \to x_0\, (S_{\theta_2}(\mathcal{I}_2))$ or $S_{\theta_2}(\mathcal{I}_2)$-$\lim_{j,k\to\infty} x_{jk} = x_0$.

Definition 2.1 ([25]). *Let $\mathcal{P}$ be a σ-algebra on a nonempty set Υ. A set function $\mathcal{Y}$ on Υ is called an uncertain measure, if it satisfies the following axioms:*

Axiom 1 (Normality): $\mathcal{Y}(\Upsilon) = 1$;

Axiom 2 (Duality): $\mathcal{Y}(\Theta) + \mathcal{Y}(\Theta^c) = 1$ for any $\Theta \in \mathcal{P}$;

(Subadditivity): For every countable sequence of $\{\Theta_j\} \in \mathcal{P}$,

$$\mathcal{Y}\left(\bigcup_{j=1}^{\infty} \Theta_j\right) \le \bigcup_{j=1}^{\infty} \mathcal{Y}(\Theta_j).$$

Each element Θ in $\mathcal{P}$ is called an event and triplet $(\Upsilon, \mathcal{P}, \mathcal{Y})$ is called an uncertainty space.

Definition 2.2 ([25]). *A variable $\varpi = \xi + \mathrm{i}\eta$ from an uncertainty space $(\Upsilon, \mathcal{P}, \mathcal{Y})$ to the set of complex numbers is a complex uncertain variable if and only if ξ and η are uncertain variables, where ξ and η are the real and imaginary parts of ϖ, respectively.*

Definition 2.3 ([32]). *Let $\varpi = \xi + \mathrm{i}\eta$ be a complex uncertain variable, where ξ and η are real and imaginary part of ϖ, respectively. Then, the complex uncertainty distribution of ϖ is a function from $\mathbb{C}$ to $[0,1]$ defined by $\Phi(z) = \mathcal{Y}\{\xi \le s, \eta \le t\}$ for any complex number $z = s + \mathrm{i}t$.*

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Definition 2.4 ([32])**.** *Let $\varpi = \xi + \mathrm{i}\eta$ be a complex uncertain variable. If the expected value of ξ and η i.e., $E[\xi]$ and $E[\eta]$ exists, then the expected value of ϖ is defined by*

$$E[\varpi] = E[\xi] + \mathrm{i}E[\eta].$$

Definition 2.5 ([34])**.** *Let ϖ and ϖ^* be two complex uncertain variables. Then, the p-distance between them is defined as*

$$d_p(\varpi, \varpi^*) = \left(E\left[\|\varpi - \varpi^*\|^p\right]\right)^{\frac{1}{p+1}}, \ p > 0.$$

Definition 2.6 ([35])**.** *A complex uncertain sequence (ϖ_s) is considered statistically convergent in p-distance to ϖ if*

$$\lim_{s\to\infty} \frac{1}{s} \left| \left\{ u \leq s : \left(E\left[\|\varpi_u - \zeta\|^p\right]\right)^{\frac{1}{p+1}} \geq \rho \right\} \right| = 0 \text{ for each } \rho > 0.$$

Definition 2.7 ([6])**.** *Let ϖ and ϖ^* be two complex uncertain variables, then the metric between them is defined as follows*

$$D(\varpi, \varpi^*) = \inf\left\{t : \mathcal{X}\left\{\|\varpi - \varpi^*\| \leq t\right\} = 1\right\}.$$

Definition 2.8 ([6])**.** *If the condition $\lim_{n\to\infty} D(\varpi_n, \varpi) = 0$ is hold for a complex uncertain sequence (ϖ_n), then (ϖ_n) is called convergent in metric to ϖ.*

3. Main Results

Definition 3.1. *A CUDS (ϖ_{uv}) is considered to be rough $\mathcal{I}_2$-convergent almost surely to ϖ, if for every small positive value σ, and for any event Θ where $\mathcal{Y}\{\Theta\} = 1$ we have the following condition satisfied for every element $\tau \in \Theta$:*

$$\left\{(u,v) \in \mathbb{N}^2 : \|\varpi_{uv}(\tau) - \varpi(\tau)\| \geq r + \sigma\right\} \in \mathcal{I}_2,$$

where r is called the roughness degree.

Definition 3.2. *A CUDS (ϖ_{uv}) is deemed to be rough lacunary statistically convergent almost surely to ϖ, if for every small positive value σ, and for any event Θ where $\mathcal{Y}\{\Theta\} = 1$, the following condition holds for every element $\tau \in \Theta$:*

$$\lim_{m,n\to\infty} \frac{1}{h_{mn}} \left|\left\{(u,v) \in I_{mn} : \|\varpi_{uv}(\tau) - \varpi(\tau)\| \geq r + \sigma\right\}\right| = 0,$$

where r is termed as the roughness degree.

Definition 3.3. *A CUDS (ϖ_{uv}) is deemed to be rough $\mathcal{I}_2$-lacunary statistically convergent almost surely to ϖ , if for every small positive value σ and κ, and for any event Θ with $\mathcal{Y}\{\Theta\} = 1$, the following condition holds for every element $\tau \in \Theta$:*

$$\left\{(m,n) \in \mathbb{N}^2 : \frac{1}{h_{mn}} \left|\left\{(u,v) \in I_{mn} : \|\varpi_{uv}(\tau) - \varpi(\tau)\| \geq r + \sigma\right\}\right| \geq \kappa \right\} \in \mathcal{I}_2,$$

where r represents the roughness degree. If we take $r = 0$ we obtain the notion of $\mathcal{I}_2$-lacunary statistical convergence almost surely of CUDS.

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Definition 3.4. *A CUDS (ϖ_{uv}) is termed as rough $\mathcal{I}_2$-lacunary statistically convergent in measure to ϖ if, for every given small positive values ρ, σ and κ, there exists a set satisfying the condition*

$$\left\{(m,n)\in\mathbb{N}^2 : \frac{1}{h_{mn}}\left|\{(u,v)\in I_{mn} : \mathcal{Y}\left(\|\varpi_{uv}-\varpi\|\geq\rho\right)\geq r+\sigma\}\right|\geq\kappa\right\}\in\mathcal{I}_2,$$

where r is referred to as the roughness degree.

Definition 3.5. *A CUDS (ϖ_{uv}) is regarded as rough $\mathcal{I}_2$-lacunary statistically convergent in mean to ϖ , if for every given small positive values σ and κ, there exists a set that fulfills the condition*

$$\left\{(m,n)\in\mathbb{N}^2 : \frac{1}{h_{mn}}\left|\{(u,v)\in I_{mn} : E\left[\|\varpi_{uv}-\varpi\|\right]\geq r+\sigma\}\right|\geq\kappa\right\}\in\mathcal{I}_2,$$

where r is termed as the roughness degree.

Definition 3.6. *Let Φ,Φ_{uv} represent the complex uncertainty distributions of complex uncertain variables ϖ,ϖ_{uv}, respectively. The CUDS (ϖ_{uv}) is considered rough $\mathcal{I}_2$-lacunary statistically convergent in distribution to ϖ , if for any given small positive values σ and κ, there exists a set satisfies the condition:*

$$\left\{(m,n)\in\mathbb{N}^2 : \frac{1}{h_{mn}}\left|\{(u,v)\in I_{mn} : \|\Phi_{uv}(z)-\Phi(z)\|\geq r+\sigma\}\right|\geq\kappa\right\}\in\mathcal{I}_2,$$

where r is termed as the roughness degree, and this holds for all z at which $\Phi(z)$ is continuous.

Theorem 3.7. *Consider a CUDS (ϖ_{uv}) having real part and imaginary part α_{uv} and β_{uv}, respectively. If the uncertain double sequences (α_{uv}) and (β_{uv}) rough $\mathcal{I}_2$-lacunary statistically converges in measure to α and β, respectively, then the CUDS (ϖ_{uv}) also rough $\mathcal{I}_2$-lacunary statistically converges in measure to $\varpi=\alpha+i\beta$.*

Proof. Let the uncertain sequences (α_{uv}) and (β_{uv}) be rough $\mathcal{I}_2$-lacunary statistically convergent in measure to α and β respectively. The definition implies that for any arbitrarily small positive values ρ, σ and κ,

$$\left\{(m,n)\in\mathbb{N}^2 : \tfrac{1}{h_{mn}}\left|\left\{(u,v)\in I_{mn} : \mathcal{Y}\left(\|\alpha_{uv}-\alpha\|\geq\tfrac{\rho}{\sqrt{2}}\right)\geq\tfrac{r+\sigma}{\sqrt{2}}\right\}\right|\geq\kappa\right\}\in\mathcal{I}_2,$$

$$\left\{(m,n)\in\mathbb{N}^2 : \tfrac{1}{h_{mn}}\left|\left\{(u,v)\in I_{mn} : \mathcal{Y}\left(\|\beta_{uv}-\beta\|\geq\tfrac{\rho}{\sqrt{2}}\right)\geq\tfrac{r+\sigma}{\sqrt{2}}\right\}\right|\geq\kappa\right\}\in\mathcal{I}_2.$$

It should be noted that $\|\varpi_{uv}-\varpi\|=\sqrt{|\alpha_{uv}-\alpha|^2+|\beta_{uv}-\beta|^2}$.
Thus, we have

$$\{\|\varpi_{uv}-\varpi\|\geq\rho\}\subset\left\{|\alpha_{uv}-\alpha|\geq\tfrac{\rho}{\sqrt{2}}\right\}\cup\left\{|\beta_{uv}-\beta|\geq\tfrac{\rho}{\sqrt{2}}\right\}.$$

$$\Rightarrow\mathcal{Y}\{\|\varpi_{uv}-\varpi\|\geq\rho\}\leq\mathcal{Y}\left\{\|\alpha_{uv}-\alpha\|\geq\tfrac{\rho}{\sqrt{2}}\right\}+\mathcal{Y}\left\{\|\beta_{uv}-\beta\|\geq\tfrac{\rho}{\sqrt{2}}\right\}.$$

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Then, for each $\sigma > 0$, $r \geq 0$

$$\{(u,v) \in I_{mn} : \mathcal{Y}(\|\varpi_{uv} - \varpi\| \geq \rho) \geq r + \sigma\}$$

$$\subseteq \left\{(u,v) \in I_{mn} : \mathcal{Y}\left(\|\alpha_{uv} - \alpha\| \geq \tfrac{\rho}{\sqrt{2}}\right) \geq \tfrac{r+\sigma}{2}\right\}$$

$$\cup \left\{(u,v) \in I_{mn} : \mathcal{Y}\left(\|\beta_{uv} - \beta\| \geq \tfrac{\rho}{\sqrt{2}}\right) \geq \tfrac{r+\sigma}{2}\right\}.$$

So we have

$$\tfrac{1}{h_{mn}} \left|\{(u,v) \in I_{mn} : \mathcal{Y}(\|\varpi_{uv} - \varpi\| \geq \rho) \geq r + \sigma\}\right|$$

$$\leq \tfrac{1}{h_{mn}} \left|\left\{(u,v) \in I_{mn} : \mathcal{Y}\left(\|\alpha_{uv} - \alpha\| \geq \tfrac{\rho}{\sqrt{2}}\right) \geq \tfrac{r+\sigma}{\sqrt{2}}\right\}\right|$$

$$+ \tfrac{1}{h_{mn}} \left|\left\{(u,v) \in I_{mn} : \mathcal{Y}\left(\|\beta_{uv} - \beta\| \geq \tfrac{\rho}{\sqrt{2}}\right) \geq \tfrac{r+\sigma}{\sqrt{2}}\right\}\right|.$$

Exploiting the subadditivity axiom of uncertain measure, we obtain

$$\left\{(m,n) \in \mathbb{N}^2 : \tfrac{1}{h_{mn}} \left|\{(u,v) \in I_{mn} : \mathcal{Y}(\|\varpi_{uv} - \varpi\| \geq \rho) \geq r + \sigma\}\right| \geq \kappa\right\}$$

$$\subseteq \left\{(m,n) \in \mathbb{N}^2 : \tfrac{1}{h_{mn}} \left|\left\{(u,v) \in I_{mn} : \mathcal{Y}\left(\|\alpha_{uv} - \alpha\| \geq \tfrac{\rho}{\sqrt{2}}\right) \geq \tfrac{r+\sigma}{\sqrt{2}}\right\}\right| \geq \kappa\right\}$$

$$\cup \left\{(m,n) \in \mathbb{N}^2 : \tfrac{1}{h_{mn}} \left|\left\{(u,v) \in I_{mn} : \mathcal{Y}\left(\|\beta_{uv} - \beta\| \geq \tfrac{\rho}{\sqrt{2}}\right) \geq \tfrac{r+\sigma}{\sqrt{2}}\right\}\right| \geq \kappa\right\} \in \mathcal{I}_2,$$

for every $\kappa > 0$. Thus, (ϖ_{uv}) is rough $\mathcal{I}_2$-lacunary statistically convergent in measure to ϖ. □

Theorem 3.8. *If a CUDS (ϖ_{uv}) is rough $\mathcal{I}_2$-lacunary statistically convergent in mean to ϖ, then it is rough $\mathcal{I}_2$-lacunary statistically convergent in measure to ϖ.*

Proof. Let (ϖ_{uv}) be rough $\mathcal{I}_2$-lacunary statistically convergent in mean to ϖ. Following the definition of rough $\mathcal{I}_2$-lacunary statistical convergence in mean of CUDS, it can be deduced that for each $\sigma, \kappa > 0$ and $r \geq 0$,

$$\left\{(m,n) \in \mathbb{N}^2 : \frac{1}{h_{mn}} \left|\{(u,v) \in I_{mn} : E[\|\varpi_{uv} - \varpi\|] \geq r + \sigma\}\right| \geq \kappa\right\} \in \mathcal{I}_2.$$

By applying the Markov inequality, it is clear that for given $\rho \geq 1$, we get

$$\mathcal{Y}\{\|\varpi_{uv} - \varpi\| \geq \rho\} \leq \frac{E[\|\varpi_{uv} - \varpi\|]}{\rho} \leq E[\|\varpi_{uv} - \varpi\|].$$

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For each $\sigma > 0$ and $r \geq 0$,

$$\{(u,v) \in I_{mn} : \mathcal{Y}\left(\|\varpi_{uv} - \varpi\| \geq \rho\right) \geq r + \sigma\}$$

$$\subseteq \{(u,v) \in I_{mn} : E\left[\|\varpi_{uv} - \varpi\|\right] \geq r + \sigma\}.$$

Therefore $\frac{1}{h_{mn}} \left|\{(u,v) \in I_{mn} : \mathcal{Y}\left(\|\varpi_{uv} - \varpi\| \geq \rho\right) \geq r + \sigma\}\right|$

$$\leq \frac{1}{h_{mn}} \left|\{(u,v) \in I_{mn} : E\left[\|\varpi_{uv} - \varpi\|\right] \geq r + \sigma\}\right|.$$

For all $\kappa > 0$,

$$\left\{(m,n) \in \mathbb{N}^2 : \frac{1}{h_{mn}} \left|\{(u,v) \in I_{mn} : \mathcal{Y}\left(\|\varpi_{uv} - \varpi\| \geq \rho\right) \geq r + \sigma\}\right| \geq \kappa\right\}$$

$$\subseteq \left\{(m,n) \in \mathbb{N}^2 : \frac{1}{h_{mn}} \left|\{(u,v) \in I_{mn} : E\left[\|\varpi_{uv} - \varpi\|\right] \geq r + \sigma\}\right| \geq \kappa\right\} \in \mathcal{I}_2.$$

Thus, (ϖ_{uv}) is rough $\mathcal{I}_2$-lacunary statistically convergent in measure to ϖ. □

Remark 3.9. *The converse of the above theorem is not true in general.*

Example 3.10. *Consider the uncertainty space $(\Upsilon, \mathcal{P}, \mathcal{Y})$ to be $\{\tau_1, \tau_2, \cdots\}$ with power set and $\mathcal{Y}\{\Upsilon\} = 1$, $\mathcal{Y}\{\Theta\} = 0$ and*

$$\mathcal{Y}(\Theta) = \begin{cases} \sup_{\tau_{u+v} \in \Theta} \frac{u+v}{2(u+v)+1}, & \text{if } \sup_{\tau_{u+v} \in \Theta} \frac{u+v}{2(u+v)+1} < \frac{1}{2} \\ 1 - \sup_{\tau_{u+v} \in \Theta^c} \frac{u+v}{2(u+v)+1}, & \text{if } \sup_{\tau_{u+v} \in \Theta^c} \frac{u+v}{2(u+v)+1} < \frac{1}{2} \\ \frac{1}{2}, & \text{otherwise}, \end{cases}$$

for $u, v = 1, 2, ...$

Consider the double sequence of complex uncertain variables ϖ_{uv} as follows:

$$\varpi_{uv}(\tau) = \begin{cases} (u+v)\,\mathrm{i}, & \text{if } \tau = \tau_{u+v} \\ 0, & \text{otherwise} \end{cases} \quad \text{for } u, v = 1, 2, \cdots$$

and $\varpi \equiv 0$. Take $\mathcal{I}_2 = \mathcal{I}_2^d$. For any preassigned ρ, σ, $\kappa > 0$ and $u, v \geq 2$ we have

$$\left\{(m,n) \in \mathbb{N}^2 : \frac{1}{h_{mn}} \left|\{(u,v) \in I_{mn} : \mathcal{Y}\left(\|\varpi_{uv} - \varpi\| \geq \rho\right) \geq r + \sigma\}\right| \geq \kappa\right\}$$

$$= \left\{(m,n) \in \mathbb{N}^2 : \frac{1}{h_{mn}} \left|\{(u,v) \in I_{mn} : \mathcal{Y}\left(\tau : \|\varpi_{uv}(\tau) - \varpi(\tau)\| \geq \rho\right) \geq r + \sigma\}\right| \geq \kappa\right\}$$

$$= \left\{(m,n) \in \mathbb{N}^2 : \frac{1}{h_{mn}} \left|\{(u,v) \in I_{mn} : \mathcal{Y}\left\{\tau = \tau_{u+v}\right\} \geq r + \sigma\}\right| \geq \kappa\right\}$$

$$= \left\{(m,n) \in \mathbb{N}^2 : \frac{1}{h_{mn}} \left|\left\{(u,v) \in I_{mn} : \frac{u+v}{2(u+v)+1} \geq r + \sigma\right\}\right| \geq \kappa\right\} \in \mathcal{I}_2.$$

Thus, the CUDS (ϖ_{uv}) is rough $\mathcal{I}_2$-lacunary statistically convergent in measure to ϖ for $r = \frac{1}{2}$. However, for each u, v, we have the complex uncertainty distributions of uncertain variable $\|\varpi_{uv} - \varpi\|$

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is

$$\Phi_{uv}(t)=\begin{cases}0, & \text{if } t<0\\ 1-\frac{u+v}{2(u+v)+1}, & \text{if } 0\leq t<u+v \qquad \text{for } u,v=1,2,\cdots\\ 1, & \text{if } t\geq u+v\end{cases}$$

Now the expected value of $\|\varpi_{uv}-\varpi\|$ is calculated as

$$E\left[\|\varpi_{uv}-\varpi\|\right]=\int_0^{+\infty}\left(1-\Phi_{uv}(t)\right)dt=\int_0^{u+v}\frac{u+v}{2\left(u+v\right)+1}dt=\frac{\left(u+v\right)^2}{2\left(u+v\right)+1}.$$

As a result, for any given σ and κ both greater than zero, and $r=\frac{1}{2}$,

$$\left\{(m,n)\in\mathbb{N}^2:\tfrac{1}{h_{mn}}\left|\{(u,v)\in I_{mn}:E\left[\|\varpi_{uv}(\tau)-\varpi(\tau)\|\right]\geq r+\sigma\}\right|\geq\kappa\right\}$$

$$=\left\{(m,n)\in\mathbb{N}^2:\tfrac{1}{h_{mn}}\left|\left\{(u,v)\in I_{mn}:\tfrac{(u+v)^2}{2(u+v)+1}\geq r+\sigma\right\}\right|\geq\kappa\right\}\notin\mathcal{I}_2.$$

Hence, the CUDS (ϖ_{uv}) is not rough $\mathcal{I}_2$-statistically convergent in mean to ϖ for $r=\frac{1}{2}$.

Theorem 3.11. *Let (ξ_{uv}), (η_{uv}) be the real and imaginary part of a CUDS (ϖ_{uv}) are considered to be rough $\mathcal{I}_2$-statistical convergence in measure to ξ and η respectively. then (ϖ_{uv}) is deemed rough $\mathcal{I}_2$-lacunary statistically convergent in distribution to $\varpi=\xi+i\eta$.*

Proof. Let $z=s+it$ be a continuous point of the complex uncertainty distribution Φ. For any $\alpha>s$ and $\beta>t$, we have

$$\begin{aligned}\{\xi_{uv}\leq s,\eta_{uv}\leq t\} &=\{\xi_{uv}\leq s,\ \eta_{uv}\leq t,\ \xi\leq\alpha,\ \eta\leq\beta\}\\ &\cup\{\xi_{uv}\leq s,\ \eta_{uv}\leq t,\ \xi>\alpha,\eta>\beta\}\\ &\cup\{\xi_{uv}\leq s,\eta_{uv}\leq t,\xi\leq\alpha,\eta>\beta\}\\ &\cup\{\xi_{uv}\leq s,\eta_{uv}\leq t,\xi>\alpha,\eta\leq\beta\}\\ &\subset\{\xi\leq\alpha,\eta\leq\beta\}\cup\{|\xi_{uv}-\xi|\geq\alpha-s\}\\ &\cup\{|\eta_{uv}-\eta|\geq\beta-t\}.\end{aligned}$$

Applying the subadditivity axiom, we can infer

$$\begin{aligned}\Phi_{uv}(z) &=\Phi_{uv}(s+\mathrm{i}t)\leq\Phi(\alpha+\mathrm{i}\beta)+\mathcal{Y}\{|\xi_{uv}-\xi|\geq\alpha-s\}\\ &+\mathcal{Y}\{|\eta_{uv}-\eta|\geq\beta-t\}.\end{aligned}$$

Suppose that (ξ_{uv}) and (η_{uv}) are rough $\mathcal{I}_2$-lacunary statistically convergent in measure to ξ and η respectively, it follows that for any given σ,κ and $r\geq0$, we can deduce that

$$\left\{(m,n)\in\mathbb{N}^2:\frac{1}{h_{mn}}\left|\{(u,v)\in I_{mn}:\mathcal{Y}\left(|\xi_{uv}-\xi|\geq\alpha-s\right)\geq r+\sigma\}\right|\geq\kappa\right\}\in\mathcal{I}_2$$

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and

$$\left\{(m,n)\in\mathbb{N}^2:\frac{1}{h_{mn}}\left|\left\{(u,v)\in I_{mn}:\mathcal{Y}\left(|\eta_{uv}-\eta|\geq\beta-t\right)\geq r+\sigma\right\}\right|\geq\kappa\right\}\in\mathcal{I}_2.$$

Then, for any $\alpha>s$, $\beta>t$ and letting $\alpha+\mathrm{i}\beta\to s+\mathrm{i}t$, we have

$$\|\Phi_{uv}(z)-\Phi(z)\|\leq\mathcal{Y}\left\{|\xi_{uv}-\xi|\geq\alpha-s\right\}+\mathcal{Y}\left\{|\eta_{uv}-\eta|\geq\beta-t\right\}$$

Then, for each $\sigma>0$ and $r\geq0$

$$\{(u,v)\in I_{mn}:\|\Phi_{uv}(z)-\Phi(z)\|\geq r+\sigma\}$$

$$\subseteq\{(u,v)\in I_{mn}:\mathcal{Y}\{|\xi_{uv}-\xi|\geq\alpha-s\}\geq r+\sigma\}$$

$$\cup\{(u,v)\in I_{mn}:\mathcal{Y}\{|\eta_{uv}-\eta|\geq\beta-t\}\geq r+\sigma\}$$

Therefore $\frac{1}{h_{mn}}\left|\{(u,v)\in I_{mn}:\|\Phi_{uv}(z)-\Phi(z)\|\geq r+\sigma\}\right|$

$$\leq\frac{1}{h_{mn}}\left|\{(u,v)\in I_{mn}:\mathcal{Y}\{|\xi_{uv}-\xi|\geq\alpha-s\}\geq r+\sigma\}\right|$$

$$+\frac{1}{h_{mn}}\left|\{(u,v)\in I_{mn}:\mathcal{Y}\{|\eta_{uv}-\eta|\geq\beta-t\}\geq r+\sigma\}\right|$$

For every $\kappa>0$,

$$\left\{(m,n)\in\mathbb{N}^2:\frac{1}{h_{mn}}\left|\{(u,v)\in I_{mn}:\|\Phi_{uv}(z)-\Phi(z)\|\geq r+\sigma\}\right|\geq\kappa\right\}$$

$$\subseteq\left\{(m,n)\in\mathbb{N}^2:\frac{1}{h_{mn}}\left|\{(u,v)\in I_{mn}:\mathcal{Y}\{|\xi_{uv}-\xi|\geq\alpha-s\}\geq r+\sigma\}\right|\geq\kappa\right\}$$

$$\cup\left\{(m,n)\in\mathbb{N}^2:\frac{1}{h_{mn}}\left|\{(u,v)\in I_{mn}:\mathcal{Y}\{|\eta_{uv}-\eta|\geq\beta-t\}\geq r+\sigma\}\right|\geq\kappa\right\}\in\mathcal{I}_2.$$

Hence, the CUDS (ϖ_{uv}) is rough $\mathcal{I}_2$-lacunary statistically convergent in distribution to ϖ. □

Remark 3.12. *However, in general, the converse of the above theorem does not hold.*

Example 3.13. *Consider the uncertainty space $(\Upsilon,\mathcal{P},\mathcal{Y})$ to be $\{\tau_1,\tau_2\}$ with $\mathcal{X}(\tau_1)=\mathcal{X}(\tau_2)=\frac{1}{2}$. We consider a complex uncertain variable as*

$$\varpi(\tau)=\begin{cases}\mathrm{i}, & \text{if } \tau=\tau_1\\ -\mathrm{i}, & \text{if } \tau=\tau_2.\end{cases}$$

We also take $\varpi_{uv}=-\varpi$ for $u,v=1,2,\cdots$ and take $\mathcal{I}_2=\mathcal{I}_2^d$.
Thus, the sequence (ϖ_{uv}) and ϖ have the same distribution as:

$$\Phi_{uv}(z)=\Phi_{uv}(s+\mathrm{i}t)=\begin{cases}0, & \text{if } s<0,-\infty<t<+\infty\\ 0, & \text{if } s\geq0,t<-1\\ \frac{1}{2}, & \text{if } s\geq0,-1\leq t<1\\ 1, & \text{if } s\geq0,t\geq1.\end{cases}$$

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So, the CUDS (ϖ_{uv}) is rough $\mathcal{I}_2$-lacunary statistically convergent in distribution to ϖ.
However, for a given $\rho, \sigma, \kappa > 0$ and $r \geq 0$, we obtain

$$\left\{(m,n) \in \mathbb{N}^2 : \frac{1}{h_{mn}} \left|\{(u,v) \in I_{mn} : \mathcal{Y}\left(\|\varpi_{uv} - \varpi\| \geq \rho\right) \geq r + \sigma\}\right| \geq \kappa\right\} \notin \mathcal{I}_2.$$

Thus, the sequence (ϖ_{uv}) is not rough $\mathcal{I}_2$-lacunary statistically convergent in measure to ϖ for $r \geq \frac{1}{2}$.

Definition 3.14. *A CUDS (ϖ_{uv}) is defined as rough $\mathcal{I}_2$-lacunary statistically convergent in p-distance to ϖ, if for every $\sigma, \kappa > 0$ such that*

$$\left\{(m,n) \in \mathbb{N}^2 : \frac{1}{h_{mn}} \left|\left\{(u,v) \in I_{mn} : \left(E\left[\|\varpi_{uv} - \varpi\|^p\right]\right)^{\frac{1}{p+1}} \geq r + \sigma\right\}\right| \geq \kappa\right\} \in \mathcal{I}_2,$$

where r represents the roughness degree.

Theorem 3.15. *Let $\varpi, \varpi_1, \varpi_2, \cdots$ be complex uncertain variables defined on uncertainty space $(\Upsilon, \mathcal{P}, \mathcal{Y})$. Then, (ϖ_{uv}) is considered to be rough $\mathcal{I}_2$-lacunary statistically convergent in measure to ϖ, if it is rough $\mathcal{I}_2$-lacunary statistically convergent in p-distance to ϖ.*

Proof. Let the CUDS (ϖ_{uv}) be rough $\mathcal{I}_2$-lacunary statistically convergent in p distance to ϖ, then for every $\sigma, \kappa > 0$, we have

$$\left\{(m,n) \in \mathbb{N}^2 : \frac{1}{h_{mn}} \left|\left\{(u,v) \in I_{mn} : \left(E\left[\|\varpi_{uv} - \varpi\|^p\right]\right)^{\frac{1}{p+1}} \geq r + \sigma\right\}\right| \geq \kappa\right\} \in \mathcal{I}_2.$$

For any given ρ, $p > 0$, we have

$$\mathcal{Y}\left(\|\varpi_{uv} - \varpi\| \geq \rho\right) \leq \frac{E\left[\|\varpi_{uv} - \varpi\|^p\right]}{\rho^p}, \text{ (Applying Markov's inequality)}.$$

So, for each $\sigma > 0$ and $r \geq 0$

$$\begin{aligned}
&\{(u,v) \in I_{mn} : \mathcal{Y}\left(\|\varpi_{uv} - \varpi\| \geq \rho\right\} \geq r + \sigma\} \\
&\subseteq \left\{(u,v) \in I_{mn} : \tfrac{E\left[\|\varpi_{uv} - \varpi\|^p\right]}{\rho^p} \geq r + \sigma\right\} \\
&= \left\{(u,v) \in I_{mn} : \left(E\left[\|\varpi_{uv} - \varpi\|^p\right]\right)^{\frac{1}{p+1}} \geq \left[(r+\sigma) \cdot \rho^p\right]^{\frac{1}{p+1}}\right\} \\
&= \left\{(u,v) \in I_{mn} : \left(E\left[\|\varpi_{uv} - \varpi\|^p\right]\right)^{\frac{1}{p+1}} \geq r' + \sigma'\right\},
\end{aligned}$$

where $r' + \sigma' = \left[(r+\sigma) \cdot \rho^p\right]^{\frac{1}{p+1}}$.

$$\begin{aligned}
&\Rightarrow \tfrac{1}{h_{mn}} \left|\{(u,v) \in I_{mn} : \mathcal{Y}\left(\|\varpi_{uv} - \varpi\| \geq \rho\right) \geq r + \sigma\}\right| \\
&\leq \tfrac{1}{h_{mn}} \left|\left\{(u,v) \in I_{mn} : \left(E\left[\|\varpi_{uv} - \varpi\|^p\right]\right)^{\frac{1}{p+1}} \geq r' + \sigma'\right\}\right|.
\end{aligned}$$

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For every $\kappa > 0$,

$$\Big\{(m,n) \in \mathbb{N}^2 : \tfrac{1}{h_{mn}} \left|\{(u,v) \in I_{mn} : \mathcal{Y}\left(\|\varpi_{uv} - \varpi\| \geq \rho\right) \geq r + \sigma\}\right| \geq \kappa\Big\}$$

$$\subseteq \Big\{(m,n) \in \mathbb{N}^2 : \tfrac{1}{h_{mn}} \Big|\Big\{(u,v) \in I_{mn} : \left(E\left[\|\varpi_{uv} - \varpi\|^p\right]\right)^{\frac{1}{p+1}} \geq r' + \sigma'\Big\}\Big| \geq \kappa\Big\} \in \mathcal{I}_2.$$

Hence, the sequence (ϖ_{uv}) is rough $\mathcal{I}_2$-lacunary statistically convergent in measure to ϖ. □

Remark 3.16. *However, in general, the converse of the above theorem does not hold.*

Example 3.17. *Let* $\mathbb{N}^2 = \bigcup_{i,j=1}^{\infty,\infty} D_{i,j}$, *where*

$$\left\{D_{i,j} = \left(2^{i-1}s, 2^{j-1}t\right) : 2 \text{ does not divide } s \text{ and } t,\ s,t^* \in \mathbb{N}\right\}$$

be the decomposition of $\mathbb{N}^2$ *such that all* $D_{i,j}$ *are infinite and* $D_{ij} \cap D_{kl} = \Phi$, *for* $(i,j) \neq (k,l)$. *Let* $\mathcal{I}_2$ *be the class of all subsets of* $\mathbb{N}^2$ *that can intersect only finite number of* $D_{i,j}$ *'s. Then,* $\mathcal{I}_2$ *is a non-trivial admissible ideal of* $\mathbb{N}^2$.
Now, we consider the uncertainty space $(\Upsilon, \mathcal{P}, \mathcal{Y})$ *to be* $\{\tau_1, \tau_2, \cdots\}$ *with power set and* $\mathcal{Y}\{\Upsilon\} = 1$, $\mathcal{Y}\{\varnothing\} = 0$ *and*

$$\mathcal{Y}\{\Theta\} = \begin{cases} \sup_{\tau_{u+v} \in \Theta} \beta_{uv}, & \text{if } \sup_{\tau_{u+v} \in \Theta} \beta_{uv} < \frac{1}{2} \\ 1 - \sup_{\tau_{u+v} \in \Theta^c} \beta_{mn}, & \text{if } \sup_{\tau_{u+v} \in \Theta^c} \beta_{uv} < \frac{1}{2} \\ \frac{1}{2}, & \text{otherwise,} \end{cases}$$

where $\beta_{uv} = \frac{1}{(i+1)(j+1)}$, *if* $(u,v) \in D_{i,j}$ *for* $u,v = 1,2,\cdots$.
Also, the complex uncertain variables are defined by

$$\varpi_{uv}(\tau) = \begin{cases} (u+v+1)\mathrm{i}, & \text{if } \tau = \tau_{u+v} \\ 0, & \text{otherwise} \end{cases} \qquad \text{for } u,v = 1,2,\cdots$$

and $\varpi \equiv 0$.
For any $\rho > 0$ *and* $(u,v) \in \mathbb{N}^2 \setminus D_{1,1}$, *we have*

$$\mathcal{Y}\left(\{\tau \in \Upsilon : \|\varpi_{uv}(\tau) - \varpi(\tau)\| \geq \rho\}\right) = \mathcal{Y}\left(\tau_{u+v}\right) = \beta_{uv}.$$

Then, for every $\sigma > 0$ *and* $r \geq 0$, *we have*

$$\left\{(u,v) \in \mathbb{N}^2 : \mathcal{Y}\left(\{\tau \in \Upsilon : \|\varpi_{uv}(\tau) - \varpi(\tau)\| \geq \rho\}\right) \geq r + \sigma\right\}$$

$$= \left\{(u,v) \in \mathbb{N}^2 : \beta_{uv} \geq r + \sigma\right\} \in \mathcal{I}_2.$$

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Now

$$\{(u,v)\in I_{mn}:\mathcal{Y}\left(\{\tau\in\Upsilon:\|\varpi_{uv}(\tau)-\varpi(\tau)\|\geq\rho\}\right)\geq r+\sigma\}$$

$$\subseteq\left\{(u,v)\in\mathbb{N}^2:\mathcal{Y}\left(\{\tau\in\Upsilon:\|\varpi_{uv}(\tau)-\varpi(\tau)\|\geq\rho\}\right)\geq r+\sigma\right\}$$

So

$$\frac{1}{h_{mn}}\left|\{(u,v)\in I_{mn}:\mathcal{Y}\left(\{\tau\in\Upsilon:\|\varpi_{uv}(\tau)-\varpi(\tau)\|\geq\rho\}\right)\geq r+\sigma\}\right|$$

$$\leq\left|\left\{(u,v)\in\mathbb{N}^2:\mathcal{Y}\left(\{\tau\in\Upsilon:\|\varpi_{uv}(\tau)-\varpi(\tau)\|\geq\rho\}\right)\geq r+\sigma\right\}\right|$$

Then, for every $\kappa>0$, we have

$$\left\{(m,n)\in\mathbb{N}^2:\frac{1}{h_{mn}}\left|\{(u,v)\in I_{mn}:\mathcal{Y}\left(\{\tau\in\Upsilon:\|\varpi_{uv}(\tau)-\varpi(\tau)\|\geq\rho\}\right)\geq r+\sigma\}\right|\geq\kappa\right\}$$

$$\subseteq\left\{(u,v)\in\mathbb{N}^2:\mathcal{Y}\left(\{\tau\in\Upsilon:\|\varpi_{uv}(\tau)-\varpi(\tau)\|\geq\rho\}\right)\geq r+\sigma\right\}\in\mathcal{I}_2.$$

Therefore, the sequence (ϖ_{uv}) is rough $\mathcal{I}_2$-lacunary statistically convergent in measure to $\varpi\equiv 0$.
Now for $p>0$, we have

$$\|\varpi_{uv}(\tau)-\varpi(\tau)\|^p=\begin{cases}(u+v+1)^p, & \text{if } \tau=\tau_{u+v}\\ 0, & \text{otherwise}\end{cases}\qquad \text{for } u,v=1,2,\cdots.$$

Then, for each $(u,v)\in\mathbb{N}^2\setminus D_{1,1}$, we have the uncertainty distribution of uncertain variable $\|\varpi_{uv}(\tau)-\varpi(\tau)\|^p$ is

$$\Phi_{uv}(t)=\begin{cases}0, & \text{if } t<0\\ 1-\beta_{uv}, & \text{if } 0\leq t<(u+v+1)^p\\ 1, & \text{if } t\geq(u+v+1)^p\end{cases}\qquad \text{for } u,v=1,2,\cdots \text{ and } p>0$$

So for $(u,v)\in\mathbb{N}^2\setminus D_{1,1}$, we have

$$E\left[\|\varpi_{uv}-\varpi\|^p\right]=\int_0^{(u+v+1)^p}\left(1-(1-\beta_{uv})\right)dt=(u+v+1)^p\beta_{uv}$$

$$\Rightarrow\left(E\left[\|\varpi_{uv}-\varpi\|^p\right]\right)^{\frac{1}{p+1}}=\left((u+v+1)^p\beta_{uv}\right)^{\frac{1}{p+1}}.$$

Then, for any $\sigma,\kappa>0$ and $r\geq 0$, we have

$$\left\{(m,n)\in\mathbb{N}^2:\frac{1}{h_{mn}}\left|\left\{(u,v)\in I_{mn}:\left(E\left[\|\varpi_{uv}-\varpi\|^p\right]\right)^{\frac{1}{p+1}}\geq r+\sigma\right\}\right|\geq\kappa\right\}$$

$$=\left\{(m,n)\in\mathbb{N}^2:\frac{1}{h_{mn}}\left|\left\{(u,v)\in I_{mn}:\left((u+v+1)^p\beta_{uv}\right)^{\frac{1}{p+1}}\geq r+\sigma\right\}\right|\geq\kappa\right\}\notin\mathcal{I}_2.$$

Hence, the sequence (ϖ_{uv}) is not rough $\mathcal{I}_2$-lacunary statistically convergent in p-distance to $\varpi\equiv 0$.

Theorem 3.18. *Let $\varpi,\varpi_1,\varpi_2,\cdots$ be complex uncertain variables defined on uncertainty space $(\Upsilon,\mathcal{P},\mathcal{Y})$. Then (ϖ_{uv}) is rough $\mathcal{I}_2$-lacunary statistically convergent in distribution to ϖ if it is rough $\mathcal{I}_2$-lacunary statistically convergent in p-distance to ϖ.*

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Proof. If the sequence (ϖ_{uv}) displays rough $\mathcal{I}_2$-lacunary statistically convergent in p-distance to ϖ, then, as per Theorems 3.11 and 3.15, it also shows rough $\mathcal{I}_2$-lacunary statistically convergent in distribution to the same limit ϖ. □

Remark 3.19. *However, in general, the converse of the above theorem does not hold.*

Example 3.20. *In example 3.17, the complex uncertainty distributions of (ϖ_{uv}) are*

$$\Phi_{uv}(z) = \Phi_{uv}(s+\mathrm{i}t) = \begin{cases} 0, & \text{if } s<0,\ t<\infty \\ 0, & \text{if } s\geq 0,\ t<0 \\ 1-\beta_{uv}, & \text{if } s\geq 0,\ 0\leq t<(u+v+1) \\ 1, & \text{if } s\geq 0,\ t\geq (u+v+1) \end{cases}$$

for $u,v = 1,2,\cdots$ and the complex uncertainty distributions of ϖ is

$$\Phi(z) = \Phi(s+\mathrm{i}t) = \begin{cases} 0, & \text{if } s<0,\ t<\infty \\ 0, & \text{if } s\geq 0,\ t<0 \\ 1, & \text{if } s\geq 0,\ t\geq 0. \end{cases}$$

It can be demonstrated that the CUDS (ϖ_{uv}) is rough $\mathcal{I}_2$-lacunary convergent in distribution to $\varpi\equiv 0$. However, it is not rough $\mathcal{I}_2$-lacunary convergent in p-distance to $\varpi\equiv 0$.

Definition 3.21. *A CUDS (ϖ_{uv}) is said to be rough $\mathcal{I}_2$-lacunary statistically convergent in metric to ϖ, if for every $\sigma,\kappa>0$ such that*

$$\left\{(m,n)\in\mathbb{N}^2 : \frac{1}{h_{mn}}\left|\left\{(u,v)\in I_{mn} : D\left(\varpi_{uv},\varpi\right)\geq r+\sigma\right\}\right|\geq\kappa\right\}\in\mathcal{I}_2.$$

Theorem 3.22. *Let $\varpi,\varpi_1,\varpi_2,\cdots$ be complex uncertain variables defined on uncertainty space $(\Upsilon,\mathcal{P},\mathcal{Y})$. Then (ϖ_{uv}) is rough $\mathcal{I}_2$-lacunary statistically convergent in mean to ϖ, if it is rough $\mathcal{I}_2$-lacunary statistically convergent in metric to ϖ.*

Proof. Let the CUDS (ϖ_{uv}) be rough $\mathcal{I}_2$-lacunary statistically convergent in metric to ϖ, then for each $\sigma,\kappa>0$ and $r\geq 0$ we obtain

$$\left\{(m,n)\in\mathbb{N}^2 : \frac{1}{h_{mn}}\left|\left\{(u,v)\in I_{mn} : D\left(\varpi_{uv},\varpi\right)\geq r+\sigma\right\}\right|\geq\kappa\right\}\in\mathcal{I}_2,$$

where

$$D\left(\varpi_{uv},\varpi\right) = \inf\left\{t : \mathcal{Y}\left\{\|\varpi_{uv}-\varpi\|\leq t\right\}=1\right\}.$$

Let $D\left(\varpi_{uv},\varpi\right)=D$ and $\Phi_{uv}(t)$ denote the complex uncertainty distributions of the uncertain variable $\|\varpi_{uv}-\varpi\|$. Then, we get

$$D\left(\varpi_{uv},\varpi\right) = \inf\left\{t : \Phi_{uv}(t)=1\right\}.$$

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For any $\rho' > 0$,

$$E\left[\|\varpi_{uv}-\varpi\|\right] = \int_0^{+\infty}\left(1-\Phi_{uv}(t)\right)dt = \int_0^{D+\rho'}\left(1-\Phi_{uv}(t)\right)dt + \int_{D+\rho'}^{+\infty}\left(1-\Phi_{uv}(t)\right)dt$$

$$= \int_0^{D+\rho'}\left(1-\Phi_{uv}(t)\right)dt < 1\cdot(D+\rho') = D+\rho',$$

and we have

$$E\left[\|\varpi_{uv}-\varpi\|\right] \leq D \Rightarrow E\left[\|\varpi_{uv}-\varpi\|\right] \leq D\left(\varpi_{uv},\varpi\right).$$

So, for each $\sigma > 0$ and $r \geq 0$,

$$\left\{(u,v)\in I_{mn} : E\left[\|\varpi_{uv}-\varpi\|\right] \geq r+\sigma\right\} \subseteq \left\{(u,v)\in I_{mn} : D\left(\varpi_{uv},\varpi\right) \geq r+\sigma\right\}.$$

So

$$\frac{1}{h_{mn}}\left|\left\{(u,v)\in I_{mn} : E\left[\|\varpi_{uv}-\varpi\|\right] \geq r+\sigma\right\}\right| \leq \frac{1}{h_{mn}}\left|\left\{(u,v)\in I_{mn} : D\left(\varpi_{uv},\varpi\right) \geq r+\sigma\right\}\right|.$$

Then, for all $\kappa > 0$,

$$\left\{(m,n)\in\mathbb{N}^2 : \frac{1}{h_{mn}}\left|\left\{(u,v)\in I_{mn} : E\left[\|\varpi_{uv}-\varpi\|\right] \geq r+\sigma\right\}\right| \geq \kappa\right\}$$

$$\subseteq \left\{(m,n)\in\mathbb{N}^2 : \frac{1}{h_{mn}}\left|\left\{(u,v)\in I_{mn} : D\left(\varpi_{uv},\varpi\right) \geq r+\sigma\right\}\right| \geq \kappa\right\} \in \mathcal{I}_2.$$

Hence, the CUDS (ϖ_{uv}) is rough $\mathcal{I}_2$-lacunary statistically convergent in mean to ϖ. □

Remark 3.23. *However, in general, the converse of the above theorem does not hold.*

Example 3.24. *Consider the uncertainty space $(\Upsilon,\mathcal{P},\mathcal{Y})$ to be $\{\tau_1,\tau_2,\cdots\}$ with power set and $\mathcal{Y}\{\Upsilon\} = 1$, $\mathcal{Y}\{\Phi\} = 0$ and also, the complex uncertain variables are defined by*

$$\mathcal{Y}\{\Theta\} = \begin{cases} \sup_{\tau_{u+v}\in\Theta}\frac{(u+v)\beta_{uv}}{2(u+v)+1}, & \text{if } \sup_{\tau_{u+v}\in\Theta}\frac{(u+v)\beta_{uv}}{2(u+v)+1} < \frac{1}{2} \\ 1-\sup_{\tau_{u+v}\in\Theta^c}\frac{(u+v)\beta_{uv}}{2(u+v)+1}, & \text{if } \sup_{\tau_{u+v}\in\Theta^c}\frac{(u+v)\beta_{uv}}{2(u+v)+1} < \frac{1}{2} \\ \frac{1}{2}, & \text{otherwise,} \end{cases}$$

where

$$\beta_{uv} = \begin{cases} 1, & \text{if } u=k^2,\ v=l^2,\ k,l\in\mathbb{N} \\ 0, & \text{otherwise} \end{cases} \qquad \text{for } u,v=1,2,\cdots$$

Additionally, the complex uncertain variables are characterized by

$$\varpi_{uv}(\tau) = \begin{cases} (u+v+1)\mathrm{i}, & \text{if } \tau=\tau_{u+v} \\ 0, & \text{otherwise} \end{cases} \qquad \text{for } u,v=1,2,\cdots$$

and $\varpi \equiv 0$. Let $\mathcal{I}_2 = \mathcal{I}_2^d$.
The complex uncertainty distributions associated with the uncertain variable $\|\varpi_{uv}-\varpi\|$ is

$$\Phi_{uv}(t) = \begin{cases} 0, & \text{if } t<0 \\ 1-\frac{(u+v)\beta_{uv}}{2(u+v)+1}, & \text{if } 0\leq t<(u+v+1) \\ 1, & \text{if } t\geq(u+v+1) \end{cases} \qquad \text{for } u,v=1,2,\cdots$$

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Now

$$E\left[\|\varpi_{uv}-\varpi\|\right]=\int_0^{+\infty}\left(1-\Phi_{uv}(t)\right)dt=\int_0^{(u+v+1)}\frac{(u+v)\,\beta_{uv}}{2\,(u+v)+1}dt=\frac{(u+v)\,(u+v+1)\beta_{uv}}{2\,(u+v)+1}.$$

Then, for all $\sigma,\kappa>0$ and $r\geq 0$, we obtain

$$\left\{(m,n)\in\mathbb{N}^2:\tfrac{1}{h_{mn}}\left|\left\{(u,v)\in I_{mn}:E\left[\|\varpi_{uv}-\varpi\|\right]\geq r+\sigma\right\}\right|\geq\kappa\right\}$$

$$=\left\{(m,n)\in\mathbb{N}^2:\tfrac{1}{h_{mn}}\left|\left\{(u,v)\in I_{mn}:\tfrac{(u+v)(u+v+1)\beta_{uv}}{2(u+v)+1}\geq r+\sigma\right\}\right|\geq\kappa\right\}\in\mathcal{I}_2.$$

Once more, the metric between complex uncertain variables ϖ_{mn} and ϖ is given by

$$D\left(\varpi_{uv},\varpi\right)=\inf\left\{t:\mathcal{Y}\left\{\|\varpi_{uv}-\varpi\|\leq t\right\}=1\right\}=\inf\left\{t:\Phi_{uv}(t)=1\right\}=u+v+1.$$

So, for each $\sigma,\kappa>0$ and $r\geq 0$

$$\left\{(m,n)\in\mathbb{N}^2:\tfrac{1}{h_{mn}}\left|\left\{(u,v)\in I_{mn}:D\left(\varpi_{uv},\varpi\right)\geq r+\sigma\right\}\right|\geq\kappa\right\}$$

$$=\left\{(m,n)\in\mathbb{N}^2:\tfrac{1}{h_{mn}}\left|\left\{(u,v)\in I_{mn}:(u+v+1)\geq r+\sigma\right\}\right|\geq\kappa\right\}\notin\mathcal{I}_2.$$

As a result, the CUDS (ϖ_{uv}) is rough $\mathcal{I}_2$-lacunary statistically convergent in mean to $\varpi\equiv 0$, but it is not rough $\mathcal{I}_2$-lacunary statistically convergent in metric to $\varpi\equiv 0$.

Theorem 3.25. *If (ϖ_{uv}) is rough $\mathcal{I}_2$-lacunary statistically convergent in metric to ϖ, then it is rough $\mathcal{I}$ statistically convergent in measure to ϖ.*

Proof. Let (ϖ_{uv}) be rough $\mathcal{I}_2$-lacunary statistically convergent in metric to ζ, then it is rough $\mathcal{I}_2$-lacunary statistically convergent in measure to ϖ by Theorems 3.8 and 3.22. □

Remark 3.26. *But the converse of the above theorem is not true in general.*

Example 3.27. *Based on example 3.24, it can be demonstrated that the complex uncertain sequence (ϖ_{uv}) is rough $\mathcal{I}_2$-lacunary statistically convergent in measure to $\varpi\equiv 0$, but it is not rough $\mathcal{I}_2$-lacunary statistically convergent in metric to $\varpi\equiv 0$.*

4. Interrelationships among all convergence concepts

1. Rough $\mathcal{I}_2$-lacunary statistically convergence in measure;
2. Rough $\mathcal{I}_2$-lacunary statistically convergence in metric;
3. Rough $\mathcal{I}_2$-lacunary statistically convergence in mean;
4. Rough $\mathcal{I}_2$-lacunary statistically convergence in distribution;
5. Rough $\mathcal{I}_2$-lacunary statistically convergence in p-distance.

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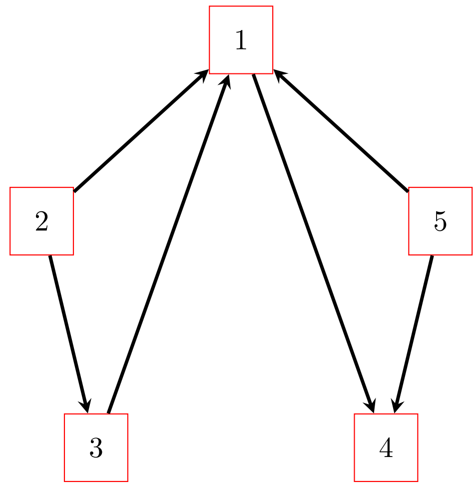

FIGURE 1. Interrelationships among all convergence concepts

5. Conclusion

In summary, introducing rough $\mathcal{I}_2$-lacunary statistical convergence adds a significant dimension to the study of convergence theory, particularly in uncertain contexts. By expanding upon existing concepts and exploring convergence across various dimensions of uncertainty, this research contributes valuable insights into the behavior of complex uncertain sequences. Furthermore, examining rough $\mathcal{I}_2$-lacunary statistical convergence in metric spaces and under p-distance broadens our understanding of convergence beyond traditional settings, paving the way for future research in diverse mathematical structures. The interconnectedness of convergence concepts highlighted in this study underscores the complex nature of convergence in uncertain environments, offering avenues for further exploration and deepening our understanding of convergence theory.

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