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DECOMPOSITION OF COMPLETE TRIPARTITE GRAPHS INTO TRIANGLES AND CLAWS

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Abstract. Let $K_{r,s,t}$ be a complete tripartite graph with $r \leq s \leq t$. Let C_k and S_k respectively denote a cycle and a star with k edges. In this paper, we show that the necessary and sufficient conditions for the existence of $\{pC_3, qS_3\}$ -decomposition of $K_{r,s,t}$ for all possible values of $p, q \geq 0$.

1. Introduction

The graphs considered here are finite, undirected and simple graphs. Let $K_{n_1,n_2,\ldots,n_r}$ denotes a complete r-partite graph with part sizes n_1, n_2,...,n_r, where each $n_i > 0$ is an integer. Partition of a graph G into edge-disjoint subgraphs $G_1, G_2, G_3, \ldots, G_n$, is called *decomposition* of G. We say that G has a $\{pC_3, qS_3\}$-decomposition, if G can be decomposed into p copies of C_3 and q copies of S_3. If G can be decomposed into p copies of C_3 and q copies of S_3, for all possible choices of the integers p, $q \geq 0$, then we say that G has a complete $\{C_3, S_3\}$-decomposition. Throughout this paper, the partite sets of the complete tripartite graph $K_{r,s,t}$, $1 \leq r \leq s \leq t$, are assumed to be $\{a_1, a_2, \ldots, a_r\}$, $\{b_1, b_2, \ldots, b_s\}$ and $\{c_1, c_2, \ldots, c_t\}$. Let C_k, P_k and S_k, respectively, denote a cycle, path and star with k edges. The definitions and notations not defined here can be found in [11]. The problem of finding necessary and sufficient conditions to decompose the complete $n-$partite graph into C_k, had been considered for many values of n and k. The case n= 2 was completely solved by Sotteau, [17]. Smith [16] proved that the necessary conditions for the decomposition of complete equipartite graphs into cycles of length $2p$(where $p \geq 3$ is a prime) are also sufficient. In the case of complete tripartite graphs,

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Cavenagh [5] has shown that $K_{m,m,m}$ can be decomposed into C_k if and only if $k \leq 3m$ and k divides $3m^2$. Billington [2] gave necessary and sufficient conditions for the existence of a decomposition of any complete tripartite graph into specified number of C_3 and C_4. Billington, et.al in [4] gave necessary and sufficient conditions for the existence of path and cycle decompositions of complete equipartite graphs with 3 and 5 parts. Mahmoodian and Mirzakhani [12] proved the existence of a C_5-decomposition of $K_{r,s,t}$, whenever the necessary conditions are satisfied and two of the partite sets are equal in size, except when $r = s \equiv 0(mod\ 5)$ and $t \not\equiv 0(mod\ 5)$. The authors of [1,3,6,7] also studied this problem. Fu, et.al [8], showed that for any nonnegative integers p and q and any positive integer n, there exists a decomposition of K_n into p copies of C_3 and q copies of S_3, if and only if $3(p+q)=\frac{n(n+1)}{2}$, $q \neq 1, 2$ if n is odd, $q = 1$ if $n = 4$; and $q \geq \max\left\{3, \left\lceil \frac{n}{4} \right\rceil\right\}$ if $n \geq 6$ is even. Shyu [14] proved that $K_{m,n}$ has a complete $\{P_k, S_k\}$-decomposition for some m and n. He also obtained a necessary and sufficient conditions on (p, q) for the existence of complete $\{P_4, S_4\}$-decomposition of $K_{m,n}$. Shyu [15] proved that K_n has a complete $\{C_4, S_4\}$-decomposition, if and only if $4(p+q)=\binom{n}{2}$, $q \neq 1$, if n is odd and $q \geq max\left\{3, \left\lceil \frac{n}{4} \right\rceil\right\}$, if n is even. Jeevadoss and Muthusamy [10] gave necessary and sufficient conditions for the existence of complete $\{P_k, C_k\}$-decomposition of $K_{m,n}$ and K_n, when $m \geq \frac{k}{2}$, $n \geq \left\lceil \frac{k+1}{2} \right\rceil$ for $k \equiv 0(mod\ 4)$ and when $m, n \geq 2k$ for $k \equiv 2(mod\ 4)$. Ganeshamurthy and Paulraja [9] gave necessary and sufficient conditions for the existence of a complete $\{C_3, C_6\}$-decomposition of $K_{a,b,c}$, $a \leq b \leq c$. Recently, Priyadarsini and Muthusamy [13] have proved the existence of decomposition of $K_{r,s,t}$ into paths and cycles of length 3.

In this paper, we show that the necessary conditions are also sufficient for the existence of $\{pC_3, qS_3\}$-decomposition of $K_{r,s,t}$ with some conditions on r, s and t. The obvious necessary conditions for such existence are:

(1) $(rs + st + tr) \equiv 0\ (mod\ 3)$;
(2) $3(p + q) = rs + st + tr$.

We prove that $K_{r,s,t}$ has a $\{pC_3, qS_3\}$-decomposition, if one of the following conditions holds:

(1) at least any two of r, s, t are congruent to 0 $(mod\ 3)$;
(2) all the integers r, s, t are congruent to 1 $(mod\ 3)$;
(3) all the integers r, s, t are congruent to 2 $(mod\ 3)$.

Also we prove that the above conditions are sufficient.

Definition 1.1 ([7])**.** *A rectangular array of order $r \times s$ with entries from the set T=$\{1, 2, \ldots, t\}$, is called a **Latin rectangle** of order $r \times s$ on t elements, if each element of T appears at most once in each row and each column.*

Lemma 1.2 ([7])**.** *Let r, s and t be integers such that $r \leq s \leq t$. A Latin rectangle of order $r \times s$ based on t elements gives rs edge-disjoint triangles in the complete tripartite graph $K_{r,s,t}$.*

The triangle (a_i, b_j, c_k) in the 3-partite graph $K_{r,s,t}$ is the subgraph of $K_{r,s,t}$ induced by the ith vertex of part 1, jth vertex of part 2 and kth vertex of part 3, and $S(a_i : b_l, b_j, c_k)$ denotes a claw

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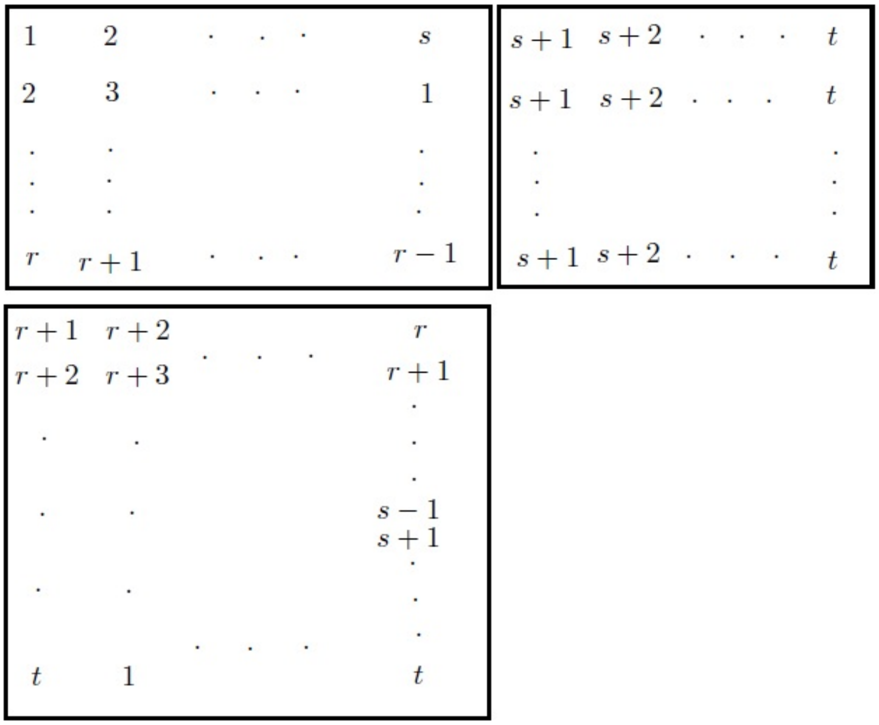

Figure 1.

with center vertex a_i in one part and end vertices b_l,b_j and c_k in other two parts, where, $1 \leq i \leq r$, $1 \leq l, j \leq s$, $1 \leq k \leq t$.

Definition 1.3 ([7]). *Let r, s and t be integers such that $r \leq s \leq t$. A **Latin representation** of the complete tripartite graph $K_{r,s,t}$ is a Latin rectangle of order $r \times s$ on t elements, together with a set of $t-s(respectively, t-r)$ remaining elements at the end of each row(respectively, column) so that each element from the set T=$\{1, 2, 3, .., t\}$ occurs exactly once in each row and in each column.*

Remark 1.4. *To construct a Latin representation of the complete tripartite graph $K_{r,s,t}$, we first take a Latin rectangle of order $r \times s$ on t elements. We then adjoin at the end of each row (respectively, column) by a set of remaining elements from the set $\{1, 2, 3, ..., t\}$ not already used in that row (respectively, column).*

Each entry k from the set appended at the end of the ith row, represents an edge from the ith element of the partite set of size r to the element k of the partite set of size t. Similarly, each entry k from the set appended at the end of the jth column represents an edge from the jth element of the partite set of size s to the element k of the partite set of size t. So a Latin representation of $K_{r,s,t}$ gives a decomposition of $K_{r,s,t}$ into rs triangles and $rK_{1,t-s} + sK_{1,t-r}$.

Definition 1.5. *An S-trade is a set of elements in the Latin representation, corresponding to a set of triangles and edges in $K_{r,s,t}$ which are S_3-decomposable.*

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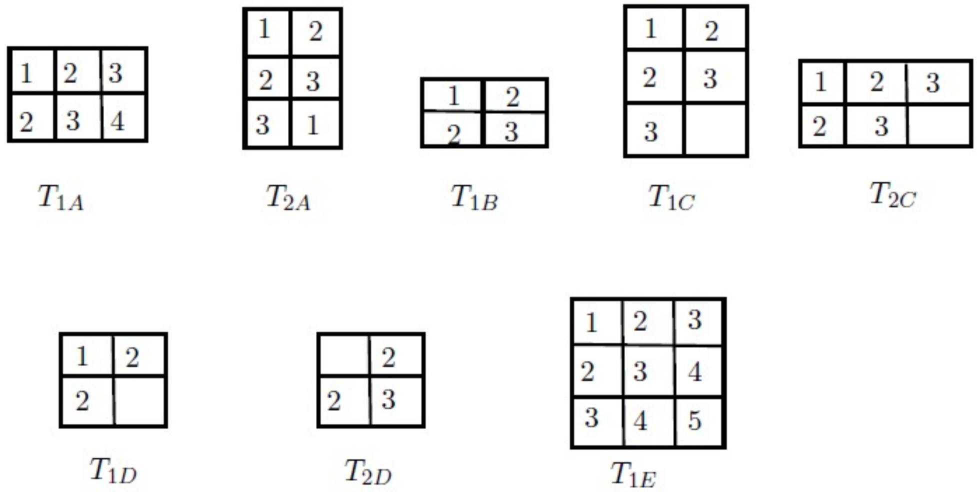

FIGURE 2.

Form a Latin representation of order $r \times s$ on t elements as shown in Figure 1. By Lemma 1.2, the Latin rectangle inside the Latin representation, gives rs triangles. Here we define 3 types of S-trades. We use these three types of trades suitably to get the required decomposition of $K_{r,s,t}$.

Definition 1.6. *The S-trade of type 1 can be obtained from the triangles in the Latin rectangle, which are S_3-decomposable. The S-trade of type 2 can be obtained from the newly added elements in the right side and the newly added elements at the bottom of the Latin rectangle which are S_3-decomposable. The S-trade of type 3 can be obtained from the newly added elements in the right side or at the bottom of the Latin rectangle along with the triangles inside the Latin rectangle, which are S_3-decomposable.*

Construction 1.7. *Define S- trade of type 1 as T_{ij}; $i \in 1, 2$, $j \in \{A, B, C, D, E\}$, where A represents S_3 decomposition using 6 triangles, B represents S_3 decomposition using 4 triangles, C represents S_3 decomposition using 5 triangles, D represents S_3 decomposition using 3 triangles, E represents S_3 decomposition using 9 triangles.*

(i) *Take six triangles: (a_1, b_1, c_1), (a_1, b_2, c_2), (a_1, b_3, c_3), (a_2, b_1, c_2), (a_2, b_2, c_3), (a_2, b_3, c_4) as shown in the trade T_{1A} of Figure 2. The union of these six triangles give six claws:*

$$S(b_1 : c_1, a_1, c_2),\ S(b_2 : c_2, c_3, a_1),\ S(b_3 : c_3, a_1, c_4),\ S(a_1 : c_1, c_2, c_3),$$
$$S(a_2 : c_2, c_3, c_4),\ S(a_2 : b_1, b_2, b_3).$$

Similarly, for trade T_{2A}.

(ii) *Take four triangles: (c_1, a_1, b_1), (c_3, a_2, b_2), (c_2, a_1, b_2), (c_2, a_2, b_1) as shown in the trade T_{1B} of Figure 2. The union of these four triangles give four claws:*

$$S(a_1 : c_1, b_2, c_2),\ S(b_1 : a_1, c_1, c_2),\ S(b_2 : c_2, a_2, c_3),\ S(a_2 : c_2, b_1, c_3).$$

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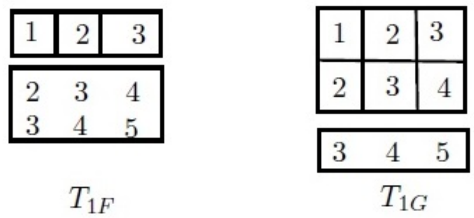

Figure 3.

(iii) *Take five triangles:* (a_1, b_1, c_1), (a_2, b_1, c_2), (a_3, b_1, c_3), (a_1, b_2, c_2), (a_2, b_2, c_3) *as shown in the trade* T_{1C} *of Figure 2, the union of these five triangles give five claws:*

$$S(b_1 : c_1, c_2, a_3),\ S(a_1 : c_1, b_1, c_2),\ S(a_2 : b_1, c_2, b_2),\ S(c_3 : a_3, b_1, a_2),\ S(b_2 : c_2, a_1, c_3).$$

Similarly, for trade T_{2C}.

(iv) *Take three triangles:* (a_1, b_1, c_1), (a_1, b_2, c_2), (a_2, b_1, c_2) *as shown in the trade* T_{1D} *of Figure 2. The union of these three triangles give three claws:*

$$S(a_1 : c_1, b_2, c_2),\ S(b_1 : a_1, c_1, a_2),\ S(c_2 : b_2, b_1, a_2).$$

Similarly, for trade T_{2D}.

(v) *Take nine triangles:* (a_1, b_1, c_1), (a_1, b_2, c_2), (a_1, b_3, c_3), (a_2, b_1, c_2), (a_2, b_2, c_3), (a_2, b_3, c_4), (a_3, b_1, c_3), (a_3, b_2, c_4), (a_3, b_3, c_5) *as shown in the trade* T_{1E} *of Figure 2. The union of these nine triangles can be written as nine claws:*

$$S(a_1 : c_1, b_1, b_2),\ S(a_1 : b_3, c_2, c_3),\ S(a_2 : c_2, b_1, b_2),\ S(a_2 : c_3, b_3, c_4),$$
$$S(a_3 : c_3, b_1, b_2),\ S(a_3 : c_4, b_3, c_5),\ S(b_1 : c_1, c_2, c_3),\ S(b_2 : c_2, c_3, c_4),\ S(b_3 : c_3, c_4, c_5).$$

Construction 1.8. *The S-trade of type* 3 *are of* 2 *kinds as follows:* T_{1F}, T_{1G}, *where* F *represents three triangles along with the edges corresponding to the elements of two rows at the bottom of the Latin rectangle and* G *represents six triangles along with edges corresponding to the elements of one row at the bottom of the Latin rectangle.* **(i)** *The triangles inside the Latin rectangle along with the edges corresponding to the elements at the bottom of the Latin rectangle can be decomposed into claws. For example, take three triangles:* (a_1, b_1, c_1), (a_1, b_2, c_2), (a_1, b_3, c_3) *with a common vertex* a_1 *along with the edges corresponding to the elements outside the Latin rectangle:* (b_1, c_2), (b_1, c_3), (b_2, c_3), (b_2, c_4), (b_3, c_4), (b_3, c_5) *as shown in the trade* T_{1F} *of Figure 3, can be decomposed into* 5 *claws:*

$$S(a_1 : c_1, b_1, b_2),\ S(a_1 : b_3, c_2, c_3),\ S(b_1 : c_2, c_3, c_1),\ S(b_2 : c_2, c_3, c_4),\ S(b_3 : c_3, c_4, c_5).$$

(ii) *Take six triangles:* (a_1, b_1, c_1), (a_1, b_2, c_2), (a_1, b_3, c_3), (a_2, b_1, c_2), (a_2, b_2, c_3), (a_2, b_3, c_4) *along with the edges corresponding to the elements outside the Latin rectangle:* (b_1, c_3), (b_2, c_4), (b_2, c_5) *as shown in the trade* T_{1G} *of Figure 3 gives the choices of* p *and* q.

When $p = 6$, *the edges corresponding to the elements at the bottom of the Latin rectangle cannot be decomposed into claws, and hence* $p \neq 6$. *Similarly when* $p = 5$, *the remaining triangle along with the edges corresponding to the elements at the bottom of the Latin rectangle, cannot be decomposed into claws.*

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When $p = 4$, we have four triangles: (a_1, b_1, c_1), (a_1, b_2, c_2), (a_2, b_1, c_2), (a_2, b_3, c_4). The remaining triangles: (a_1, b_3, c_3), (a_2, b_2, c_3) along with the edges corresponding to the elements at the bottom of the Latin rectangle, gives three claws:

$$S(b_2 : a_2, c_3, c_4),\ S(b_3 : c_5, c_3, a_1),\ S(c_3 : a_2, a_1, b_1).$$

When $p = 3$, we have three triangles: (a_1, b_1, c_1), (a_1, b_2, c_2), (a_1, b_3, c_3). The remaining triangles: (a_2, b_1, c_2), (a_2, b_2, c_3), (a_2, b_3, c_4) along with the edges corresponding to the elements at the bottom of the Latin rectangle, give 4 claws:

$$S(b_1 : a_2, c_2, c_3),\ S(b_2 : a_2, c_3, c_4),\ S(b_3 : a_2, c_5, c_4),\ S(a_2 : c_2, c_3, c_4).$$

When $p = 2$, we have two triangles: (a_1, b_1, c_1), (a_1, b_2, c_2). The remaining triangles: (a_1, b_3, c_3), (a_2, b_1, c_2), (a_2, b_2, c_3), (a_2, b_3, c_4) along with the edges corresponding to the elements at the bottom of the Latin rectangle, give 5 claws:

$$S(b_1 : a_2, c_2, c_3),\ S(b_3 : a_1, c_3, c_5),\ S(c_4 : a_2, b_3, b_2),\ S(c_3 : a_2, b_2, a_1),\ S(a_2 : c_2, b_2, b_3).$$

When $p = 1$, we have the triangle (a_1, b_1, c_1). The remaining triangles: (a_1, b_2, c_2), (a_1, b_3, c_3), (a_2, b_1, c_2), (a_2, b_2, c_3), (a_2, b_3, c_1) along with the edges corresponding to the elements at the bottom of the Latin rectangle, give 6 claws:

$$S(b_1 : c_3, c_2, a_2),\ S(b_2 : a_2, c_2, c_3),\ S(b_3 : c_5, a_1, c_3),\ S(a_2 : c_2, c_3, b_3),\ S(a_1 : b_2, c_2, c_3),\ S(c_4 : b_3, b_2, a_2).$$

When $p = 0$, the six triangles along with the edges corresponding to the elements at the bottom of the Latin rectangle, gives 7 claws:

$$S(b_1 : c_3, c_1, c_2),\ S(b_2 : c_4, c_2, c_3),\ S(b_3 : c_5, c_4, c_3),\ S(a_1 : b_1, c_1, b_2),\ S(a_1 : c_2, b_3, c_3),\ S(a_2 : b_1, c_2, b_2),\ S(a_2 : c_3, b_3, c_4).$$

Hence, we have the required decomposition, when $0 \leq p \leq 4$, $3 \leq q \leq 7$.

2. **Necessary conditions**

In this section, we obtain the necessary conditions for the existence of $\{pC_3, qS_3\}$-decomposition in $K_{r,s,t}$ as follows:

Theorem 2.1. *Let r,s and t be positive integers with $r \leq s \leq t$ and p, q be non negative integers. If $K_{r,s,t}$, has a $\{pC_3, qS_3\}$-decomposition, then the following holds:*

1. *$(rs + st + tr) \equiv 0 \ (mod\ 3)$;*
2. *$3(p + q) = rs + st + tr$;*
3. *$q \neq 1, 2$ when $r = s = t$ and $q \geq (t(r + s) - 2rs)/3$.*

Proof. By the counting argument, we get the required conditions (1) and (2). The degree of each vertex in the partite set of size r is $s + t$. To prove (3) by contrary, assume that $q = 1$. It implies that the end vertices of S_3 have odd degree in $K_{r,s,t} - E(S_3)$. Similarly, if $q = 2$, then the end vertices of 2 copies of S_3 have odd degree in $K_{r,s,t} - E(2S_3)$. Therefore the resulting graph $K_{r,s,t} - E(zS_3)$, where $z = 1, 2$ cannot be decomposed into C_3, a contradiction to the hypothesis, and hence $q \neq 1, 2$. □

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1	2	3
2	3	1
3	1	2

Figure 4.

Theorem 2.2. *Let r, s and t be positive integers with $r \leq s \leq t$. If there exists a $\{pC_3, qS_3\}$-decomposition of $K_{r,s,t}$, then r, s and t must satisfy one of the following:*

(1) *at least two of r, s, t are congruent to 0 (mod 3);*
(2) *all of r, s, t are congruent to 1 (mod 3);*
(3) *all of r, s, t are congruent to 2 (mod 3).*

Proof. Follows from Theorem 2.1 and the number of edges in $K_{r,s,t}$. □

3. Small cases

In this section, we obtain $\{pC_3, qS_3\}$-decomposition of $K_{r,s,t}$ when $(r, s, t) \in \{(3, 3, 3), (3, 3, 4), (4, 4, 4)\}$.

Example 3.1. *The graph $K_{3,3,3}$ has a $\{pC_3, qS_3\}$-decomposition, where $p + q = 9$ and $q \neq 1, 2$.*

Proof. Form a Latin square of order 3×3 on 3 elements as shown in Figure 4. By Lemma 1.2, we have nine edge-disjoint triangles:

$$(a_1, b_1, c_1),\ (a_1, b_2, c_2),\ (a_1, b_3, c_3),\ (a_2, b_1, c_2),\ (a_2, b_2, c_3),$$
$$(a_2, b_3, c_1),\ (a_3, b_1, c_3),\ (a_3, b_2, c_1),\ (a_3, b_3, c_2).$$

This gives us the required decomposition when p=9, q=0. When $p = 6$, we have six triangles: (a_1, b_1, c_1), (a_1, b_2, c_2), (a_1, b_3, c_3), (a_2, b_1, c_2), (a_3, b_1, c_3), (a_3, b_3, c_2). By using trade of type T_{1D}, the remaining three triangles: (a_2, b_2, c_3),(a_2, b_3, c_1),(a_3, b_2, c_1) give three claws:

$$S(a_2 : c_3, b_2, b_3),\ S(b_2 : c_3, a_3, c_1),\ S(c_1 : b_3, a_2, a_3).$$

When $p = 5$, we have five triangles: (a_1, b_3, c_3), (a_2, b_3, c_1), (a_3, b_1, c_3), (a_3, b_2, c_1), (a_3, b_3, c_2). By using trade of type T_{1B}, the remaining four triangles: (a_1, b_1, c_1),(a_1, b_2, c_2),(a_2, b_1, c_2),(a_2, b_2, c_3) give four claws:

$$S(a_1 : c_1, b_2, c_2),\ S(b_1 : c_1, a_1, c_2),\ S(b_2 : c_2, c_3, a_2),\ S(a_2 : c_2, b_1, c_3).$$

When $p = 4$, we have four triangles: (a_2, b_3, c_1), (a_3, b_1, c_3), (a_3, b_2, c_1), (a_3, b_3, c_2).
By using trade T_{1C}, the remaining five triangles: (a_1, b_1, c_1),(a_1, b_2, c_2),(a_1, b_3, c_3),(a_2, b_1, c_2),(a_2, b_2, c_3) we get five claws:

$$S(a_1 : c_1, c_2, b_3),\ S(b_1 : c_1, a_1, c_2),\ S(b_2 : c_2, a_1, c_3),\ S(a_2 : c_2, b_1, b_2),\ S(c_3 : b_3, a_1, a_2).$$

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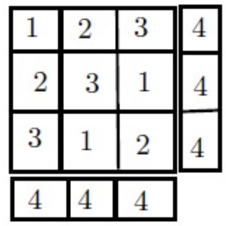

FIGURE 5.

When $p = 3$, we have three triangles: (a_3, b_1, c_3), (a_3, b_2, c_1), (a_3, b_3, c_2).
By using trade of type T_{1A}, the remaining six triangles: (a_1, b_1, c_1),(a_1, b_2, c_2),(a_1, b_3, c_3),(a_2, b_1, c_2), (a_2, b_2, c_3),(a_2, b_3, c_1) give six claws:

$$S(a_1 : c_1, b_1, b_2),\ S(a_1 : c_2, b_3, c_3),\ S(b_1 : c_1, c_2, a_2),\ S(b_2 : c_2, c_3, a_2),\ S(b_3 : c_3, a_2, c_1),\ S(a_2 : c_2, c_3, c_1).$$

When $p = 2$, we have two triangles: (a_3, b_2, c_1), (a_2, b_3, c_1). By using trade T_{1B}, the four triangles: (a_1, b_1, c_1),(a_1, b_3, c_3),(a_3, b_3, c_2),(a_3, b_1, c_3) give four claws:

$$S(a_1 : c_1, b_3, c_3),\ S(b_1 : c_1, a_1, c_3),\ S(b_3 : c_3, c_2, a_3),\ S(a_3 : c_3, b_1, c_2)$$

and by using trade T_{1D}, the three triangles: (a_1, b_2, c_2),(a_2, b_1, c_2),(a_2, b_2, c_3)give three claws:

$$S(b_2 : c_2, a_1, c_3),\ S(c_2 : a_1, b_1, a_2),\ S(a_2 : b_1, b_2, c_3).$$

When $p = 1$, we have the triangle:(a_3, b_1, c_3). By using trade T_{1C}, the five triangles: (a_1, b_1, c_1),(a_1, b_2, c_2), (a_1, b_3, c_3),(a_2, b_2, c_3),(a_2, b_1, c_2) give five claws:

$$S(a_1 : c_1, c_2, b_3),\ S(b_1 : c_1, a_1, c_2),\ S(b_2 : c_2, a_1, c_3),\ S(c_3 : b_3, a_1, a_2),\ S(a_2 : c_2, b_1, b_2)$$

and by using trade T_{1D}, the three triangles: (a_2, b_3, c_1),(a_3, b_2, c_1),(a_3, b_3, c_2) give three claws:

$$S(c_1 : b_2, a_3, a_2),\ S(a_3 : b_2, b_3, c_2),\ S(b_3 : c_2, a_2, c_1).$$

When $p = 0$, by using trade of type T_{1E}, we get nine claws:

$$S(a_1 : c_1, b_1, b_2),\ S(a_1 : b_3, c_2, c_3),\ S(a_2 : c_2, b_1, b_2),\ S(a_2 : c_3, b_3, c_1),\ S(a_3 : c_3, b_1, b_2),$$
$$S(a_3 : c_1, b_3, c_2),\ S(b_1 : c_1, c_2, c_3),\ S(b_2 : c_2, c_3, c_1),\ S(b_3 : c_3, c_1, c_2).$$

Hence the graph $K_{3,3,3}$, has the required $\{pC_3, qS_3\}$-decomposition. □

Example 3.2. *The graph $K_{3,3,4}$ can be decomposed into pC_3 and qS_3, where $0 \leq p \leq 9$, $2 \leq q \leq 11$ and $p + q = 11$.*

Proof. Form a Latin representation of order 3×3 on 4 elements as shown in Figure 5. By Example 3.1, the Latin square of order 3×3 on 3 elements give the choices of p and q, where $0 \leq p \leq 9$, $0 \leq q \leq 9$, $q \neq 1, 2$. The edges correspond to the newly added elements at the right side and the bottom of the Latin rectangle give two copies of claws $S(c_4 : a_1, a_2, a_3)$ and $S(c_4 : b_1, b_2, b_3)$. Hence we get the required decomposition. □

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1	2	3	4
2	3	4	1
3	4	1	2
4	1	2	3

Figure 6.

Example 3.3. *The graph $K_{4,4,4}$ can be decomposed into pC_3 and qS_3, where $0 \leq p \leq 16$, $0 \leq q \leq 16$, $q \neq 1, 2$ and $p + q$=16.*

Proof. Form a Latin square of order 4×4 on 4 elements as shown in Figure 6. By Lemma 1.2, we have 16 edge-disjoint triangles.

When $p = 13$, by using Trade T_{1D}, the three triangles: (a_1, b_1, c_1), (a_1, b_2, c_2), (a_2, b_1, c_2) give three claws: $S(a_1 : c_1, b_1, b_2)$, $S(c_2 : b_2, a_1, a_2)$, $S(b_1 : c_1, a_2, c_2)$.

When p=12, by using Trade T_{1B}, the four triangles: (a_1, b_1, c_1), (a_1, b_2, c_2),(a_2, b_1, c_2),(a_2, b_2, c_3) give four claws:

$$S(a_1 : c_1, b_1, c_2),\ S(a_2 : c_2, b_2, c_3),\ S(b_1 : c_1, a_2, c_2),\ S(b_2 : a_1, c_2, c_3).$$

When p=11, by using Trade T_{1C}, the five triangles: (a_1, b_1, c_1),(a_2, b_1, c_2),(a_3, b_1, c_3),(a_1, b_2, c_2),(a_2, b_2, c_3) give five claws:

$$S(a_1 : c_1, b_1, c_2),\ S(a_2 : b_1, c_2, b_2),\ S(c_3 : a_3, b_1, a_2),\ S(a_1 : c_1, b_1, c_2),\ S(b_1 : c_1, c_2, a_3).$$

When p=10, by using Trade T_{1A}, the six triangles: (a_1, b_1, c_1),(a_2, b_1, c_2),(a_3, b_1, c_3),(a_2, b_1, c_2),(a_2, b_2, c_3), (a_2, b_3, c_4) give six claws:

$$S(b_1 : c_1, a_1, c_2),\ S(b_2 : c_2, c_3, a_1),\ S(b_3 : c_3, a_1, c_4),\ S(a_1 : c_1, c_2, c_3),\ S(a_2 : c_2, c_3, c_4),\ S(a_2 : b_1, b_2, b_3).$$

When p=9, by using Trade T_{1B}, the four triangles: (a_1, b_1, c_1),(a_1, b_2, c_2),(a_2, b_1, c_2),(a_2, b_2, c_3) give four claws:

$$S(a_1 : c_1, b_1, c_2),\ S(a_2 : c_2, b_2, c_3),\ S(b_1 : c_1, a_2, c_2),\ S(b_2 : a_1, c_2, c_3).$$

Using Trade T_{1D}, the three triangles: (a_1, b_3, c_3),(a_1, b_4, c_4),(a_2, b_3, c_4) give three claws:

$$S(a_1 : c_3, b_3, b_4),\ S(c_4 : b_4, a_1, a_2),\ S(b_3 : c_3, a_2, c_4).$$

When p=8, by using Trade T_{1C}, the five triangles: (a_1, b_1, c_1),(a_2, b_1, c_2),(a_3, b_1, c_3),(a_1, b_2, c_2),(a_2, b_2, c_3) give five claws:

$$S(a_1 : c_1, b_1, c_2),\ S(a_2 : b_1, c_2, b_2),\ S(c_3 : a_3, b_1, a_2),S(a_1 : c_1, b_1, c_2),\ S(b_1 : c_1, c_2, a_3).$$

Using Trade T_{1D}, the three triangles: (a_1, b_3, c_3),(a_1, b_4, c_4),(a_2, b_3, c_4) give three claws:

$$S(a_1 : c_3, b_3, b_4),\ S(c_4 : b_4, a_1, a_2),\ S(b_3 : c_3, a_2, c_4).$$

When $p = 7$, by using Trade T_{1E}, the nine triangles:

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$$(a_1,b_1,c_1),(a_1,b_2,c_2),(a_1,b_3,c_3),(a_2,b_1,c_2),(a_2,b_2,c_3),$$
$$(a_2,b_3,c_4),(a_3,b_1,c_3),(a_3,b_2,c_4),(a_3,b_3,c_1)$$

give nine claws:

$$S(a_1:c_1,b_1,b_2),\ S(a_1:b_3,c_2,c_3),\ S(a_2:c_2,b_1,b_2),\ S(a_2:c_3,b_3,c_4),\ S(a_3:c_3,b_1,b_2),S(a_3:c_4,b_3,c_1),$$
$$S(b_1:c_1,c_2,c_3),\ S(b_2:c_2,c_3,c_4),\ S(b_3:c_3,c_4,c_1).$$

When p=6, by using Trade T_{1C}, the five triangles: (a_1,b_1,c_1),(a_2,b_1,c_2),(a_3,b_1,c_3),(a_1,b_2,c_2),(a_2,b_2,c_3) give five claws:

$$S(a_1:c_1,b_1,c_2),\ S(a_2:b_1,c_2,b_2),\ S(c_3:a_3,b_1,a_2),\ S(a_1:c_1,b_1,c_2),\ S(b_1:c_1,c_2,a_3).$$

Using Trade T_{2C}, the five triangles: (a_3,b_1,c_3),(a_3,b_2,c_4),(a_3,b_3,c_1),(a_4,b_1,c_4),
(a_4,b_2,c_1) give five claws:

$$S(a_3:c_3,c_4,b_3),\ S(a_4:c_4,b_1,b_2),\ S(b_1:a_3,c_3,c_4),\ S(b_2:a_3,c_4,c_1),\ S(c_1:a_1,b_3,a_2).$$

When p=5, by using Trade T_{1C}, the five triangles: (a_1,b_1,c_1),(a_2,b_1,c_2),(a_3,b_1,c_3),(a_1,b_2,c_2),(a_2,b_2,c_3) give five claws:

$$S(a_1:c_1,b_1,c_2),\ S(a_2:b_1,c_2,b_2),\ S(c_3:a_3,b_1,a_2),\ S(a_1:c_1,b_1,c_2),\ S(b_1:c_1,c_2,a_3).$$

Using Trade T_{2A}, the six triangles: (a_1,b_3,c_3),(a_1,b_4,c_4),(a_2,b_3,c_4),(a_2,b_4,c_1),
(a_3,b_3,c_1),(a_3,b_4,c_2) give six claws:

$$S(b_3:c_3,c_4,c_1),\ S(b_4:a_1,a_2,a_3),\ S(a_1:c_3,b_3,c_4),\ S(a_2:c_4,b_3,c_1),\ S(a_3:c_1,b_3,c_2),\ S(b_4:c_4,c_1,c_2).$$

When $p=4$, by using Trade T_{1A}, the twelve triangles:

$$(a_1,b_1,c_1),(a_1,b_2,c_2),(a_1,b_3,c_3),(a_2,b_1,c_2),(a_2,b_2,c_3),(a_3,b_3,c_4),$$
$$(a_3,b_1,c_3),(a_3,b_2,c_4),(a_3,b_3,c_1),(a_4,b_1,c_4),(a_4,b_2,c_1),(a_4,b_3,c_2)$$

give twelve claws:

$$S(a_1:c_1,c_2,c_3),\ S(b_1:a_1,c_1,c_2),\ S(b_2:a_1,c_2,c_3),\ S(b_3:a_1,c_3,c_4),\ S(a_2:c_2,c_3,c_4)\ S(a_2:b_1,b_2,b_3),$$
$$S(a_3:c_3,c_4,c_1),\ S(b_1:a_3,c_3,c_4),\ S(b_2:a_3,c_4,c_1),\ S(b_3:a_3,c_1,c_2),S(a_4:c_4,c_1,c_2)\ S(a_2:b_1,b_2,b_3).$$

When p=3, by using Trade T_{2C}, the five triangles: (a_1,b_1,c_1),(a_1,b_2,c_2),(a_1,b_3,c_3),(a_2,b_1,c_2),(a_2,b_2,c_3) give five claws:

$$S(a_1:c_1,b_2,b_3),\ S(b_1:a_1,c_1,a_2),\ S(b_2:c_2,a_2,c_3),\ S(c_2:a_1,a_2,b_1),\ S(c_3:b_3,a_1,a_2).$$

Using Trade T_{1B}, the eight triangles: (a_3,b_1,c_3),(a_3,b_2,c_4),(a_4,b_1,c_4),(a_4,b_2,c_1),
(a_3,b_1,c_3),(a_3,b_2,c_4),(a_4,b_1,c_4),(a_4,b_2,c_1) give eight claws:

$$S(a_3:c_3,b_2,c_4),\ S(a_4:c_4,b_1,c_1),\ S(b_1:a_3,c_3,c_4),\ S(b_2:c_4,a_4,c_1),$$
$$S(a_3:c_1,b_4,c_2),\ S(a_4:c_2,b_3,c_3),\ S(b_3:a_3,c_1,c_2),\ S(b_4:c_2,a_4,c_3).$$

When p=2, by using Trade T_{1A}, the six triangles: (a_1,b_1,c_1),(a_1,b_2,c_2),(a_1,b_3,c_3),(a_2,b_1,c_2),(a_2,b_2,c_3),
(a_2,b_3,c_4) give six claws:

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$S(a_1 : c_1, c_2, c_3)$, $S(b_1 : a_1, c_1, c_2)$, $S(b_2 : a_1, c_2, c_3)$, $S(b_3 : a_1, c_3, c_4)$, $S(a_2 : c_2, c_3, c_4)$, $S(a_2 : b_1, b_2, b_3)$.

Using Trade T_{1B}, the eight triangles: (a_3, b_1, c_3),(a_3, b_2, c_4),(a_4, b_1, c_4),(a_4, b_2, c_1),
(a_3, b_1, c_3),(a_3, b_2, c_4),(a_4, b_1, c_4),(a_4, b_2, c_1) give eight claws:

$$S(a_3 : c_3, b_2, c_4),\ S(a_4 : c_4, b_1, c_1),\ S(b_1 : a_3, c_3, c_4),\ S(b_2 : c_4, a_4, c_1),\ S(a_3 : c_1, b_4, c_2),$$
$$S(a_4 : c_2, b_3, c_3),\ S(b_3 : a_3, c_1, c_2),\ S(b_4 : c_2, a_4, c_3).$$

When p=1, by using Trade T_{1B}, the four triangles: (a_1, b_1, c_1),(a_1, b_2, c_2),(a_2, b_1, c_2),(a_2, b_2, c_3) give four claws:

$$S(a_1 : c_1, b_2, c_2),\ S(b_1 : a_1, c_1, c_2),\ S(b_2 : c_2, a_2, c_3),\ S(a_2 : c_2, b_1, c_3).$$

Using Trade T_{1D}, the three triangles: (a_1, b_3, c_3),(a_1, b_4, c_4),(a_2, b_3, c_4) give three claws: $S(a_1 : c_3, b_4, c_4)$, $S(b_3 : a_1, c_3, a_2)$, $S(c_4 : b_4, a_2, b_3)$. Using Trade T_{1B}, the eight triangles: (a_3, b_1, c_3),(a_3, b_2, c_4),(a_4, b_1, c_4),
(a_4, b_2, c_1),
(a_3, b_1, c_3),(a_3, b_2, c_4),(a_4, b_1, c_4),(a_4, b_2, c_1) give eight claws:

$$S(a_3 : c_3, b_2, c_4),\ S(a_4 : c_4, b_1, c_1),\ S(b_1 : a_3, c_3, c_4),$$
$$S(b_2 : c_4, a_4, c_1),\ S(a_3 : c_1, b_4, c_2),\ S(a_4 : c_2, b_3, c_3),\ S(b_3 : a_3, c_1, c_2),\ S(b_4 : c_2, a_4, c_3).$$

When $p = 0$, by using Trade T_{1B}, 16 triangles give 16 copies of claws:

$$S(a_1 : c_1, b_2, c_2),\ S(b_1 : a_1, c_1, c_2),\ S(b_2 : c_2, a_2, c_3),\ S(a_2 : c_2, b_1, c_3),$$
$$S(a_1 : c_3, b_4, c_4),\ S(b_3 : a_1, c_3, c_4),\ S(b_4 : c_4, a_2, c_1),\ S(a_2 : c_4, b_3, c_1),$$
$$S(a_3 : c_3, b_2, c_4),\ S(a_4 : c_4, b_1, c_1),\ S(b_1 : a_3, c_3, c_4),\ S(b_2 : c_4, a_4, c_1),$$
$$S(a_3 : c_1, b_4, c_2),\ S(a_4 : c_2, b_3, c_3),\ S(b_3 : a_3, c_1, c_2),\ S(b_4 : c_2, a_4, c_3).$$

Thus $K_{4,4,4}$, has the required $\{pC_3, qS_3\}$-decomposition. □

4. Sufficient conditions

In this section, we show that the necessary conditions for the existence of $\{pC_3, qS_3\}$-decomposition in $K_{r,s,t}$ are sufficient.

Lemma 4.1. *If all of $r, s \equiv 0 \pmod 3$ and $t \equiv 0$ or 1 or $2 \pmod 3$ with $r \leq s \leq t$, then $K_{r,s,t}$ has a $\{pC_3, qS_3\}$-decomposition.*

Proof. Form a Latin representation of order $r \times s$ on t elements as shown in Figure 1. The Latin rectangle of order $r \times s$ on s elements give $(r/3)(s/3)$ copies of trade T_{1E}. By Example 3.1, we can get the required decomposition for the permissible choices of p and q, where $0 \leq p \leq rs$, $0 \leq q \leq rs$ and $q \neq 1, 2$. By using S-trade of type 2, the edges corresponding to $r(t-s)$ elements in the right side of the Latin rectangle and the edges corresponding to $s(t-r)$ elements at the bottom of the Latin rectangle give $r(t-s)/3$ and $s(t-r)/3$ claws respectively. Hence we have the required decomposition. □

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Lemma 4.2. *If all of* $r, s, t \equiv 1$ *or* 2 $(mod\ 3)$ *with* $r \leq s \leq t$*, then* $K_{r,s,t}$ *has a* $\{pC_3, qS_3\}$*-decomposition.*

Proof. **Case** 1:
If $r, s, t \equiv 1$ or 5 $(mod\ 6)$, where $r > 1$, then the elements inside the Latin rectangle give $\frac{(r-3)}{2}\frac{(s-3)}{2}$ copies of trade T_{1B}, and $(r-3)/2$ copies of trade T_{1A}, and $(s-3)/2$ copies of trade T_{2A}, and one copy of trade T_{1E}. By Construction 1.7, we can get the required number of claws. This gives the required decomposition for all permissible choices of p and q inside the Latin rectangle. By using S-trade of type 2, the edges correspond to $r(t-s)$ elements in the right side of the Latin rectangle and the edges correspond to $s(t-r)$ elements at the bottom of the Latin rectangle give $r(t-s)/3$, and $s(t-r)/3$ claws respectively. Hence we have the required decomposition.
Case 2:
If $r, s, t \equiv 4$ or 8 $(mod\ 12)$, then the edges corresponding to the elements inside the Latin rectangle represent $(r/4)(s/4)$ copies of the graph $K_{4,4,4}$. By Example 3.3, we get the required decomposition for all permissible choices of p and q inside the Latin rectangle. By using S-trade of type 2, the edges corresponding to $r(t-s)$ elements in the right side of the Latin rectangle and the edges corresponding to $s(t-r)$ elements at the bottom of the Latin rectangle give $r(t-s)/3$ and $s(t-r)/3$ claws respectively. Hence we have the required decomposition.
Case 3:
If $r, s, t \equiv 10$ or 2 $(mod\ 12)$, then the edges corresponding to the elements inside the Latin rectangle give $\frac{(r-6)}{2}\frac{(s-6)}{2}$ copies of trade T_{1B} and $2(r-6)/2$ copies of trade T_{1A} and $2(s-6)/2$ copies of trade T_{2A} and four copies of trade T_{1E}. By Construction 1.7, we can get the required number of claws. This gives the required decomposition for all permissible choices of p and q inside the Latin rectangle . By using S-trade of type 2, the edges corresponding to $r(t-s)$ elements in the right side of the Latin rectangle and the edges corresponding to $s(t-r)$ elements at the bottom of the Latin rectangle give $r(t-s)/3$ and $s(t-r)/3$ claws respectively. Hence we have the required decomposition. □

Lemma 4.3. *If* $r \equiv 1$ $(mod\ 3)$*,* $s, t \equiv 0$ $(mod\ 3)$*, then* $K_{r,s,t}$ *has a* $\{pC_3, qS_3\}$*-decomposition.*

Proof. Form a Latin representation of order $r \times s$ on t elements as shown in Figure 1. The elements inside the Latin rectangle give $(s/3)(r-1)/3$ copies of trade T_{1E}. By Example 3.1, the elements inside the Latin rectangle give p copies of C_3 and q copies of S_3, where $0 \leq p \leq s(r-1)$, $0 \leq q \leq s(r-1)$ and $q \neq 1, 2$. By Construction 1.8, the last row inside the Latin rectangle along with the first two rows at the bottom of the Latin rectangle give $(s/3)$ copies of trade T_{1F}. Therefore, we get $5s/3$ of copies of claws. The edges corresponding to the remaining elements at the bottom of the Latin rectangle, give $s(t-(r+2))/3$ copies of claws and the edges corresponding to the elements at the right side of the Latin rectangle give $r(t-s)/3$ copies of claws. Hence, we get the required decomposition. □

Lemma 4.4. *If* $r \equiv 2$ $(mod\ 3)$*,* $s, t \equiv 0$ $(mod\ 3)$ *with* $r \leq s \leq t$*, then* $K_{r,s,t}$*, has a* $\{pC_3, qS_3\}$*-decomposition.*

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Proof. Form a Latin representation of order $r \times s$ on t elements as shown in Figure 1. The elements inside the Latin rectangle give $(s/3)(r-2)/3$ copies of trade T_{1E}. By Example 3.1, the edges corresponding to the elements inside the Latin rectangle give p copies of C_3 and q copies of S_3, where $0 \leq p \leq s(r-2)$, $0 \leq q \leq s(r-2)$ and $q \neq 1, 2$. The last two rows inside the Latin rectangle along with the first row at the bottom of the Latin rectangle give $(s/3)$ copies of trade T_{1G}. By Construction 1.8, we get p copies of C_3 and q copies of S_3, where $0 \leq p \leq 4s/3$, $3s/3 \leq q \leq 7s/3$. The edges corresponding to the remaining elements at the bottom of the Latin rectangle give $s(t-(r+1))/3$ copies of claws and the edges corresponding to the elements at the right side of the Latin rectangle give $r(t-s)/3$ copies of claws. Hence, we get the required decomposition. □

Lemma 4.5. *If $r, t \equiv 0$ $(mod\ 3)$ and $s \equiv 1$ $(mod\ 3)$ with $r \leq s \leq t$, then $K_{r,s,t}$ has a $\{pC_3, qS_3\}$-decomposition.*

Proof. Form a Latin representation of order $r \times s$ on t elements as shown in Figure 1. The elements inside the Latin rectangle give $(r/3)(s-4)/3$ copies of trade T_{1E} and $2(r/3)$ copies of trade T_{2A}. By Construction 1.7, we get the required number of claws. The edges correspond to the elements at the right side of the Latin rectangle give $(r/3)(t-s)$ copies of claws and the edges correspond to the elements at the bottom of the Latin rectangle give $s(t-r)/3$ copies of claws. Hence, we get the required decomposition. □

Lemma 4.6. *If $r, t \equiv 0$ $(mod\ 3)$, $s \equiv 2$ $(mod\ 3)$ with $r \leq s \leq t$, then $K_{r,s,t}$ has a $\{pC_3, qS_3\}$-decomposition.*

Proof. Form a Latin representation of order $r \times s$ on t elements as shown in Figure 1. The elements inside the Latin rectangle give $(r/3)(s-2)/3$ copies of trade T_{1E} and $(r/3)$ copies of trade T_{2A}. By Construction 1.7, we get the required number of claws. The edges corresponding to the elements at the right side of the Latin rectangle give $(r/3)(t-s)$ copies of claws and the edges corresponding to the elements at the bottom of the Latin rectangle give $s(t-r)/3$ copies of claws. Hence, we get the required decomposition. □

Main Theorem. *Let p and q be non negative integers and let r, s and t be positive integers. There exists a decomposition of $K_{r,s,t}$, $r \leq s \leq t$ into pC_3 and qS_3, if and only if $(rs+st+tr) \equiv 0$ $(mod\ 3)$, $3(p+q) = rs+st+tr$, $q \neq 1, 2$.*

Proof. Necessity: by Theorem 2.2, the integers r, s and t satisfying the given necessary conditions, hold in one of the following cases:

(1) at least any two of r, s, t are congruent to 0 $(mod\ 3)$
(2) all of r, s, t are congruent to 1 $(mod\ 3)$
(3) all of r, s and t are congruent to 2 $(mod\ 3)$.

Sufficiency: Follows from the Lemmas 4.1-4.6 in Section 4. □

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