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COMMUTATIVITY OF PRIME RING WITH GENERALIZED SKEW DERIVATIONS HAVING A LIE-TYPE BEHAVIOUR

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Abstract. Let R be a prime ring of characteristic different from 2 and 3, Q_r its right Martindale quotient ring and C its the extended centroid. Suppose that F is a non-zero generalized skew derivation of R such that

$$F([x,y]_k)=[F(x),y]_k+[x,F(y)]_k$$

for all $x,y\in R$, with $k>1$ fixed integer. In this paper we will showw that, then R is commutative.

1. Introduction

Throughout the paper, R is always an associative prime ring, Q_r its right Martindale quotient ring, and $C=Z(Q_r)$, the center of Q_r, called the *extended centroid* of R; when R is a prime ring, C is a field. We refer the reader to [2] for details.

We recall that, for all $x,y\in R$, $[x,y]=xy-yx$ and, for a fixed integer $k>1$, $[x,y]_k=[[x,y]_{k-1},y]=\sum_{i=1}^{k}(-1)^i\binom{k}{i}y^ixy^{k-i}$.
An additive mapping $d:R\to R$ is said to be a *derivation* of R, if $d(xy)=d(x)y+xd(y)$, for all $x,y\in R$. If, for any $x,y\in S\subseteq R$, $d([x,y])=[d(x),y]+[x,d(y)]$, then the mapping d is called a *Lie derivation* on S. Obviously any derivation of R is a Lie derivation on any arbitrary subset S of R.
Let now $F:R\to R$ be an additive map on the prime ring R, defined as follows: $F(xy)=F(x)y+xd(y)$, for all $x,y\in R$ and for some derivation d of R. Such mapping F is called *generalized derivation* of R with associated derivation d. We remark that any derivation of R is a generalized derivation of R and

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any mapping of R of the form $F(x) = ax + xb$, for some $a, b \in R$, is a generalized derivation, usually called *inner generalized derivation*.

The well-known Posner's first theorem states that if δ and d are two non-zero derivations of R, then the composition $(d\delta)$ cannot be a non-zero derivation of R [18, Theorem 1]. An analogue of Posner's result for Lie derivations was proved by Lanski in [15]. More precisely Lanski showed that if δ and d are two non-zero derivations of R and L is a Lie ideal of R, then $(d\delta)$ cannot be a Lie derivation of L into R unless $char(R) = 2$ and either R satisfies $s_4(x_1, \dots, x_4)$, the standard identity of degree 4, or $d = \alpha\delta$, for $\alpha \in C$.

Starting from the previously cited results, several authors studied various kinds of additive mappings which satisfy particular conditions on suitable subsets of prime and semiprime rings. The conclusions of these research works indicate how the structure of the ring is closely related to the behavior of additive mappings defined on it.

In [10], De Filippis proves that, if the composition of d and g, two non-zero derivations of R, acts as a Lie derivation on $[I, I]$, where I is a non-zero right ideal of R, then $s_4(I, I, I, I)I = 0$ or $d(I)I = g(I)I = 0$.

In [11], the author and De Filippis proved that the composition of two generalized derivations of R, acting as a Lie derivation, must be a derivation of R and described all the possible forms of the generalized derivations involved in the composition.

More recently, Argaç and Yarbil, in [1, Theorem 3], study the form of a non-zero generalized derivation F of a prime ring R of characteristic different from 2, that acts as a Lie-type map. More precisely they proved that, if d is a derivation of R such that $F([x, y]) = [F(x), y] + [x, d(y)]$, for any $x, y \in L$ a non-central Lie ideal of R, then $F(x) = \gamma x + d(x)$, for all $x \in R$, where $\gamma \in C$.

Our work is then motivated by the previous results. In the current presentation , we will continue the study of additive maps which act as Lie-type maps. We will now recall the definition of generalized skew derivations of the ring R.

Let R be an associative ring and α an automorphism of R. An additive mapping $d : R \to R$ is called a *skew derivation* of R if $d(xy) = d(x)y + \alpha(x)d(y)$, for all $x, y \in R$, and α is called an associated automorphism of d. An additive mapping $G : R \to R$ is said to be a *generalized skew derivation* of R , if there exists a skew derivation d of R with associated automorphism α such that $G(xy) = G(x)y + \alpha(x)d(y)$, for all $x, y \in R$, d is said to be an associated skew derivation of G and α is called an associated automorphism of G. Any mapping of R with form $G(x) = ax + \alpha(x)b$, for some $a, b \in R$ and $\alpha \in Aut(R)$, is called *inner generalized skew derivation*. In particular, if $a = -b$, then G is called *inner skew derivation*. If a generalized skew derivation (respectively, a skew derivation) is not inner, then it is usually called *outer*. The concept of generalized skew derivation unifies the notions of skew derivation and generalized derivation.

Here we will examine what happens when there exists a fixed integer $k \geq 2$, such that the non-zero generalized skew derivation F of R satisfies the following condition:

$$F([x, y]_k) = [F(x), y]_k + [x, F(y)]_k \quad \forall x, y \in R. \tag{1.1}$$

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We prove the following result:

Theorem 1.1. *Let R be a prime ring of characteristic different from 2 and 3. If there exists a non-zero generalized skew derivation F of R satisfying relation (1.1), then R is commutative.*

Let us point out that here one may exclude the case when $k = 1$ since, under this assumption, the relation (1.1) would be reduced to the simplest

$$F([x, y]) = [F(x), y] + [x, F(y)] \quad \forall x, y \in R \tag{1.2}$$

that is F would be a Lie derivation of R. In this case it is well known that $F = \delta + \tau$, where δ is a derivation of R and $\tau : R \to C$ is an additive mapping sending commutators to zero (see [3, Theorem 4.3]). Therefore there would be nothing to discuss, since relation (1.1) would become a trivial identity.

2. Preliminaries

Now, to prove our result, we need to recall the following well-known Facts on skew derivations and automorphisms involved in generalized polynomial identities for prime rings:

Fact 2.1. Let R be a prime ring and I be a two-sided ideal of R. Then I, R, and Q_r satisfy the same generalized polynomial identities with coefficients in Q_r (see [5]). Furthermore, I, R, and Q_r satisfy the same generalized polynomial identities with automorphisms (see [7, Theorem 1]).

Fact 2.2. Let R be a prime ring, then the following statements hold:

1. Every generalized derivation of R can be uniquely extended to Q_r ([16, Theorem 3]).
2. Any automorphism of R can be uniquely extended to Q_r ([6, Fact 2]).
3. Every generalized skew derivation of R can be uniquely extended to Q_r ([4, Lemma 2]) as follows: by a (right) generalized skew derivation of Q_r, we mean an additive mapping $G : Q_r \longrightarrow Q_r$ such that $G(xy) = G(x)y + \alpha(x)d(y)$, for all $x, y \in Q_r$, where d is a skew derivation of R and α is an automorphism of R. Moreover, there exists $G(1) = a \in Q_r$ such that $G(x) = ax + d(x)$, for all $x \in R$.

Fact 2.3. If $\Phi(x_i, D(x_i))$ is a generalized polynomial identity for R, where R is a prime ring and D is an outer skew derivation of R, then R also satisfies the generalized polynomial identity $\Phi(x_i, y_i)$, where x_i and y_i are distinct indeterminates. Furthermore, if $\Phi(x_i, D(x_i), \alpha(x_i))$ is a generalized polynomial identity for a prime ring R, D is an outer skew derivation of R and α is an outer automorphism of R, then R also satisfies the generalized polynomial identity $\Phi(x_i, y_i, z_i)$, where x_i, y_i, and z_i are distinct indeterminates (see [8]).

Fact 2.4. Let R be a prime ring and $n, k \geq 1$ fixed integers. If $([x, y]_k)^n = 0$, for any $x, y \in R$, then R is commutative (it follows from [9, Theorem 1] in its bounded version for prime rings).

Fact 2.5. Let R be a prime ring, $a \in R$, and $k \geq 1$ a fixed integer. If $[a, x]_k = 0$, for any $x \in R$, then $a \in Z(R)$ (it follows from [13, Theorem 2] in its bounded version for prime rings).

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The next Lemma has a value independent of the particular context of this paper, nonetheless, it will be useful for us to face the study in relation to the case in which R should be a matrix algebra:

Lemma 2.6. *Let $R = M_2(C)$ be the matrix ring over C, with characteristic of C different from 2 and 3. Let p, q be elements of R, φ an inner automorphism of R and denote $p = \sum_{r,s=1}^{2} p_{rs}e_{rs}$, $q = \sum_{r,s=1}^{2} q_{rs}e_{rs}$, $\varphi(p) = \sum_{r,s=1}^{2} p'_{rs}e_{rs}$ and $\varphi(q) = \sum_{r,s=1}^{2} q'_{rs}e_{rs}$, for $p_{rs}, q_{rs}, p'_{rs}, q'_{rs} \in C$, where e_{rs} is the usual matrix unit with 1 in the (r, s)-entry and zero elsewhere.*
Suppose that, for any inner automorphism φ of R, the following condition holds:

$$p_{ij}q_{ij} = 0, \qquad \forall i \neq j \qquad \Longrightarrow \qquad p'_{ij}q'_{ij} = 0, \qquad \forall i \neq j.$$

Then either $p \in Z(R)$ or $q \in Z(R)$.

Proof. Assume $p \notin Z(R)$ and $q \notin Z(R)$, otherwise we conclude. By our hypothesis, we have

$$p_{12}q_{12} = 0 \tag{2.1}$$

and

$$p_{21}q_{21} = 0. \tag{2.2}$$

Assume, firstly, that both p and q are not diagonal matrices and suppose $q_{21} \neq 0$. Then, by (2.2), $p_{21} = 0$; moreover, since p is not a diagonal matrix, $p_{12} \neq 0$ and, by (2.1), $q_{12} = 0$.
Let φ_1 and φ_2 be two inner automorphisms of $M_2(C)$, defined as

$$\varphi_1(x) = (1 + e_{21})x(1 - e_{21}) \qquad \textit{and} \qquad \varphi_2(x) = (1 - e_{21})x(1 + e_{21})$$

for all $x \in R$. If both the $(2, 1)$ entries of $\varphi_1(q)$ and $\varphi_2(q)$ are zero, we obtain

$$[\varphi_1(q)]_{21} = q_{21} + q_{11} - q_{22} = 0 \tag{2.3}$$

and

$$[\varphi_2(q)]_{21} = q_{21} - q_{11} + q_{22} = 0. \tag{2.4}$$

Combining (2.3) and (2.4), we get the contradiction $q_{21} = 0$. Therefore, at least one of them is zero.
If $[\varphi_1(q)]_{21} \neq 0$, then $[\varphi_1(p)]_{21} = 0$, that is

$$p_{11} - p_{22} - p_{12} = 0 \qquad \Longrightarrow \qquad 0 \neq p_{12} = p_{11} - p_{22}. \tag{2.5}$$

If $[\varphi_2(q)]_{21} \neq 0$, then $[\varphi_2(p)]_{21} = 0$, that is

$$-p_{11} + p_{22} - p_{12} = 0 \qquad \Longrightarrow \qquad 0 \neq p_{12} = p_{22} - p_{11}. \tag{2.6}$$

Case I.: Assume that (2.5) holds:
Let χ_1 and χ_2 be two inner automorphisms of $M_2(C)$, defined as

$$\chi_1(x) = (1 - e_{12})x(1 + e_{12}) \qquad \textit{and} \qquad \chi_2(x) = (1 + 2e_{12})x(1 - 2e_{12})$$

Since $[\chi_1(q)]_{21} \neq 0$, then $[\chi_1(p)]_{12} \neq 0$ and $[\chi_1(q)]_{12} = 0$, that is

$$0 \neq p_{12} - p_{22} + p_{11} = 2p_{12} \tag{2.7}$$

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and

$$-q_{22}+q_{11}-q_{21}=0 \qquad \Longrightarrow \qquad q_{21}=q_{11}-q_{22}. \tag{2.8}$$

Since $[\chi_2(q)]_{21}\neq 0$, then $[\chi_2(p)]_{12}\neq 0$ and $[\chi_2(q)]_{12}=0$, that is

$$0\neq p_{12}+2p_{22}-2p_{11}=-p_{12} \tag{2.9}$$

and

$$2q_{22}-2q_{11}-4q_{21}=0 \qquad \Longrightarrow \qquad 2q_{21}=q_{22}-q_{11}. \tag{2.10}$$

Comparing (2.8) and (2.10), we get $3q_{21}=0$; since $char(R)\neq 3$, we have $q_{21}=0$, a contradiction.

Case II.: Assume that (2.6) holds:

Let χ_3 and χ_4 be two inner automorphisms of $M_2(C)$, defined as

$$\chi_1(x)=(1+e_{12})x(1-e_{12}) \qquad \text{and} \qquad \chi_2(x)=(1-2e_{12})x(1+2e_{12}).$$

Since $[\chi_3(q)]_{21}\neq 0$, then $[\chi_3(p)]_{12}\neq 0$ and $[\chi_3(q)]_{12}=0$, that is

$$0\neq p_{12}+p_{22}-p_{11}=2p_{12} \tag{2.11}$$

and

$$q_{22}-q_{11}-q_{21}=0 \qquad \Longrightarrow \qquad q_{21}=q_{22}-q_{11}. \tag{2.12}$$

Since $[\chi_4(q)]_{21}\neq 0$, then $[\chi_4(p)]_{12}\neq 0$ and $[\chi_4(q)]_{12}=0$, that is

$$0\neq p_{12}-2p_{22}+2p_{11}=-p_{12} \tag{2.13}$$

and

$$-2q_{22}+2q_{11}-4q_{21}=0 \qquad \Longrightarrow \qquad 2q_{21}=q_{11}-q_{22}. \tag{2.14}$$

Comparing (2.12) and (2.14), we get $3q_{21}=0$; since $char(R)\neq 3$, we have $q_{21}=0$, a contradiction.
Therefore, either p or q is a diagonal matrix.
Assume that $q_{21}\neq 0$, it implies $p_{21}=p_{12}=0$; then, since $[\chi_1(q)]_{21}\neq 0$, we have

$$0=[\chi_1(p)]_{21}=p_{21}$$

and

$$0=[\chi_1(p)]_{12}=-p_{22}+p_{11}$$

that is $p\in Z(R)$, a contradiction. Analogously, if $p_{21}\neq 0$, we get $q\in Z(R)$, a contradiction.
Finally, assume that p and q are both diagonal matrices; in this case

$$p_{12}=q_{12}=p_{21}=q_{21}=0$$

and

$$[\chi_1(p)]_{12}=[\chi_1(q)]_{12}=[\chi_1(p)]_{21}=[\chi_1(p)]_{21}=0$$

Since

$$0=[\chi_1(p)]_{12}=-p_{22}+p_{11}$$

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and

$$0 = [\chi_1(q)]_{12} = -q_{22} + q_{11}.$$

We obtain $p, q \in Z(R)$, again a contradiction. □

3. The case of inner generalized derivations

In this Section we prove Theorem 1.1, when F is an inner generalized derivation of R, that is $F(x) = ax + xb$, for some $a, b \in Q_r$ and R satisfies

$$a[x,y]_k + [x,y]_k b - [ax+xb,y]_k - [x,ay+yb]_k = 0 \tag{3.1}$$

with $k > 1$ fixed integer.

Lemma 3.1. *If R satisfies (3.1), for some $a, b \in C$ and a fixed integer $k > 1$, then R is commutative unless $a + b = 0$.*

Proof. To prove this result it is enough to remark that (3.1) reduces to

$$(a+b)^k [x,y]_k = 0. \tag{3.2}$$

Since $a + b \in C$ is not a zero-divisor (unless it is zero), the commutativity of R follows from the Fact 2.4. □

Lemma 3.2. *If R satisfies (3.1), for some $a, b \in Q_r$ and a fixed integer $k > 1$, then one of the following holds:*

1. *R is commutative;*
2. *(3.1) is a non-trivial generalized polynomial identity of R;*
3. *$a, b \in C$ and $a + b = 0$.*

Proof. Assume R is not commutative and, since R and Q_r satisfy the same generalized polynomial identities (see Fact 2.1), assume that (3.1) is a trivial generalized polynomial identity of Q_r.
If both $a \in C$ and $b \in C$, (3.1) becomes $(a+b)^k[x,y]_k = 0$, for all $x, y \in Q_r$; it is a trivial generalized polynomial identity of Q_r only when $a + b = 0$, as required.
If $b \in C$ and $a \notin C$, Q_r satisfies the following generalized polynomial identity:

$$a[x,y]_k - [ax,y]_k - [x,py]_k$$

where $p = a + b \notin C$. Since $k \geq 2$ and $\{1, a\}$ is linearly C-independent, Q_r satisfies the following:

$$\sum_{i=1}^{k} (-1)^i \binom{k}{i} y^i x y^{k-i} - \sum_{i=1}^{k} (-1)^i \binom{k}{i} y (py)^{i-1} x (py)^{k-i}$$

which is a trivial generalized polynomial identity of Q_r only when $p \in Z(R)$ and $p^{k-1} = 1_C$, a contradiction (we omit the symmetrical case, when $a \in C$ and $b \notin C$).

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Therefore, we can assume that $\{1, a\}$ and $\{1, b\}$ is linearly C-independent; in this case Q_r satisfies the following trivial generalized polynomial identity:

$$\sum_{i=1}^{k-1}(-1)^i\binom{k}{i}y(ay+yb)^{i-1}x(ay+yb)^{k-i-1}y$$

It is trivial only if $a = -b \in C$, a contradiction. □

Lemma 3.3. *If $R = M_2(C)$, the ring of 2×2 matrices over C, with $char(C) \neq 2$, satisfies (3.1), for some $a, b \in Q_r$, and a fixed integer $k > 1$, then $a, b \in C$ and $a + b = 0$.*

Proof. By our assumption, R satisfies (3.1). Let e_{rs} be the usual matrix unit with 1 in the (r, s)-entry and zero elsewhere and denote $a = \sum_{hl} = a_{hl}e_{hl}$ and $b = \sum_{hl} = b_{hl}e_{hl}$, for $a_{hl}, b_{hl} \in C$.
Fix, for any $r \neq s$, $x = y = e_{rs}$ in (3.1), then we get

$$\sum_{i=0}^{k}(-1)^i\binom{k}{i}(e_{rs}b)^i e_{rs}(ae_{rs})^{k-i} = 0$$

It means $(a_{sr} - b_{sr})^k = 0$, for all $s \neq r$, that is $a - b$ is a diagonal matrix. Let φ be the automorphism of $M_m(C)$ defined as $\varphi(x) = (1 + e_{sr})x(1 - e_{sr})$, for all $x \in M_2(C)$, with $r \neq s$; notice that, by (3.1), R satisfies

$$\varphi(a)[x, y]_k + [x, y]_k\varphi(b) - [\varphi(a)x + x\varphi(b), y]_k - [x, \varphi(a)y + y\varphi(b)]_k$$

Therefore, also $\varphi(a - b) = (a - b) - (a - b)e_{sr} + e_{sr}(a - b) - e_{sr}(a - b)e_{sr}$ is a diagonal marix; in particular, the (s, r)-entry of $\varphi(a - b)$ is zero, that is $a_{ss} - b_{ss} = a_{rr} - b_{rr}$, for all $r \neq s$. It implies that $a - b \in Z(R)$.
Assume $b = a - \lambda$, for some $\lambda \in C$, then, by (3.1), R satisfies

$$a[x, y]_k + [x, y]_k a - [ax + xa, y]_k - [x, ay + ya - \lambda y]_k \tag{3.3}$$

Fix $y = y_0$, where $0 \neq y_0^2 \in Z(R)$; by easy calculations, we obtain $[[x, y_0]]_k = 2^{k-1}[x, y_0]y_0^{k-1}$ and, for $y = y_0$ in (3.3),

$$2^{k-1}\left([x, y_0][y_0^{k-1}, a] - (x[a, y_0] + [a, y_0]x)y_0^{k-1}\right) - [x, ay_0 + y_0a - \lambda y_0]_k = 0 \tag{3.4}$$

for all $x \in R$. Consider $y_0 = e_{11} - e_{22}$ and $x = e_{21}$ in (3.4), then $[x, y_0] = 2e_{21}$ and

$$2^{k-1}\left(2e_{21}[e_{11} + (-1)^{k-1}e_{22}, a] - \left(e_{21}[a, e_{11} - e_{22}] + [a, e_{11} - e_{22}]e_{21}\right)(e_{11} + (-1)^{k-1}e_{22})\right)$$
$$-[e_{21}, (a - \lambda)(e_{11} - e_{22}) + (e_{11} - e_{22})a]_k = 0$$

that is

$$X = 2^{k-1}\Big(2e_{21}a - 2e_{21}a(e_{11} + (-1)^{k-1}e_{22}) -$$
$$e_{21}a(e_{11} - e_{22})(e_{11} + (-1)^{k-1}e_{22}) + e_{21}(e_{11} - e_{22})a(e_{11} + (-1)^{k-1}e_{22}) -$$
$$a(e_{11} - e_{22})e_{21}(e_{11} + (-1)^{k-1}e_{22}) - (e_{11} - e_{22})ae_{21}(e_{11} + (-1)^{k-1}e_{22})\Big) -$$

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$$\sum_{i=0}^{k}(-1)^i\binom{k}{i}\Big((a-\lambda)(e_{11}-e_{22})+(e_{11}-e_{22})a\Big)^i e_{21}\cdot$$

$$\Big((a-\lambda)(e_{11}-e_{22})+(e_{11}-e_{22})a\Big)^{k-i}=0. \tag{3.5}$$

Notice that, if $Y=(a-\lambda)(e_{11}-e_{22})+(e_{11}-e_{22})a$, then

$$Y=\begin{pmatrix}\beta_1 & 0\\ 0 & \beta_2\end{pmatrix} \qquad \mathit{and} \qquad Y^k=\begin{pmatrix}\beta_1^k & 0\\ 0 & \beta_2^k\end{pmatrix}$$

where $\beta_1=2a_{21}-\lambda$ and $\beta_2=\lambda-2a_{22}$. Then

$$\sum_{i=0}^{k}(-1)^i\binom{k}{i}Y^ie_{21}Y^{k-i}=\begin{pmatrix}0 & 0\\ \beta_1^k & 0\end{pmatrix}+$$

$$\begin{pmatrix}0 & 0\\ (-1)^k\beta_2^k & 0\end{pmatrix}+\begin{pmatrix}0 & 0\\ \sum_{i=1}^{k-1}(-1)^i\binom{k}{i}\beta_2^i\beta_1^{k-i} & 0\end{pmatrix}$$

that is

$$\sum_{i=0}^{k}(-1)^i\binom{k}{i}Y^ie_{21}Y^{k-i}=\begin{pmatrix}0 & 0\\ (\beta_1-\beta_2)^k & 0\end{pmatrix}. \tag{3.6}$$

On the other hand, if

$$Z=2e_{21}a-2e_{21}a(e_{11}+(-1)^{k-1})-e_{21}a(e_{11}-e_{22})(e_{11}+(-1)^{k-1}e_{22})+$$

$$e_{21}(e_{11}-e_{22})a(e_{11}+(-1)^{k-1}e_{22})-a(e_{11}-e_{22})e_{21}(e_{11}+(-1)^{k-1}e_{22})-$$

$$(e_{11}-e_{22})ae_{21}(e_{11}+(-1)^{k-1}e_{22})$$

we get

$$Z=\begin{pmatrix}2a_{12} & 0\\ 0 & 2a_{12}+(-1)^k4a_{12}\end{pmatrix}. \tag{3.7}$$

Since

$$X=2^{k-1}Z-\sum_{i=0}^{k}(-1)^i\binom{k}{i}Y^ie_{21}Y^{k-i}$$

thank to (3.6) and (3.7), by (3.5), we get

$$X=\begin{pmatrix}2^ka_{12} & 0\\ -(\beta_1-\beta_2)^k & 2^ka_{12}+(-1)^k2^{k+1}a_{12}\end{pmatrix}. \tag{3.8}$$

In particular the $(1,1)$-entry of X is zero, that is, since $char(C)\neq 2$, $a_{12}=0$. Symmetrically, we obtain $a_{21}=0$, it follows a is a diagonal matrix. As above, it means $a\in Z(R)$.
Therefore $b\in Z(R)$ and, by (3.3), R satisfies

$$(2a-\lambda)^k[x,y]_k.$$

Since $[x,y]_k$ is not a polynomial identity of R, it implies

$$0 = 2a - \lambda = a + b$$

as required.

□

Proposition 3.4. *Let R be a prime ring of characteristic different from 2, Q_r its right Martindale quotient ring and C its extended centroid. Suppose there are $a, b \in Q_r$ such that $F(x) = ax + xb$, for any $x \in R$, and*

$$F([x,y]_k) - [F(x), y]_k - [x, F(y)]_k = 0$$

for all $x, y \in R$, with $k > 1$ fixed integer. Then R is commutative unless $a, b \in C$ and $a + b = 0$.

Proof. Assume R is not commutative, otherwise we are done. From our assumptions R, as well as Q_r, satisfies (3.1) and, since $F \neq 0$, by Lemma 3.2, it is a non-trivial generalized polynomial identity for Q_r. Then, because of [12, Theorems 2.5 and 3.5], we know that both Q_r and $Q_r \otimes_C \overline{C}$ are centrally closed, where $\overline{C}$ is the algebraic closure of C. We may replace Q_r by itself or $Q_r \otimes_C \overline{C}$ according as C is finite or infinite. Therefore we may assume that Q_r is centrally closed over C, which is either finite or algebraically closed. By Martindale's theorem [17], Q_r is a primitive ring having a non-zero socle H, with C as the associated division ring. In light of Jacobson's theorem [14, p. 75], Q_r is isomorphic to a dense ring of linear transformations on some vector space V over C. Since R is not commutative, $dim_C V > 1$.

Suppose $dim_C V = m \geq 2$, then $Q \cong M_m(C)$; for $m = 2$, since $F \neq 0$, by Lemma 3.3, we get a contradiction, therefore assume $m \geq 3$.

Let e_{rs} be the usual matrix unit with 1 in the (r,s)-entry and zero elsewhere , and denote $a = \sum_{hl} = a_{hl}e_{hl}$, $b = \sum_{hl} = b_{hl}e_{hl}$, for $a_{hl}, b_{hl} \in C$.

For any $r \neq s \neq t$, fix $x = e_{rs}$ and $y = e_{ts}$, then left multiplying by e_{rr} in (3.1), we get

$$e_{rs}(ae_{ts})^k = 0, \qquad \forall r \neq t \neq s$$

It implies $a_{st} = 0$, for all $s \neq t$, it means that a is a diagonal matrix. As above, it implies that $a \in Z(R)$.

Now (3.1) becomes

$$[x,y]_k b - [xb, y]_k - [x, yp]_k = 0 \tag{3.9}$$

for all $x, y \in M_m(C)$, with $p = a + b = \sum_{hl} = p_{hl}e_{hl}$, for $p_{hl} \in C$. Now, fixing $x = y = e_{ts}$, for any $s \neq t$, we get

$$(e_{ts}p)^k e_{ts} = 0, \qquad \forall t \neq s$$

so $p_{st} = 0$, for all $s \neq t$, it means that p is a diagonal matrix; it follows that $p \in Z(R)$. Therefore $b \in Z(R)$ and, by (3.9), $p^k[x,y]_k = 0$, for all $x, y \in R$. Since $[x,y]_k$ is not a polynomial identity of

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$M_m(C)$, it implies that $p=0$, that is a contradiction .
Now let $dim_C V=\infty$. We know that Q_r satisfies

$$a[x,y]_k+[x,y]_k b-[ax+xb,y]_k-[x,ay+yb]_k. \tag{3.10}$$

Let $x=1_C$. By (3.10), we get $[a+b,y]_k=0$, for all $y\in Q_r$. Since R is not commutative, by Fact 2.5, $a+b\in C$, so there exists $\lambda\in C$, such that $b=\lambda-a$.
Now (3.10) turneds into

$$(a+\lambda)[x,y]_k-[x,y]_k a-[(a+\lambda)x-xa,y]_k-[x,(a+\lambda)y-ya]_k=0 \tag{3.11}$$

for all $x,y\in Q_r$. Notice that by fixing $x=y=r_0$, with $0\neq r_0\in Q_r$ such that $r_0^2=0$, and (3.11), we get

$$\sum_{i=0}^{k}(-1)^i\binom{k}{i}(-r_0a)^i r_0(ar_0)^{k-i}=0 \tag{3.12}$$

By easy calculations on (3.12) we have,

$$2^k(r_0a)^k r_0=0 \tag{3.13}$$

Now, fix a minimal idempotent element $e\in Soc(R)$ and consider $r_0=ey(1-e)$. In this case, by (3.13),

$$\Big(ey(1-e)a\Big)^k ey(1-e)=0$$

for all $y\in Q_r$. The primeness of R yields $(1-e)ae=0$, that is $eae=ae$.
Analogously, for $r_0=(1-e)ye$, we get $eae=ea$. It means that for any minimal idempotent $e\in Soc(R)$, $[a,e]=0$; since $Soc(R)$ is generated by these idempotent elements, $[a,Soc(R)]=0$, that is, $a\in C$. In this case, we have also $b\in C$ and, by (3.11), Q_r satisfies $\lambda^k[x,y]_k=0$. It follows $\lambda=0$, otherwise, by Fact 2.4, we get the commutativity of R, that is a contradiction. ☐

4. The case of inner generalized skew derivations

In this section we prove Theorem 1.1, when F is an inner generalized skew derivation of R with associated automorphism α, that is $F(x)=ax+\alpha(x)b$, for some $a,b\in Q_r$. In this case R satisfies

$$a[x,y]_k+\alpha([x,y]_k)b-[ax+\alpha(x)b,y]_k-[x,ay+\alpha(y)b]_k=0 \tag{4.1}$$

for fixed integer $k>1$. We assume firstly that α is an inner automorphism of R, that is there exists an invertible element $q\in Q_r$, such that $\alpha(x)=qxq^{-1}$, for all $x\in R$. In this case, by (4.1), R satisfies the following generalized polynomial identity:

$$\Phi(x,y)=a[x,y]_k+q[x,y]_kq^{-1}b-[ax+qxq^{-1}b,y]_k-[x,ay+qyq^{-1}b]_k. \tag{4.2}$$

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Lemma 4.1. *Let R be a non-commutative prime ring of characteristic different from 2, which is isomorphic to a dense ring of linear transformations on some vector space V over C,and let q be an invertible element of R and $a, b \in R$ such that R satisfies (4.2), for a fixed integer $k > 1$. If $q \notin C$ and $q^{-1}b \notin C$, then $dim_C V = 2$.*

Proof. Assume $dim_D V \geq 3$. By Fact 2.1, also Q_r satisfies (4.2). Since $q^{-1}b \notin C$, there exists $v \in V$ such that $\{q^{-1}bv, v\}$ is linearly C-independent; since $dim_C V \geq 3$, so there exists $w \in V$ such that $\{q^{-1}bv, v, w\}$ is linearly C-independent.

Now we divided our proof in three steps:

STEP 1.: $\{q^{-1}bv, v, w, qw\}$ is linearly C-dependent.

Proof. If $\{q^{-1}bv, v, w, qw\}$ is linearly C-independent, by density of Q_r, there exist $r, s \in Q_r$ such that

$$\begin{array}{ccc} rv = 0 & rq^{-1}bv = w & sw = 0 \\ sv = 0 & sqw = qw & sq^{-1}bv = 0. \end{array}$$

Hence, by main assumption, we get the contradiction

$$0 = \Big(a[r,s]_k + q[r,s]_k q^{-1}b - [ar + qrq^{-1}b, s]_k - [r, as + qsq^{-1}b]_k\Big)v = -(-1)^k qw \neq 0.$$

Then $\{q^{-1}bv, v, w, qw\}$ is linearly C-dependent, that is there exist $\lambda_1, \lambda_2, \lambda_3 \in C$ such that

$$qw = \lambda_1 q^{-1}bv + \lambda_2 v + \lambda_3 w.$$

□

STEP 2.: $\{w, qw\}$ is linearly C-dependent.

Proof. Since $\{q^{-1}bv, v, w\}$ is linearly C-independent, by density of Q_r, there exist $r, s \in Q_r$ such that

$$rv = 0 \qquad rq^{-1}bv = w \qquad sv = 0 \qquad sq^{-1}bv = 0 \qquad sw = w$$

Hence, by main assumption, we get $sqw = \lambda_3 w$ and

$$\begin{aligned} 0 &= \Big(a[r,s]_k + q[r,s]_k q^{-1}b - [ar + qrq^{-1}b, s]_k - [r, as + qsq^{-1}b]_k\Big)v \\ &= (-1)^k qw - (-1)^k \lambda_3 w \\ &= (-1)^k \Big(\lambda_1 q^{-1}bv + \lambda_2 v\Big). \end{aligned}$$

Since $\{q^{-1}bv, v\}$ is linearly C-independent, it implies $\lambda_1 = \lambda_2 = 0$, that is

$$qw = \lambda_3 w.$$

□

STEP 3.: For any $u \in V$, $\{u, qu\}$ is linearly C-dependent.

Proof. Suppose that there exists $u_0 \in V$ such that $\{u_0, qu_0\}$ is linearly C-independent.
If $\{q^{-1}bv, v, u_0\}$ is linearly C-independent, as in the previous steps, we get $\{u_0, qu_0\}$ is linearly C-dependent, a contradiction. So $\{q^{-1}bv, v, u_0\}$ is linearly C-dependent, that is

$$u_0 = \eta_1 q^{-1}bv + \eta_2 v \qquad for\ some\ \eta_1, \eta_2 \in C$$

Since $dim_C V \geq 3$, there exists $w_0 \in V$ such that $\{q^{-1}bv, v, w_0\}$ is linearly C-independent. As above, $\{w_0, qw_0\}$ is linearly C-dependent.
Notice that if $u_0 + w_0 \in Span\{q^{-1}bv, v\}$, then

$$u_0 + w_0 = \theta_1 q^{-1}bv + \theta_2 v \qquad for\ some\ \theta_1, \theta_2 \in C \tag{4.3}$$

and

$$w_0 = (\theta_1 - \eta_1)q^{-1}bv + (\theta_2 - \eta_2)v. \tag{4.4}$$

It means that $\{q^{-1}bv, v, w_0\}$ is linearly C-dependent, a contradiction (Analogously, we get a contradiction, if we suppose $u_0 - w_0 \in Span\{q^{-1}bv, v\}$).
Then $\{q^{-1}bv, v, u_0 + w_0\}$ and $\{q^{-1}bv, v, u_0 - w_0\}$ are linearly C-independent. As in the previous steps, $\{u_0 + w_0, q(u_0 + w_0)\}$ and $\{u_0 - w_0, q(u_0 - w_0)\}$ is linearly C-dependent. Then

$$q(u_0 + w_0) = \gamma_1(u_0 + w_0) \qquad for\ some\ \gamma_1 \in C \tag{4.5}$$

$$q(u_0 - w_0) = \gamma_2(u_0 - w_0) \qquad for\ some\ \gamma_2 \in C. \tag{4.6}$$

Comparing both (4.5) and (4.6), we get

$$2qw_0 = (\gamma_1 - \gamma_2)u_0 + (\gamma_1 + \gamma_2)w_0 \tag{4.7}$$

and

$$2qu_0 = (\gamma_1 + \gamma_2)u_0 + (\gamma_1 - \gamma_2)w_0. \tag{4.8}$$

Moreover, since $\{w_0, qw_0\}$ is linearly C-dependent, there exists $\lambda \in C$ such that $qw_0 = \lambda w_0$, so that (4.7) becomes

$$2\lambda w_0 = (\gamma_1 - \gamma_2)u_0 + (\gamma_1 + \gamma_2)w_0. \tag{4.9}$$

If $\{u_0, w_0\}$ is linearly C-dependent, that is $u_0 = \mu w_0$, for some $\mu \in C$, then

$$qu_0 = \mu q w_0 = \lambda\mu w_0 = \lambda u_0$$

Since $\{u_0, qu_0\}$ is linearly C-independent, it is a contradiction.
Then $\{u_0, w_0\}$ is linearly C-independent; therefore, by (4.9), $\lambda = \gamma_1 = \gamma_2$ and, by (4.8), since $char(R) \neq 2$, $qu_0 = \lambda u_0$, a contradiction. □

The conclusion of the last step leads to a contradiction, $q \in C$. □

Lemma 4.2. *Let R be a non-commutative prime ring of characteristic different from 2, α an automorphism of R and $a, b \in R$ such that R satisfies (4.1), for a fixed integer $k > 1$.*
Then, for any idempotent element $e^2 = e \in R$, one of the following holds:

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(1) $\alpha(1-e)be = 0$
(2) $\alpha^{-1}(1-e)e = 0$

Proof. Replacing x with $x+1$ in (4.1), we get $[a+b, y]_k = 0$, for all $y \in R$, that is, by Fact 2.5, $a + b \in Z(R)$.
Then, for any idempotent element $e^2 = e \in R$,

$$(1-e)(a+b)e = 0 \tag{4.10}$$

and, replacing x with $ex\alpha^{-1}(1-e)$ and y with $y = \alpha^{-1}(e)y(1-e)$, (4.1) becomes

$$\Big[aex\alpha^{-1}(1-e) + \alpha(e)\alpha(x)(1-e)b, \alpha^{-1}(e)y(1-e)\Big]_k$$

$$+\Big[ex\alpha^{-1}(1-e), a\alpha^{-1}(e)y(1-e) + e\alpha(y)\alpha(1-e)b\Big]_k = 0 \tag{4.11}$$

for all $x, y \in R$. Right multiplying by e and left multiplying by $\alpha^{-1}(1-e)$ in (4.11), R satisfies

$$\alpha^{-1}(1-e)\sum_{i=0}^{k}(-1)^i\binom{k}{i}\Big(e\alpha(y)\alpha(1-e)b\Big)^i ex\alpha^{-1}(1-e)\Big(e\alpha(y)\alpha(1-e)b\Big)^{k-i}e$$

that is

$$\alpha^{-1}(1-e)e\Big[x\alpha^{-1}(1-e)e, \alpha(y)\alpha(1-e)be\Big]_k = 0 \tag{4.12}$$

for all $x, y \in R$.
Fix $x = x_0$ and denote $u_0 = \alpha^{-1}(1-e)e$ and $v_0 = x_0\alpha^{-1}(1-e)e$, then

$$u_0\Big[v_0, \alpha(y)\alpha(1-e)be\Big]_k = 0$$

for all $y \in R$. Assume $\alpha(1-e)be \neq 0$, otherwise we are done.
Consider $I = \alpha(y)\alpha(1-e)be$ and $\delta(x) = [v_0, x]$, the derivation of R induced by v_0, then

$$u_0\Big[\delta(z), z\Big]_{k-1} = 0$$

for all $z \in I$. Since I is a non-zero left ideal of R, by [19], one of the following holds:

(1) u_0;
(2) $\delta = 0$;
(3) $Iv_0 = 0$ and $u_0v_0 = 0$.

Assume $u_0 \neq 0$, otherwise we are done. Then, for all $x, y \in R$, either

$$x\alpha^{-1}(1-e)e \in Z(R) \tag{4.13}$$

or both the following hold:

$$\alpha(y)\alpha(1-e)bex\alpha^{-1}(1-e)e = 0 \tag{4.14}$$

$$\alpha^{-1}(1-e)ex\alpha^{-1}(1-e)e = 0. \tag{4.15}$$

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If R satisfies (4.13), then, for all $x \in R$, $[x\alpha^{-1}(1-e)e, 1-e] = 0$, that is $(1-e)x\alpha^{-1}(1-e)e = 0$. It implies $\alpha^{-1}(1-e)e = 0$, a contradiction.
If R satisfies (4.14) and (4.15), we get again the contradiction $\alpha^{-1}(1-e)e = 0$.
Assume there exists $x_1 \in R$ such that $x_1\alpha^{-1}(1-e)e \in Z(R)$ and, for $x = x_1$, at least one of (4.14) and (4.15) does not hold; analogously, assume there exists $x_2 \in R$ such that, for $x = x_2$, R satisfies both (4.14) and (4.15) and $x_2\alpha^{-1}(1-e)e \notin Z(R)$.
By repeating a similar argument with $x = x_1 - x_2$ in (4.12), we get either

$$(x_1 - x_2)\alpha^{-1}(1-e)e \in Z(R) \ \Rightarrow \ x_2\alpha^{-1}(1-e)e \in Z(R)$$

that is a contradiction, or

$$\alpha(y)\alpha(1-e)be(x_1-x_2)\alpha^{-1}(1-e)e = \alpha^{-1}(1-e)e(x_1-x_2)\alpha^{-1}(1-e)e = 0$$

for all $y \in R$. In this case we get

$$\alpha(y)\alpha(1-e)bex_1\alpha^{-1}(1-e)e = 0 \ and \ \alpha^{-1}(1-e)ex_1\alpha^{-1}(1-e)e = 0$$

At least one of them is a contradiction. □

Proposition 4.3. *Let R be a prime ring of characteristic different from 2 and 3, Q_r its right Martindale quotient ring and C its an or the extended centroid. Suppose that F is a non-zero generalized skew derivation of R, defined as $F(x) = ax + qxq^{-1}b$, for all $x \in R$, where $a, b \in Q_r$ and q is an invertible element of Q_r, such that*

$$F([x,y]_k) - [F(x),y]_k - [x,F(y)]_k = 0$$

for all $x, y \in R$, with $k > 1$ fixed integer. Then R is commutative.

Proof. In all that follows we assume $q \notin C$ and $q^{-1}b \notin C$, otherwise we conclude the result, by Proposition 3.4.

Now, to reach a contradiction, suppose that R is not commutative .
Hence, $\Phi(x,y)$ is a non-trivial generalized polynomial identity for R, as well as Q_r. By Martindale's theorem [17], Q_r is a primitive ring having a non-zero socle H, with C as the associated division ring. In light of Jacobson's theorem [14, p. 75], Q_r is isomorphic to a dense ring of linear transformations on some vector space V over C. Since R is not commutative, by Lemma 4.1, we get $dim_C V = 2$, that is $Q \cong M_2(C)$.
Since e_{22} is an idempotent element of R and $\alpha(x) = qxq^{-1}$, for all $x \in R$, by Lemma 4.2, one of the following holds:

(1) $qe_{11}q^{-1}be_{22} = 0$;
(2) $q^{-1}e_{11}qe_{22} = 0$.

If we denote $q^{-1}b = \sum_{h,l=1}^{2} p_{hl}e_{hl}$ and $q = \sum_{h,l=1}^{2} q_{hl}e_{hl}$, for $p_{hl}, q_{hl} \in C$, it follows

(1) either $p_{12} = 0$;
(2) or $q_{12} = 0$.

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It means $p_{12}q_{12}=0$; since, analogously, we have $p_{21}q_{21}=0$, it follows

$$p_{ij}q_{ij}=0 \qquad \textit{for all} \;\; i\neq j$$

By Lemma 2.6, either $q\in C$ or $q^{-1}b\in C$; in both cases, we have a contradiction.

□

Now we are ready to prove the main Proposition of the Section:

Proposition 4.4. *Let R be a prime ring of characteristic different from 2 and 3, Q_r its right Martindale quotient ring and C its extended centroid. Suppose that $\alpha\in Aut(R)$ and F is a non-zero generalized skew derivation of R, defined as $F(x)=ax+\alpha(x)b$, for all $x\in R$, where $a,b\in Q_r$, such that*

$$F([x,y]_k)-[F(x),y]_k-[x,F(y)]_k=0$$

for all $x,y\in R$, with $k>1$ fixed integer. Then R is commutative.

Proof. In this case R satisfies (4.1).
If one of the following holds:

(1) α is the identity map of R;
(2) α is an inner automorphism of R;
(3) $b=0$,

then we conclude by Propositions 3.4 and 4.3.
Assume $\alpha\neq id_R$, α outer and $b\neq 0$.
Consider a fixed element $0\neq y_0\in R$ and denote $p_1=\alpha(y_0)$ and $p_2=ay_0+p_1b$; for $y=y_0$, then (4.1) becomes

$$a[x,y_0]_k+[\alpha(x),p_1]_kb-[ax+\alpha(x)b,y_0]_k-[x,p_2]_k=0$$

for all $x\in R$. Since $\alpha(x)$-word degree is 1, by [7, Theorem 3], R satisfies

$$a[x,y_0]_k+[z,p_1]_kb-[ax+zb,y_0]_k-[x,p_2]_k=0. \tag{4.16}$$

For $x=0$, and $z=p_1$,

$$[p_1b,y_0]_k=0. \tag{4.17}$$

By repeating the same argument, we notice that the condition in (4.17) holds for all $y_0\in R$ (also for $y_0=0$), therefore $[[\alpha(y)b,y]]_k=0$ for all $y\in R$. Since $\alpha(y)$-word degree is 1, then R satisfies $[[zb,y]]_k$ By Fact 2.5 and since $b\neq 0$, it follows $zb\in Z(R)$ for any $z\in R$. It is easy to see that R must be commutative, as required. □

5. The proof of Theorem 1.1

Now, we are ready to prove the main Theorem of the paper. As remarked in Fact 2.2, we assume that there exists $a\in Q_r$ such that $F(x)=ax+d(x)$, for all $x\in R$, where d is a skew derivation on R and α the automorphism associated with d; it means that $d(xy)=d(x)y+\alpha(x)d(y)$, for all $x,y\in R$.

Proof. By our assumption, R, as well as Q_r, satisfies

$$a[x,y]_k + d([x,y]_k) - [ax + d(x), y]_k - [x, ay + d(y)]_k = 0. \tag{5.1}$$

Assume firstly that d is an inner skew derivation of R, that is there exists $c \in Q_r$ such that $d(x) = cx - \alpha(x)c$, for all $x \in R$. In this case, by (5.1), Q_r satisfies

$$a'[x,y]_k + \alpha([x,y]_k)c' - [a'x + \alpha(x)c', y]_k - [x, a'y + \alpha(y)c']_k = 0 \tag{5.2}$$

where $a' = a + c$ and $c' = -c$. We conclude by Proposition 4.4.

Assume that d is an outer skew derivation of R, then, by applying Fact 2.3, replace $d(x)$ and $d(y)$ respectively with z_1 and z_2 in (5.1), then Q_r satisfies

$$a[x,y]_k + \Psi(x, y, \alpha(x), \alpha(y), z_1, z_2) - [ax + z_1, y]_k - [x, ay + z_2]_k = 0 \tag{5.3}$$

where

$$\Psi(x, y, \alpha(x), \alpha(y), z_1, z_2) = d([x,y]_k).$$

Notice that, since $k \geq 2$, for $y = 0$, we get $\Psi = 0$ and, by (5.3), $[[x, z_2]]_k = 0$ for all $x, z_2 \in Q_r$; by Fact 2.4, R is commutative, as required. ☐

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