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TSALLIS RELATIVE OPERATOR ENTROPY PROPERTIES WITH SOME WEIGHTED METRICS

A. EL HILALI AND M. CHERGUI*

Abstract. The present work attempts to provide some properties for Tsallis relative operator entropy $T_p(A|B)$, acting on positive definite matrices, with respect to weighted Hellinger and Alpha Procrustes distances. Many localizations of this operator have been determined. In particular, some estimations of the distances between $T_p(A|B)$ and some standard matrix means are outlined.

1. Introduction

Shannon's entropy [21] is defined for a discrete random variable X with a probability distribution $\{p_i\}_i$, as follows

$$S_s(X) = -\sum_{i=1}^{n} p_i \log p_i. \tag{1.1}$$

The authors in [14] provided another generalization to the entropy' Shannon $S_s(X)$. Namely, for a discrete probability distribution $p(x) = p(X = x)$ of a random variable X the Tsallis entropy is defined by the following formula

$$T_q(X) \equiv -\sum_{x} p^q(x) \log_q \big(p(x)\big),\ q \in \mathbb{R},$$

where $\log_q$ refers to the q-logarithmic function defined as follows

$$\log_q(x) = \frac{x^{1-q} - 1}{1 - q},$$

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∗Corresponding author

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for any non negative real number x and $q \neq 1$.
Tsallis entropy is a one-parameter extension of Shannon entropy (1.1). This can be shown by the fact that

$$\lim_{q \longrightarrow 1} T_q(X) = S_s(X).$$

The generalization of the Tsallis operator entropy to density matrices, i.e. whose trace is 1, was introduced by Abe [1] by the following formula

$$T_p(\rho \mid \sigma) = \frac{1 - tr(\rho^p \sigma^{1-p})}{1-p} = tr\big(\rho^p(\log_p \rho - \log_p \sigma)\big),$$

for any $0 \leq p < 1$.
$T_p(\rho \mid \sigma)$ is one-parameter extension of the following quantum relative entropy

$$S(\rho \mid \sigma) = tr\big(\rho(\log \rho - \log \sigma)\big),$$

in the meaning that

$$\lim_{p \to 1} T_p(\rho \mid \sigma) = S(\rho \mid \sigma).$$

To provide a more precise overview of this generalization process, we need to start by recalling some concepts and notations, which will also be needed in the sequel.

Let $\mathbb{M}_n$ be the algebra of $n \times n$ matrices over $\mathbb{R}$ endowed by an inner product $\langle . \rangle$, and $\mathbb{P}_n$ the cone of positive definite elements of $\mathbb{M}_n$. I will stand for the identity matrix.
We recall that for any two matrices A and B from $\mathbb{P}_n$, we set $A \leq B$ to mean that $B - A \geq 0$, i.e $B - A$ is a positive matrix.
Using this order, a real-valued function f, defined on an interval $J \subset \mathbb{R}$, is said to be matrix monotone for $n \times n$ matrices if for every selfadjoint matrices $A, B \in \mathbb{M}_n$ whose eigenvalues are in J, we have

$$A \leq B \Longrightarrow f(A) \leq f(B).$$

As a very useful example of such functions, we recall that the function defined on $[0, \infty)$ by $x \longmapsto f(x) = x^r$ is a matrix monotone function when $r \in [0, 1]$. Furthermore, as stated in [7], it is crucial to emphasize that, if $r > 1$, this monotonicity is no longer true.

In [23], Yanagi et al introduced a parametric extension of relative operator entropy for positive definite matrices as follows

$$T_p(A \mid B) = \frac{A \sharp_p B - A}{p}, \qquad p \in [-1, 1] \setminus \{0\}, \tag{1.2}$$

called Tsallis relative operator entropy.

$A \sharp_p B := A^{\frac{1}{2}} \left(A^{\frac{-1}{2}} B A^{\frac{-1}{2}}\right)^p A^{\frac{1}{2}}$, for all $p \in \mathbb{R}$, represents the p-weighted geometric mean of A and B, denoted simply $A \sharp B$, if $p = \frac{1}{2}$.

The extension (1.2) is justified by the following fact

$$\lim_{p \to 0} T_p(A \mid B) = S(A \mid B),$$

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where $S(A \mid B)$ stands for the relative operator entropy defined by Fujji and Kamei [11, 12] for any positive definite matrices A and B by the following formula

$$S(A \mid B) = A^{\frac{1}{2}} \log \left(A^{\frac{-1}{2}} B A^{\frac{-1}{2}}\right) A^{\frac{1}{2}}. \tag{1.3}$$

Recently, studies on Tsallis relative operator entropy have grown considerably, given the contribution of this notion to the field of information theory. Several research projects have been published, such as [6, 9–11, 13, 15, 17–19, 23], where the following properties can be found:

$$B \geq A \Longrightarrow T_p(A \mid B) \geq 0,$$

$$T_p(A \mid B) \leq T_q(A \mid B), \forall\, 0 < p \leq q \leq 1, \tag{1.4}$$

$$S(A \mid B) \leq T_p(A \mid B) \leq B - A, \tag{1.5}$$

$$T_p(A \mid B) \geq A \text{ for all } B \geq eA \text{ and } p \in (0, 1]. \tag{1.6}$$

$$T_p(A \mid B) \geq A^{-1} \text{ for all } B \geq A^{\frac{1}{2}} \exp(A^{-2}) A^{\frac{1}{2}} \text{ and } p \in (0, 1]. \tag{1.7}$$

The entropy is an efficient tool for measuring the variation between two states in various areas like quantum mechanics, quantum information, and learning information. The study of this evolution has profited substantially from the notion of distance of positive matrices, introduced by several authors, such as [2, 4, 5]. Using this approach, the authors have established several metric properties of entropy operators [3, 8–10].

In this article, we contribute to this axis of research by investigating further geometric properties for Tsallis relative operator entropy defined by (1.2) with respect to weighted Hellinger metric and Alpha Procrustes distance introduced very recently in [5] and [16] respectively.

Given $\alpha > 0$ and two positive definite matrices A and B, weighted Hellinger metric is defined by

$$d_{1,\alpha}(A, B) = \frac{1}{\alpha} d_1\left(A^{2\alpha}, B^{2\alpha}\right), \tag{1.8}$$

where $d_1(A, B) := \sqrt{trA + trB - 2tr\left(A^{\frac{1}{2}} B^{\frac{1}{2}}\right)}$ refers to Bhattacharya metric [22] widely used in quantum information theory.

Alpha Procrustes distance is defined by the following expression

$$d_{2,\alpha}(A, B) = \frac{1}{\alpha} d_2\left(A^{2\alpha}, B^{2\alpha}\right), \tag{1.9}$$

where $d_2(A, B) := \sqrt{trA + trB - 2tr\left(A^{\frac{1}{2}} B A^{\frac{1}{2}}\right)^{\frac{1}{2}}}$ is the Bures-Wasserstein distance [2], closely related to optimal transport theory.

$tr(A)$ stands for the trace of the matrix A, which is a linear form and satisfies the following cyclicity property [24, 25]

$$tr(AB) = tr(BA), \text{ for any matrices } A, B \in \mathbb{M}_n.$$

The distances $d_{1,\alpha}$ and $d_{2,\alpha}$ are equivalent on $\mathbb{P}_n$ [5]. That is, for any $A, B \in \mathbb{P}_n$, we have

$$d_{2,\alpha}(A, B) \leq d_{1,\alpha}(A, B) \leq \sqrt{2}\, d_{2,\alpha}(A, B). \tag{1.10}$$

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The rest of the paper is structured as follows. Section 2 is focused on some monotonicity properties related to $T_p(A \mid B)$ via the metrics $d_{i,\alpha}$, $1 \le i \le 2$, and on localizing $T_p(A \mid B)$ as well. Section 3 deals with estimating distances between Tsallis relative operator entropy and some standard matrix means.

2. Monotonicity and position properties

In this section, we study the monotonicity of the map $p \longmapsto d_{i,\alpha}\big(A, T_p(A \mid B)\big)$ $(1 \le i \le 2)$, for two positive definite matrices A and B. Throughout this paper, we stand $C := A^{\frac{-1}{2}} B A^{\frac{-1}{2}}$.
To establish our findings, we need the following auxiliary result, concerning the monotonicity of the trace function [20, Proposition 1].

Lemma 2.1. *Let $A, B \in \mathbb{P}_n$ and $f : [0, \infty) \to \mathbb{R}$ be an increasing continuous function. We have,*

$$A \le B \Longrightarrow tr[f(A)] \le tr[f(B)].$$

As a consequence of Lemma 2.1, we have for any $X \in \mathbb{P}_n$ the following assertion.

$$A \le B \Longrightarrow tr(X\,A) \le tr(X\,B). \tag{2.1}$$

We are now in a position to study the monotonicity of the map $X \longrightarrow d_{1,\alpha}(A, X)$.

Proposition 2.2. *Let A, M and N be three matrices from $\mathbb{P}_n$ such that $A \le M \le N$. For all $\alpha \in (0, 1]$, we have*

$$d_{1,\alpha}(A, M) \le d_{1,\alpha}(A, N).$$

Proof. By the monotonocity of the matrix function $\alpha \longmapsto x^\alpha$ $(x > 0)$ on $(0, 1]$, we have

$$A^\alpha \le M^\alpha \le N^\alpha.$$

These lead to

$$0 \le M^\alpha - A^\alpha \le N^\alpha - A^\alpha.$$

The use of Lemma 2.1 allows us to write the following inequality

$$tr\,(M^\alpha - A^\alpha)^2 \le tr\,(N^\alpha - A^\alpha)^2\,,$$

combined with the cyclicity property of the trace function, we obtain the desired result. □

The following theorem deals with the monotonicity of the map $p \longrightarrow d_{1,\alpha}\big(A, T_p(A \mid B)\big)$.

Theorem 2.3. *Let A and B be two matrices from $\mathbb{P}_n$ such that $B \ge eA$. For all $\alpha \in (0, 1]$, we have*

$$d_{1,\alpha}\big(A, T_p(A \mid B)\big) \le d_{1,\alpha}\big(A, T_q(A \mid B)\big), \tag{2.2}$$

for all $0 < p \le q \le 1$.

Proof. Combining inequalities (1.4) and (1.6), we obtain

$$A \le T_p(A \mid B) \le T_q(A \mid B),\ \forall\, 0 < p \le q \le 1.$$

Using Proposition 2.2, we get the inequality (2.2). □

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The monotonicity of the map $X \longrightarrow d_{2,\alpha}(A, X)$ is presented in the following proposition.

Proposition 2.4. *Let A, B and M be three matrices from $\mathbb{P}_n$ such that $I \le A^{-1} \le B \le M$. For all $\alpha \in \left(0, \frac{1}{2}\right]$, We have*

$$d_{2,\alpha}(A, B) \le d_{2,\alpha}(A, M). \tag{2.3}$$

Proof. Let $I \le A^{-1} \le B \le M$. Using the monotonicity of the map $\alpha \longmapsto x^{2\alpha}$ on $\left(0, \frac{1}{2}\right]$, we get

$$A^{-2\alpha} \le B^{2\alpha} \le M^{2\alpha}.$$

So,

$$2I \le \left(A^{\alpha} B^{2\alpha} A^{\alpha}\right)^{\frac{1}{2}} + \left(A^{\alpha} M^{2\alpha} A^{\alpha}\right)^{\frac{1}{2}}.$$

Applying the inequality (2.1), we obtain

$$2.tr\left[\left(A^{\alpha} M^{2\alpha} A^{\alpha}\right)^{\frac{1}{2}} - \left(A^{\alpha} B^{2\alpha} A^{\alpha}\right)^{\frac{1}{2}}\right] \le tr\left[\left(\left(A^{\alpha} M^{2\alpha} A^{\alpha}\right)^{\frac{1}{2}} + \left(A^{\alpha} B^{2\alpha} A^{\alpha}\right)^{\frac{1}{2}}\right)\left(\left(A^{\alpha} M^{2\alpha} A^{\alpha}\right)^{\frac{1}{2}} - \left(A^{\alpha} B^{2\alpha} A^{\alpha}\right)^{\frac{1}{2}}\right)\right].$$

By the cyclicity property of trace and remarking that $A \le I$, we get

$$\begin{aligned} 2.tr\left[\left(A^{\alpha} M^{2\alpha} A^{\alpha}\right)^{\frac{1}{2}} - \left(A^{\alpha} B^{2\alpha} A^{\alpha}\right)^{\frac{1}{2}}\right] &\le tr\left(A^{\alpha} M^{2\alpha} A^{\alpha} - A^{\alpha} B^{2\alpha} A^{\alpha}\right) \\ &= tr\left(A^{2\alpha}\left(M^{2\alpha} - B^{2\alpha}\right)\right) \\ &\le tr\left(M^{2\alpha} - B^{2\alpha}\right). \end{aligned}$$

Which is equivalent to the inequality (2.3). □

The ongoing theorem presents the monotonicity of the map $p \longmapsto d_{2,\alpha}\left(A, T_p(A \mid B)\right)$ on $(0, 1]$.

Theorem 2.5. *Let $A, B \in \mathbb{P}_n$ such that $A \le I$, and $B \ge A^{\frac{1}{2}} \exp(A^{-2}) A^{\frac{1}{2}}$. For all $\alpha \in \left(0, \frac{1}{2}\right]$ and for all $0 < p \le q \le 1$, the following inequality holds*

$$d_{2,\alpha}\left(A, T_p(A \mid B)\right) \le d_{2,\alpha}\left(A, T_q(A \mid B)\right). \tag{2.4}$$

Proof. Let $A \le I$ and $B \ge A^{\frac{1}{2}} \exp(A^{-2}) A^{\frac{1}{2}}$.
Combining the inequalities (1.7) and (1.4) we can write, for all $0 < p \le q \le 1$,

$$A^{-1} \le T_p(A \mid B) \le T_q(A \mid B).$$

To get the inequality (2.4) it suffices to employ Proposition 2.4. □

The position of $T_p(A \mid B)$ concerning the sphere centered at A with radius $d_{i,\alpha}(A, B)(1 \le i \le 2)$, is investigated in the following theorem.

Theorem 2.6. *Let $A, B \in \mathbb{P}_n$ and $p \in (0, 1]$. We have the following assertions:*

(i) *If $B \ge eA$ and $\alpha \in (0, 1]$, then*

$$d_{1,\alpha}\left(A, T_p(A \mid B)\right) \le d_{1,\alpha}(A, B). \tag{2.5}$$

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(ii) *If* $A \leq I$, $B \geq A^{\frac{1}{2}} \exp(A^{-2}) A^{\frac{1}{2}}$ *and* $\alpha \in \big(0, \frac{1}{2}\big]$*, then*

$$d_{2,\alpha}\big(A, T_p(A \mid B)\big) \leq d_{2,\alpha}(A, B). \tag{2.6}$$

Proof. Combining the inequalities (1.5) and (1.6), we get

$$A \leq T_p(A \mid B) \leq B.$$

Applying Proposition 2.2, we obtain the desired inequality (2.5).

Let $B \geq A^{\frac{1}{2}} \exp(A^{-2}) A^{\frac{1}{2}}$ and $A \leq I$. Thanks to the inequality (1.7) and the right side of the inequality (1.5), we obtain for all $p \in (0, 1]$,

$$A^{-1} \leq T_p(A \mid B) \leq B.$$

Using Proposition 2.4, we get the inequality (2.6). □

Remark 2.7. *If one of the conditions stated for the matrix* B *in Theorems 2.3, 2.5 and 2.6 is not satisfied, the inequalities (2.2), (2.4), (2.5) and (2.6) are no longer true as shown in the following example.*

Example 2.8. *Consider the following two positive definite symmetric matrices*

$$A = \begin{pmatrix} 0.4 & -0.1 & 0 \\ -0.1 & 0.3 & -0.1 \\ 0 & -0.1 & 0.5 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0.6 & -0.15 & 0 \\ -0.15 & 0.45 & -0.15 \\ 0 & -0.15 & 0.75 \end{pmatrix}.$$

We have, $A < I$ *and* $B = 1.5\,A < A^{\frac{1}{2}} \exp(A^{-2}) A^{\frac{1}{2}}$. *Calculations with Matlab software give the following values :*

- $d_{1,0.8}(A, T_{0.5}(A|B)) = 0.5065 > d_{1,0.8}(A, T_{0.6}(A|B)) = 0.4969 > d_{1,0.8}(A, B) = 0.4107.$
- $d_{2,0.4}(A, T_{0.55}(A|B)) = 0.8080 > d_{2,0.4}(A, T_{0.65}(A|B)) = 0.7895 > d_{2,0.4}(A, B) = 0.5256.$

3. Results involving $T_p(A \mid B)$ and some matrix means

In this section, we aim to estimate distances between $T_p(A \mid B)$ and some standard matrix means via $d_{1,\alpha}$ and $d_{2,\alpha}$. To this end, we need to recall the arithmetic-geometric-harmonic inequality

$$\mathcal{H}(A, B) \leq A \sharp B \leq A \nabla B, \tag{3.1}$$

where $A \nabla B := \frac{A+B}{2}$ and $\mathcal{H}(A, B) := \big(A^{-1} \nabla B^{-1}\big)^{-1}$ stand respectively for the arithmetic and the harmonic means of the positive matrices A and B.

Theorem 3.1. *Let* A *and* B *belong to* $\mathbb{P}_n$*, such that* $A \leq B \leq 4A$*. For all* $0 < \alpha \leq 1$ *and* $p \in \big(0\,, \frac{1}{2}\big]$*, we have the following inequalities*

$$d_{1,\alpha}\left(A \sharp B, T_p(A \mid B)\right) \leq d_{1,\alpha}\left(B, T_p(A \mid B)\right) \tag{3.2}$$

and

$$d_{1,\alpha}\left(T_p(A \mid B), A \sharp B\right) \leq d_{1,\alpha}\left(T_p(A \mid B), A \nabla B\right). \tag{3.3}$$

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Proof. Taking $A \leq B \leq 4A$, we get $A^{\frac{-1}{2}} B A^{\frac{-1}{2}} \leq 4I$. So, $C^{\frac{1}{2}} \leq 2I$, or equivalently $C^{\frac{1}{2}} \geq 2\left(C^{\frac{1}{2}} - I\right)$. Thus,

$$T_{\frac{1}{2}}(A \mid B) \leq A\sharp B.$$

The inequalities (3.1) and (1.4) combined with the fact that $A\sharp B \leq B$, we can write for every $p \in \left(0, \frac{1}{2}\right]$,

$$T_p(A \mid B) \leq T_{\frac{1}{2}}(A \mid B) \leq A\sharp B \leq A\nabla B \leq B.$$

Applying Proposition 2.2, we get the inequalities (3.2) and (3.3). □

Theorem 3.2. *Let A and B be two positive definite matrices such that $B \geq 4A$. For all $0 < \alpha \leq 1$ and $p \in \left[\frac{1}{2}, 1\right]$, the following inequalities are satisfied*

$$d_{1,\alpha}\left(A\sharp B, T_p(A \mid B)\right) \leq d_{1,\alpha}\left(A\sharp B, B\right) \tag{3.4}$$

and

$$d_{1,\alpha}\left(\mathcal{H}(A,B), A\sharp B\right) \leq d_{1,\alpha}\left(\mathcal{H}(A,B), T_p(A \mid B)\right). \tag{3.5}$$

Proof. If $B \geq 4A$, $C^{\frac{1}{2}} \leq 2\left(C^{\frac{1}{2}} - I\right)$. Thus,

$$A\sharp B \leq T_{\frac{1}{2}}(A \mid B).$$

Taking into consideration the inequalities (3.1), (1.4) and (1.5), we obtain the following chain of inequalities for all $p \in \left[\frac{1}{2}, 1\right]$,

$$\mathcal{H}(A,B) \leq A\sharp B \leq T_p(A \mid B) \leq B. \tag{3.6}$$

Using Proposition 2.2, we deduce (3.4) and (3.5). □

To give another localization of $T_p(A \mid B)$ involving the geometric mean using the distance $d_{1,\alpha}$, we provide the following lemma.

Lemma 3.3. *We have the following two assertions:*

(i) *The real function defined by $f(x) = \frac{1}{4}x - \frac{8}{5}x^{\frac{5}{8}} + \frac{8}{5}$ is strictly increasing on $[123, \infty)$ and there exists a unique β_1 satisfying $f(\beta_1) = 0$ and $123,39 < \beta_1 < 123,4$.*

(ii) *The function $g(x) = \frac{1}{8}x - \frac{8}{5}(x^{\frac{5}{8}} - 1)$ is strictly increasing on $[861, \infty)$ and there exists a unique β_2 such that $g(\beta_2) = 0$ and $861,96 < \beta_2 < 861,97$.*

Proof. The proof involves usual real-analysis techniques and has therefore been omitted. □

Theorem 3.4. *Let A and B be two positive definite matrices. If $\left(p \in \left(0, \frac{1}{4}\right] \text{ and } B \geq 16A\right)$ or $\left(p \in \left[\frac{1}{2}, \frac{5}{8}\right] \text{ and } B \geq \beta_1 A\right)$, then for all $\alpha \in \left[\frac{1}{2}, 1\right]$, we have*

$$d_{1,\alpha}\big(A\sharp B, T_p(A \mid B)\big) \leq \frac{1}{2} d_{1,\alpha}(A,B), \tag{3.7}$$

where β_1 refers to the real number defined in the first assertion in Lemma 3.3.

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Proof. Let us take $B \geq 16A$, that is $C \geq 16I$. So,

$$4(C^{\frac{1}{4}} - I) \leq C^{\frac{1}{2}} \leq \frac{1}{4}C.$$

Whence,

$$T_{\frac{1}{4}}(A \mid B) \leq A\sharp B \leq \frac{1}{4}B.$$

Thanks to the inequalities (1.4) and (1.6), and by the monotonicity of the matrix function $\alpha \longmapsto x^{\alpha}$ on $\left[\frac{1}{2}, 1\right]$, we obtain for all $p \in \left(0, \frac{1}{4}\right]$

$$\frac{1}{2}A^{\alpha} \leq T_p^{\alpha}(A \mid B) \leq T_{\frac{1}{4}}^{\alpha}(A \mid B) \leq (A\sharp B)^{\alpha} \leq \frac{1}{2}B^{\alpha}.$$

Thus,

$$\frac{1}{2}A^{\alpha} + (A\sharp B)^{\alpha} \leq \frac{1}{2}B^{\alpha} + T_p^{\alpha}(A \mid B),$$

which is equivalent to

$$(A\sharp B)^{\alpha} - T_p^{\alpha}(A \mid B) \leq \frac{1}{2}\big(B^{\alpha} - A^{\alpha}\big).$$

Applying Lemma 2.1 with the square function, we obtain

$$tr\bigg(\big((A\sharp B)^{\alpha} - T_p^{\alpha}(A \mid B)\big)^2\bigg) \leq \frac{1}{4}tr\bigg(\big(B^{\alpha} - A^{\alpha}\big)^2\bigg),$$

which is equivalent to (3.7).
Now, consider $B \geq \beta_1 A$. So, $C \geq \beta_1 I$, and we have

$$\frac{1}{4}I \leq C^{\frac{1}{2}} \leq 2(C^{\frac{1}{2}} - I).$$

Hence,

$$\frac{1}{4}A \leq A\sharp B \leq T_{\frac{1}{2}}(A \mid B).$$

From the inequality (1.4) and Lemma 3.3, we have for all $p \in [\frac{1}{2}, \frac{5}{8}]$

$$\frac{1}{4}A \leq A\sharp B \leq T_{\frac{1}{2}}(A \mid B) \leq T_p(A \mid B) \leq T_{\frac{5}{8}}(A \mid B) \leq \frac{1}{4}B.$$

The monotonicity of the map $\alpha \longmapsto x^{\alpha}$ on $\left[\frac{1}{2}, 1\right]$ allows us to deduce that

$$T_p^{\alpha}(A \mid B) + \frac{1}{2}A^{\alpha} \leq (A\sharp B)^{\alpha} + \frac{1}{2}B^{\alpha},$$

which can be rephrased as follows

$$T_p^{\alpha}(A \mid B) - (A\sharp B)^{\alpha} \leq \frac{1}{2}B^{\alpha} - \frac{1}{2}A^{\alpha}.$$

By Lemma 2.1, we get

$$tr\bigg(\big(T_p^{\alpha}(A \mid B) - (A\sharp B)^{\alpha}\big)^2\bigg) \leq \frac{1}{4}tr\bigg(\big(B^{\alpha} - A^{\alpha}\big)^2\bigg).$$

This gives the desired inequality (3.7). □

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Remark 3.5. *Under the assumptions stated in Theorem 3.4, the use of the triangular inequality and the inequality 3.7 lead to the following result*

$$d_{1,\alpha}\left(T_p(A \mid B), T_q(A \mid B)\right) \leq d_{1,\alpha}(A, B), \tag{3.8}$$

where $p, q \in \left(0, \frac{1}{4}\right]$ *or* $p, q \in \left[\frac{1}{2}, \frac{5}{8}\right]$.

Theorem 3.6. *Let* A *and* B *be two positive definite matrices. If* $\left(p \in \left(0, \frac{1}{4}\right] \text{ and } B \geq 64A\right)$ *or* $\left(p \in \left[\frac{1}{2}, \frac{5}{8}\right] \text{ and } B \geq \beta_2 A\right)$, *then for all* $\alpha \in \left[\frac{1}{2}, 1\right]$, *the following inequality holds*

$$d_{2,\alpha}(A\sharp B, T_p(A|B)) \leq \frac{1}{2} d_{2,\alpha}(A, B), \tag{3.9}$$

where β_2 *is the real number defined in Lemma 3.3.*

Proof. If $B \geq 64A$, then $C \geq 64I$. Thus, we get the following inequalities

$$4(C^{\frac{1}{4}} - I) \leq C^{\frac{1}{2}} \leq \frac{1}{8}C.$$

Thus,

$$T_{\frac{1}{4}}(A|B) \leq A\sharp B \leq \frac{1}{8}B.$$

These combined with the inequalities (1.4) and (1.6) enable us to deduce for all $p \in \left(0, \frac{1}{4}\right]$,

$$A \leq T_p(A|B) \leq T_{\frac{1}{4}}(A|B) \leq A\sharp B \leq \frac{1}{8}B.$$

Therefore,

$$\frac{1}{2\sqrt{2}}A^\alpha \leq T_p^\alpha(A|B) \leq (A\sharp B)^\alpha \leq \frac{1}{2\sqrt{2}}B^\alpha.$$

So,

$$(A\sharp B)^\alpha - T_p^\alpha(A|B) \leq \frac{1}{2\sqrt{2}}\left(B^\alpha - A^\alpha\right).$$

Thanks to Lemma 2.1 applied with the square function, we obtain

$$tr\left(\left((A\sharp B)^\alpha - T_p^\alpha(A|B)\right)^2\right) \leq \frac{1}{8} tr\left(\left(B^\alpha - A^\alpha\right)^2\right), \tag{3.10}$$

If $B \geq \beta_2 A$ then $C \geq \beta_2 I$. Thus,

$$I \leq C^{\frac{1}{2}} \leq 2(C^{\frac{1}{2}} - I).$$

So,

$$A \leq A\sharp B \leq T_{\frac{1}{2}}(A|B).$$

By virtue of Lemma 3.3, we have

$$\frac{8}{5}(C^{\frac{5}{8}} - I) \geq \frac{1}{8}C.$$

Then,

$$T_{\frac{5}{8}}(A \mid B) \leq \frac{1}{8}B.$$

Employing the inequality (1.4), we deduce for any $\frac{1}{2} \leq p \leq \frac{5}{8}$, the following chain of inequalities

$$\frac{1}{2\sqrt{2}}A \leq A\sharp B \leq T_{\frac{1}{2}}(A|B) \leq T_p(A|B) \leq T_{\frac{5}{8}}(A|B) \leq \frac{1}{8}B,$$

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So, for any $\alpha \in \left[\frac{1}{2}, 1\right]$, we have

$$T_p^{\alpha}(A|B) + \frac{1}{2\sqrt{2}} A^{\alpha} \leq (A\sharp B)^{\alpha} + \frac{1}{2\sqrt{2}} B^{\alpha},$$

that is,

$$T_p^{\alpha}(A|B) - (A\sharp B)^{\alpha} \leq \frac{1}{2\sqrt{2}} \big(B^{\alpha} - A^{\alpha}\big).$$

By Lemma 2.1, we obtain

$$tr\Big(\big(T_p^{\alpha}(A|B) - \big(A\sharp B\big)^{\alpha}\big)^2\Big) \leq \frac{1}{8} tr\Big(\big(B^{\alpha} - A^{\alpha}\big)^2\Big). \tag{3.11}$$

From inequalities (3.10) and (3.11), and using the inequality (1.10), we get

$$\begin{aligned} d_{2,\alpha}(A\sharp B, T_p(A|B)) &\leq d_{1,\alpha}(A\sharp B, T_p(A|B)) \\ &\leq \frac{1}{2\sqrt{2}} d_{1,\alpha}(A, B) \\ &\leq \frac{1}{2} d_{2,\alpha}(A, B). \end{aligned}$$

So, the proof is completed. □

Remark 3.7. *Under the assumptions stated in Theorem 3.6, the use of the triangular inequality and the inequality (3.9) lead to the following result*

$$d_{2,\alpha}\left(T_p(A \mid B), T_q(A \mid B)\right) \leq d_{2,\alpha}(A, B), \tag{3.12}$$

where $p, q \in \left(0, \frac{1}{4}\right]$ *or* $p, q \in \left[\frac{1}{2}, \frac{5}{8}\right]$.

For the distance inequalities involving Tsallis operator entropy $T_p(A \mid B)$ and the arithmetic mean $A\nabla B$, we have the following theorem.

Theorem 3.8. *Let* A *and* B *be two positive definite matrices such that* $B \geq A$. *For all* $0 < p \leq \frac{1}{2}$ *and for all* $0 < \alpha \leq 1$, *the following inequality is satisfied*

$$d_{1,\alpha}\left(A\nabla B, T_p(A \mid B)\right) \leq d_{1,\alpha}\left(B, T_p(A \mid B)\right) \tag{3.13}$$

Proof. For any strictly positive real x, we have

$$2(x^{\frac{1}{2}} - 1) \leq \frac{1 + x}{2},$$

which leads to

$$2(C^{\frac{1}{2}} - I) \leq \frac{I + C}{2}. \tag{3.14}$$

Multiplying left and right both sides of the inequality (3.14) by $A^{\frac{1}{2}}$, we obtain

$$T_{\frac{1}{2}}(A \mid B) \leq A\nabla B.$$

Combining with the inequality (1.4) and by the fact that $A\nabla B \leq B$, we get for any $0 < p \leq \frac{1}{2}$

$$0 < T_p(A \mid B) \leq A\nabla B \leq B.$$

Employing Proposition 2.2, we find the result (3.13). □

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To estimate distances $d_{i,\alpha}$, $1 \leq i \leq 2$ involving Tsallis operator entropy $T_p(A \mid B)$ and the harmonic mean $\mathcal{H}(A,B)$, we need the following lemma which is simple to prove.

Lemma 3.9. *The function h defined by $h(x) = \log x - 2\left(1 + x^{-1}\right)^{-1}$ is strictly increasing on $(0, \infty)$. Moreover, there exists a unique real number β such that $h(\beta) = 0$ and $5.40 < \beta < 5.41$.*

Theorem 3.10. *Let A and B be two positive definite matrices such that $B \geq \beta A$. For all $p \in (0\,,\,1]$ and for all $0 < \alpha \leq 1$, the following inequality holds*

$$d_{1,\alpha}\left(T_p(A \mid B), \mathcal{H}(A,B)\right) \leq d_{1,\alpha}\left(\mathcal{H}(A,B), B\right), \tag{3.15}$$

where β is the real defined in Lemma 3.9.

Proof. If $B \geq \beta A$, then $C \geq \beta I$. By using Lemma 3.9, we get

$$\log C \geq 2\left(I + C^{-1}\right)^{-1} > 0.$$

Multiplying by $A^{\frac{1}{2}}$ both sides of the left inequality, we obtain

$$S(A \mid B) \geq \mathcal{H}(A,B).$$

So, by the inequality (1.5), we have

$$\mathcal{H}(A,B) \leq T_p(A \mid B) \leq B.$$

Employing Proposition 2.2, we find the inequality (3.15). □

To present other estimations of the distances involving the harmonic mean, we will use the following lemma.

Lemma 3.11. *i) For the function defined on $[46,\,\infty)$ by $l(x) = \frac{1}{4}x - 2(x^{\frac{1}{2}} - 1)$, there exists a unique real number β_3 satisfying $l(\beta_3) = 0$ such that $46.62 < \beta_3 < 46.63$. Moreover, for any $x \geq \beta_3$, $l(x) \geq 0$.*
ii) For the function defined on $[222,\,\infty)$ by $L(x) = \frac{1}{8}x - 2(x^{\frac{1}{2}} - 1)$, there exists a unique real number β_4 satisfying $L(\beta_4) = 0$ such that $222.85 < \beta_4 < 222.86$. Moreover, for any $x \geq \beta_4$, $L(x) \geq 0$.

Proof. The proof involves usual real-analysis techniques and has therefore been omitted. □

Theorem 3.12. *Let A and B be two positive definite matrices, such that $B \geq \beta_3\,A$. The following inequality*

$$d_{1,\alpha}\big(\mathcal{H}(A,B), T_p(A \mid B)\big) \leq \frac{1}{2} d_{1,\alpha}(A,B) \tag{3.16}$$

is verified for every $p \in \left(0, \frac{1}{2}\right]$ and for all $\alpha \in \left[\frac{1}{2}, 1\right]$. The scalar β_3 stands for the fixed real number defined in Lemma 3.11.

Proof. Since $\beta_3 > \beta$, we have

$$\mathcal{H}(A,B) \leq T_p(A \mid B). \tag{3.17}$$

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Now, let us take $B \geq \beta_3 A$ and $p \in \left(0, \frac{1}{2}\right]$. We have

$$\begin{aligned} C \geq \beta_3 I &\implies 2(C^{\frac{1}{2}} - I) \leq \frac{1}{4}C \quad \text{(By Lemma 3.11)} \\ &\implies T_{\frac{1}{2}}(A \mid B) \leq \frac{1}{4}B \\ &\implies T_p(A \mid B) \leq \frac{1}{4}B. \quad \text{(Using the inequality (1.4))} \end{aligned} \tag{3.18}$$

Combining the inequalities (3.17) and (3.18), we get

$$\mathcal{H}(A, A) \leq \mathcal{H}(A, B) \leq T_p(A \mid B) \leq \frac{1}{4}B.$$

Thanks to the monotonicity of the matrix function $\alpha \longmapsto x^\alpha$ on $\left[\frac{1}{2}, 1\right]$, we find

$$\frac{1}{2}A^\alpha \leq \mathcal{H}^\alpha(A, B) \leq T_p^\alpha(A \mid B) \leq \frac{1}{2}B^\alpha.$$

Whence,

$$0 < T_p^\alpha(A \mid B) - \mathcal{H}^\alpha(A, B) \leq \frac{1}{2}B^\alpha - \frac{1}{2}A^\alpha.$$

Employing Lemma 2.1, we can state

$$tr\left(\left(T_p^\alpha(A \mid B) - \mathcal{H}^\alpha(A, B)\right)^2\right) \leq \frac{1}{4}tr\left(\left(B^\alpha - A^\alpha\right)^2\right),$$

which is equivalent to the desired inequality (3.16). □

Remark 3.13. *Using Theorem 3.12, the inequality (3.8) may be extended to $p, q \in \left(0, \frac{1}{2}\right]$ and $\alpha \in \left[\frac{1}{2}, 1\right]$ for matrices A and B with $B \geq \beta_3 A$.*

For the distance inequality with respect to $d_{2,\alpha}$, we have the following result.

Theorem 3.14. *Let $A, B \in \mathbb{P}_n$ such that $B \geq \beta_4 A$. For all $\alpha \in \left[\frac{1}{2}, 1\right]$ and for all $p \in \left(0, \frac{1}{2}\right]$, we have*

$$d_{2,\alpha}\big(\mathcal{H}(A, B), T_p(A \mid B)\big) \leq \frac{1}{2}d_{2,\alpha}(A, B), \tag{3.19}$$

where β_4 is the real number defined in Lemma 3.11.

Proof. By the second statement of Lemma 3.11, we have

$$\begin{aligned} 2(C^{\frac{1}{2}} - I) \leq \frac{1}{8}C &\implies T_{\frac{1}{2}}(A \mid B) \leq \frac{1}{8}B \\ &\implies T_p(A \mid B) \leq \frac{1}{8}B. \end{aligned} \tag{3.20}$$

From the inequalities (3.17) and (3.20), we have for all $p \in \left(0, \frac{1}{2}\right]$

$$\frac{1}{2\sqrt{2}}A \leq \mathcal{H}(A, B) \leq T_p(A \mid B) \leq \frac{1}{8}B.$$

So, for any $\alpha \in \left[\frac{1}{2}, 1\right]$, we have

$$T_p^\alpha(A \mid B) + \frac{1}{2\sqrt{2}}A^\alpha \leq \frac{1}{2\sqrt{2}}B^\alpha + \mathcal{H}^\alpha(A, B),$$

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which leads by using Lemma 2.1

$$tr\Big(\big(T_p^{\alpha}(A \mid B) - \mathcal{H}^{\alpha}(A,B)\big)^2\Big) \leq \frac{1}{8} tr\Big(\big(B^{\alpha} - A^{\alpha}\big)^2\Big).$$

This combined with the inequality (1.10), allows to deduce that

$$\begin{aligned} d_{2,\alpha}\big(\mathcal{H}(A,B), T_p(A \mid B)\big) &\leq d_{1,\alpha}\big(\mathcal{H}(A,B), T_p(A \mid B)\big) \\ &\leq \frac{1}{2\sqrt{2}} d_{1,\alpha}(A,B) \\ &\leq \frac{1}{2} d_{2,\alpha}(A,B). \end{aligned}$$

Which gives the proof of (3.19). □

Remark 3.15. *If one of the conditions stated in theorems 3.4, 3.6, 3.8, 3.12 and 3.14 for the matrices A or B or for the parameter p is not satisfied, the inequalities (3.7), (3.9), (3.13), (3.16) and (3.19) are no longer true, as emphasized by the values in Table 1 corresponding to the following example.*

Example 3.16. *Let us consider the following symmetric positive definite matrix*

$$A = \begin{pmatrix} 1 & -1 & 1 & 0 \\ -1 & 2 & -2 & 1 \\ 1 & -2 & 4 & -1 \\ 0 & 1 & -1 & 5 \end{pmatrix}.$$

Calculations with Matlab software give the following values.

B	p	α	$d_{i,\alpha}\left(\mathcal{H}(A,B), T_p\right)$	$d_{i,\alpha}\left(A \sharp B, T_p\right)$	$d_{i,\alpha}\left(A \nabla B, T_p\right)$	$\frac{1}{2}\, d_{i,\alpha}(A,B)$
$2A$	0.6	0.7	2.1926	2.5413	2.9047	2.1155
$16A$	0.92	0.8	42.3214	32.7185	15.1681	28.6442
$64A$	0.86	0.6	51.8430	38.5095	7.5911	37.4508
$120A$	0.88	0.55	63.4567	48.0117	8.4772	43.8827
$224A$	0.9	0.9	630.5535	560.1438	127.5723	476.4588
$865A$	0.95	0.65	441.8901	391.8421	104.3842	269.5102

TABLE 1. Some numerical examples

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Abdenbi El Hilali

LAGA-Laboratory, Science Faculty, University Ibn Tofail, Kenitra, Morocco.

Email: abdenbielhilali1976@gmail.com

Mohamed Chergui

Department of Mathematics, CRMEF-RSK, EREAM Team, Kenitra, Morocco.

Email: chergui_m@yahoo.fr

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