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Journal of the Iranian Mathematical Society
ISSN (on-line): 2717-1612
J. Iran. Math. Soc. 5 (2024), no. 1, 45-53
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# THE UNIT GROUP OF THE GROUP ALGEBRA $\mathbb{F}_{q} D_{36}$ 

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#### Abstract

Let $p$ be a prime number and $\mathbb{F}_{q}$ be a finite field having $q=p^{n}$ elements and $D_{36}$ be the dihedral group of order 36 . In this paper, the unit group $\mathcal{U}\left(\mathbb{F}_{q} D_{36}\right)$, of the group algebra $\mathbb{F}_{q} D_{36}$, is completely characterized.


## 1. Introduction

Due to the importance of group rings and units in group rings in both pure mathematics and applied sciences, determining the structure of units group of a group ring has always been a fascinating and challenging problem. While the significance of group rings in pure mathematics is widely acknowledged, its practical applications may not be as commonly recognized. In this context, the work of Hurley $[12,13]$ are noteworthy instances, as they establish a link between matrix rings and group rings, offering a methodology for constructing convolutional codes in the field of telecommunication engineering through the utilization of units of group rings (see also [2]). More specifically, finite group rings and group rings of finite groups over fields occupy a distinctive and crucial role. Hence extensive research has been conducted to investigate the algebraic structure of the unit group $\mathcal{U}\left(\mathbb{F}_{q} G\right)$ of a group algebra $\mathbb{F}_{q} G$ when $G$ is a finite non-abelian group ([6-13], [15-21], [26-28]).

In 2014, Makhijani et al. [17] characterized the unitary subgroup of the group of units of $\mathbb{F}_{p^{m}} D_{2 p^{n}}$, considering the canonical involution $*$, for $p>2$. The unit group of the group algebra $\mathbb{F}_{2^{k}} D_{2 n}$ was discussed in [20], specifically for odd values of $n$.

[^0]Furthermore, in 2015, Makhijani, in [21], determined the unit group of the group algebra $\mathbb{F}_{q} D_{30}$, while Gildea provided a presentation of the unit group $\mathcal{U}\left(\mathbb{F}_{5^{k}} D_{20}\right)$, [4]. For unit groups of circulant matrices and their algebraic structure, see Makhijani et al. and Sharma works, [18] and [26]. Based on the current literature survey, the unit group structure of group rings has been established for certain cases such as $\mathbb{F}_{2^{n}} D_{2 m}$ (where $m$ is odd), $\mathbb{F}_{p^{n}} D_{2 p^{m}}$ ( $p$ is odd prime), $\mathbb{F}_{q} D_{60}$, and $\mathbb{F}_{q} D_{40}$. However, as of the present knowledge, the unit group structure of the group algebra $\mathbb{F}_{q} D_{36}$ remains undetermined.

Let $p$ be a prime number. This paper provides a comprehensive characterization of the unit group of the group algebra $\mathbb{F}_{q} D_{36}$, where $\mathbb{F}_{q}$ represents any finite field with $q=p^{n}$ for some $n \in \mathbb{N}$.

Throughout the paper, we will use the following notations:

- J: Jacobson radical;
- $\mathcal{U}(G)$ : unit group of $G$;
- $\mathcal{C}_{r}$ : cyclic group of order $r$;
- $\mathbb{F}_{q}$ : finite field of order $q$;
- $G^{\prime}$ : commutator subgroup of $G$;
- $\operatorname{Dim}_{\mathbb{F}_{q}}$ : dimension over the field $\mathbb{F}_{q}$;
- $\omega(G, N)$ : the augmentation ideal corresponding to normal subgroup $N$ of $G$;
- $S_{\mathbb{F}_{q}\left(\gamma_{g}\right)}$ : cyclotomic $\mathbb{F}_{q}$-class corresponding to element $g$;
- $\mathcal{Z}(G)$ : Center of group $G$;
- $\widehat{G}=\sum_{g \in G} g$ for finite order group $G$.

We utilized the presentation of $D_{36}$ as follows:

$$
D_{36}=\left\langle a, b \mid a^{18}, b^{2},(b a)^{2}\right\rangle
$$

and it should be pointed out that this group possesses 12 distinct conjugacy classes, which are listed as follows: $[e],[a],\left[a^{2}\right],\left[a^{3}\right],\left[a^{4}\right],\left[a^{5}\right],\left[a^{6}\right],\left[a^{7}\right],\left[a^{8}\right],\left[a^{9}\right],[b]$, and $\left[a^{9} b\right]$.
Any unfamiliar terminology and notation can be referenced in [14, 24]. Throughout this paper, all rings are assumed to be associative with identity.

## 2. Main results

Lemma 2.1. Let $\mathbb{F}_{q}$ be a finite field and $G$ be a finite group. Then

$$
\mathcal{U}\left(\mathbb{F}_{q} G\right) \cong\left(1+\mathcal{J}\left(\mathbb{F}_{q} G\right)\right) \rtimes \mathcal{U}\left(\frac{\mathbb{F}_{q} G}{\mathcal{J}\left(\mathbb{F}_{q} G\right)}\right)
$$

Proof. If $f: 1+\mathcal{J}\left(\mathbb{F}_{q} G\right) \rightarrow \mathcal{U}\left(\mathbb{F}_{q} G\right)$ is the inclusion map and $g: \mathcal{U}\left(\mathbb{F}_{q} G\right) \rightarrow \mathcal{U}\left(\frac{\mathbb{F}_{q} G}{\mathcal{J}\left(\mathbb{F}_{q} G\right)}\right)$ is defined as $g(x)=x+\mathcal{J}\left(\mathbb{F}_{q} G\right)$, then the following sequence forms a short exact sequence of groups:

$$
1 \rightarrow 1+\mathcal{J}\left(\mathbb{F}_{q} G\right) \xrightarrow{f} \mathcal{U}\left(\mathbb{F}_{q} G\right) \xrightarrow{g} \mathcal{U}\left(\frac{\mathbb{F}_{q} G}{\mathcal{J}\left(\mathbb{F}_{q} G\right)}\right) \rightarrow 1 .
$$

Now, using Wedderburn-Malcev theorem [1, Chapter 10, Theorem 72.19], there exists a semisimple subalgebra $K$ of $\mathbb{F}_{q} G$, such that

$$
\mathbb{F}_{q} G=K \oplus \mathcal{J}\left(\mathbb{F}_{q} G\right),
$$

and so for each $x+\mathcal{J}\left(\mathbb{F}_{q} G\right) \in \frac{\mathbb{F}_{q} G}{\mathcal{J}\left(\mathbb{F}_{q} G\right)}$, there is a unique $x_{K} \in K$ such that $x+\mathcal{J}\left(\mathbb{F}_{q} G\right)=x_{K}+\mathcal{J}\left(\mathbb{F}_{q} G\right)$. Hence, the map

$$
h: \mathcal{U}\left(\frac{\mathbb{F}_{q} G}{\mathcal{J}\left(\mathbb{F}_{q} G\right)}\right) \rightarrow \mathcal{U}\left(\mathbb{F}_{q} G\right)
$$

defined by $h\left(x+\mathcal{J}\left(\mathbb{F}_{q} G\right)\right)=x_{K}$ is a group homomorphism.
Now, observe that $g \circ h=\mathrm{id}$ on $\mathcal{U}\left(\frac{\mathbb{F}_{q} G}{\mathcal{J}\left(\mathbb{F}_{q} G\right)}\right)$, and hence

$$
\mathcal{U}\left(\mathbb{F}_{q} G\right) \cong\left(1+\mathcal{J}\left(\mathbb{F}_{q} G\right)\right) \rtimes \mathcal{U}\left(\frac{\mathbb{F}_{q} G}{\mathcal{J}\left(\mathbb{F}_{q} G\right)}\right) .
$$

This proves the lemma.
We need the following result from [14].
Definition 2.2. Let $I$ be a nilpotent ideal of a ring $R$. By the nilpotency index of $I$ we mean the least positive integer $n$, such that $I^{n}=0$.

We write $t(G)$ for the nilpotency index of $\mathcal{J}\left(\mathbb{F}_{q} G\right)$.
Proposition 2.3. [14, Chapter 3, Corollary 1.5] Assume that $N$ is a normal p-subgroup of $G$. Then

$$
\operatorname{Dim}\left(\mathcal{J}\left(\mathbb{F}_{p^{n}} G\right)\right)=\operatorname{Dim}\left(\mathcal{J}\left(\mathbb{F}_{p^{n}}(G / N)\right)\right)+|G|-|G: N|
$$

over $\mathbb{F}_{p^{n}}$.
Firstly, let us derive some important lemmas.
Lemma 2.4. Let $q=2^{n}$ and $G=D_{18}$. Then $\operatorname{Dim}_{\mathbb{F}_{q}} \mathcal{J}\left(\mathbb{F}_{q} G\right)=1$.
Proof. We know that $D_{18}$ is a Frobenius group as the dihedral group of order $2 n$ with $n$ odd is a Frobenius group with a complement $P$ being a Sylow 2-subgroup of order 2. Therefore, by [14, Chapter 3, Corollary 7.8], we have $\operatorname{Dim}_{\mathbb{F}_{q}} \mathcal{J}\left(\mathbb{F}_{q} G\right)=|P|-1=1$.

Lemma 2.5. Let $q=2^{n}$ and $G=D_{36}$. Then $\operatorname{Dim}_{\mathbb{F}_{q}} \mathcal{J}\left(\mathbb{F}_{q} G\right)=19$ and $1+\mathcal{J}\left(\mathbb{F}_{q} G\right) \cong \mathcal{C}_{2}^{19 n}$.
Proof. Let $N=\mathcal{Z}(G)$ represent the normal 2-subgroup of the group $G$. Then by plugging in $N$ into Proposition 2.3 and the fact that $G / N \cong D_{18}$, we obtain:

$$
\operatorname{Dim}_{\mathbb{F}_{q}} \mathcal{J}\left(\mathbb{F}_{q} G\right)=\operatorname{Dim}_{\mathbb{F}_{q}} \mathcal{J}\left(\mathbb{F}_{q} D_{18}\right)+\frac{|G|}{2} .
$$

Now, applying Lemma 2.4,

$$
\operatorname{Dim}_{\mathbb{F}_{q}} \mathcal{J}\left(\mathbb{F}_{q} G\right)=1+\frac{|G|}{2}=19
$$

this proves the first statement. Commutativity of $\mathcal{J}\left(\mathbb{F}_{q} G\right)$ follows as a consequence of the result presented in [14, Chapter 3, Theorem 15.6]. Observe that

$$
\begin{aligned}
\omega\left(D_{36},\left\langle y^{9}\right\rangle\right) & =\omega\left(\left\langle y^{9}\right\rangle\right) \mathbb{F}_{q} D_{36} \\
& =\mathcal{J}\left(\mathbb{F}_{q}\left\langle y^{9}\right\rangle\right) \mathbb{F}_{q} D_{36} \\
& \subseteq \mathcal{J}\left(\mathbb{F}_{q} D_{36}\right) .
\end{aligned}
$$

Let $z=1+y^{9} \in \mathcal{Z}\left(\mathbb{F}_{q} D_{36}\right)$. Consequently, a basis for $\omega\left(D_{36},\left\langle y^{9}\right\rangle\right)$ is formed by

$$
\left\{z, y z, \ldots, y^{8} z, x z, x y z, \ldots, x y^{8} z\right\} .
$$

Any element $w$ in $\omega\left(D_{36},\left\langle y^{9}\right\rangle\right)$ can be represented as

$$
w=\left(a_{1}+a_{2} y+\cdots+a_{9} y^{8}+b_{1} x+b_{2} x y+\cdots+b_{9} x y^{8}\right) z
$$

where $a_{i}, b_{i} \in \mathbb{F}_{q}$. Now, considering $w^{2}=\left(a_{1}+a_{2} y+\cdots+a_{9} y^{8}+b_{1} x+b_{2} x y+\cdots+b_{9} x y^{8}\right)^{2} z^{2}=0$, we conclude that $1+\omega\left(D_{36},\left\langle y^{9}\right\rangle\right) \cong \mathcal{C}_{2}^{18 n}$.

Again since $\widehat{D_{18}}=\sum_{g \in D_{18}} g \in \mathcal{J}\left(\mathbb{F}_{q} D_{18}\right)$, we have $\mathcal{A}=(1+x)\left(1+y+\cdots+y^{8}\right) \in \mathcal{J}\left(\mathbb{F}_{q} D_{36}\right)$. In fact,

$$
\mathcal{J}\left(\mathbb{F}_{q} D_{36}\right)=\omega\left(D_{36},\left\langle y^{9}\right\rangle\right) \oplus \mathbb{F}_{q} \mathcal{A}
$$

as a vector space over the field $\mathbb{F}_{q}$.
Now, notice that:

$$
\begin{aligned}
\mathcal{A}^{2} & =\left((1+x)\left(1+y+\cdots+y^{8}\right)\right)^{2} \\
& =(1+x)\left(1+y+\cdots+y^{8}\right)(1+x)\left(1+y+\cdots+y^{8}\right) \\
& =(1+2 x+1)\left(1+y+\cdots+y^{8}\right)^{2} \\
& =0 .
\end{aligned}
$$

Therefore,

$$
1+\mathcal{J}\left(\mathbb{F}_{q} D_{36}\right)=\left(1+\omega\left(D_{36},\left\langle y^{9}\right\rangle\right)\right) \times\left\{1+\alpha \mathcal{A} \mid \alpha \in \mathbb{F}_{q}\right\} \cong \mathcal{C}_{2}^{19 n}
$$

This proves the lemma.
Lemma 2.6. Let $q=2^{n}, G=D_{36}$. Then

$$
\frac{\mathbb{F}_{q} G}{\mathcal{J}\left(\mathbb{F}_{q} G\right)} \cong \begin{cases}\mathbb{F}_{q} \oplus \mathbb{M}_{2}\left(\mathbb{F}_{q}\right)^{4} & \text { if } 2^{n} \equiv \pm 1 \quad(\bmod 9), \\ \mathbb{F}_{q} \oplus \mathbb{M}_{2}\left(\mathbb{F}_{q}\right) \oplus \mathbb{M}_{2}\left(\mathbb{F}_{q^{3}}\right) & \text { otherwise }\end{cases}
$$

Proof. Conjugate classes corresponding to 2 - regular elements are $[e],\left[a^{2}\right]=\left\{a^{2}, a^{16}\right\},\left[a^{4}\right]=\left\{a^{4}, a^{14}\right\}$, $\left[a^{6}\right]=\left\{a^{6}, a^{12}\right\}$ and $\left[a^{8}\right]=\left\{a^{8}, a^{10}\right\}$.

We divide the proof in two cases:
Case 1. Let $q=2^{n} \equiv \pm 1(\bmod 9)$.
If $q=2^{n} \equiv \pm 1(\bmod 9)$, then

$$
\begin{aligned}
d & =o(q) \quad(\bmod 9) \\
& =1 \text { or } 2 \quad(\bmod 9)
\end{aligned}
$$

and $T_{\mathbb{F}_{q}, G}=\{1\}$ or $\{ \pm 1\}$. But we know that conjugate class of an element and its inverse is the same. Therefore in this case, we have

$$
S_{\mathbb{F}_{q}\left(\gamma_{g}\right)}=\left\{\gamma_{g}\right\} \text { i.e., }\left|S_{\mathbb{F}_{q}\left(\gamma_{g}\right)}\right|=1,
$$

for each $2-$ regular element $g \in G$ in [3, Theorem 1.3]. This gives that

$$
\begin{equation*}
\frac{\mathbb{F}_{q} G}{\mathcal{J}\left(\mathbb{F}_{q} G\right)} \cong \mathbb{F}_{q} \oplus \mathbb{M}_{n_{1}}\left(\mathbb{F}_{q}\right) \oplus \mathbb{M}_{n_{2}}\left(\mathbb{F}_{q}\right) \oplus \mathbb{M}_{n_{1}}\left(\mathbb{F}_{q}\right) \oplus \mathbb{M}_{n_{1}}\left(\mathbb{F}_{q}\right) \tag{2.1}
\end{equation*}
$$

Now only possible integer values of $n_{i}$ in relation (2.1) are $n_{1}=n_{2}=n_{3}=n_{4}=2$, as the total dimension of $\frac{\mathbb{F}_{q} G}{\mathcal{J}\left(\mathbb{F}_{q} G\right)}$ is equal to 17 .

Thus

$$
\frac{\mathbb{F}_{q} G}{\mathcal{J}\left(\mathbb{F}_{q} G\right)} \cong \mathbb{F}_{q} \oplus \mathbb{M}_{2}\left(\mathbb{F}_{q}\right)^{4}
$$

Again, if $2^{n} \equiv \pm 2, \pm 4(\bmod 5)$, then $d=$ order of $2^{n}=3 \operatorname{or} 6(\bmod 9)$, and hence $T_{\mathbb{F}_{q}, G}=$ $\{1,2,4,5,7,8\}$ or $T_{\mathbb{F}_{q}, G}=\{1,4,7\}$. Therefore in this case, we have

$$
\begin{gathered}
S_{\mathbb{F}_{q}\left(\gamma_{e}\right)}=\left\{\gamma_{e}\right\} \Longrightarrow\left|S_{\mathbb{F}_{q}\left(\gamma_{e}\right)}\right|=1 \Longrightarrow K_{i} \cong \mathbb{F}_{q}, \\
S_{\mathbb{F}_{q}\left(\gamma_{a^{2}}\right)}=\left\{\gamma_{a^{2}}, \gamma_{a^{4}}, \gamma_{a^{8}}\right\} \Longrightarrow\left|S_{\mathbb{F}_{q}\left(\gamma_{a}\right)}\right|=3 \Longrightarrow K_{i} \cong \mathbb{F}_{q^{3}}, \\
S_{\mathbb{F}_{q}\left(\gamma_{a^{6}}\right)}=\left\{\gamma_{a^{6}}\right\} \Longrightarrow\left|S_{\mathbb{F}_{q}\left(\gamma_{a^{6}}\right)}\right|=1 \Longrightarrow K_{i} \cong \mathbb{F}_{q} .
\end{gathered}
$$

This gives

$$
\begin{equation*}
\frac{\mathbb{F}_{q} G}{\mathcal{J}\left(\mathbb{F}_{q} G\right)} \cong \mathbb{F}_{q} \oplus \mathbb{M}_{n_{1}}\left(\mathbb{F}_{q}\right) \oplus \mathbb{M}_{n_{2}}\left(\mathbb{F}_{q^{3}}\right) . \tag{2.2}
\end{equation*}
$$

From relation (2.2), $n_{i}$ must be equal to 2 as $\operatorname{Dim}_{\mathbb{F}_{q}}\left(\mathcal{J}\left(\mathbb{F}_{q} G\right)\right)=19$. This proves the lemma.
Lemma 2.7. Let $q=3^{n}$ and $G=D_{36}$. Then $\operatorname{Dim}_{\mathbb{F}_{q}} \mathcal{J}\left(\mathbb{F}_{q} G\right)=32$.
Proof. By repeatedly applying Proposition 2.3, and considering the normal 3-subgroup as the cyclic group $\mathcal{C}_{3}$ of the group $G$, we have

$$
\operatorname{Dim}_{\mathbb{F}_{q}} \mathcal{J}\left(\mathbb{F}_{q} G\right)=\operatorname{Dim}_{\mathbb{F}_{q}} \mathcal{J}\left(\mathbb{F}_{q} D_{12}\right)+24=12-4+24=32 .
$$

This proves the lemma.
Now, we will proceed to prove our main theorems.
Theorem 2.8. Let $q=3^{n}, G=D_{36}, \mathcal{V}_{1}=1+\mathcal{J}\left(\mathbb{F}_{q} G\right)$ and $\mathcal{V}_{2}=1+\omega\left(G^{\prime}\right)$. Then
(1) $\frac{\mathbb{F}_{q} G}{\mathcal{J}\left(\mathbb{F}_{q} G\right)} \cong \mathbb{F}_{q} \oplus \mathbb{F}_{q} \oplus \mathbb{F}_{q} \oplus \mathbb{F}_{q}$;
(2) $\mathcal{U}\left(\mathbb{F}_{q} G\right) \cong \mathcal{V}_{1} \rtimes\left(\mathbb{F}_{q}^{*} \times \mathbb{F}_{q}^{*} \times \mathbb{F}_{q}^{*} \times \mathbb{F}_{q}^{*}\right)$;
(3) $\mathcal{V}_{1}$ is a non-abelian maximal normal 3 -subgroup of $\mathcal{U}\left(\mathbb{F}_{q} G\right)$, with order $3^{32 n}$ and exponent 9 ;
(4) $\mathcal{V}_{1}$ is a non-abelian normal subgroup of $\mathcal{U}\left(\mathbb{F}_{q} G\right)$, with nilpotency class 8.
(5) $\mathcal{V}_{1}=\mathcal{V}_{2}$;
(6) $\mathcal{Z}\left(\mathcal{V}_{1}\right)$ is an abelian group of exponent 9 .

Proof. (1). The three regular elements of $D_{36}$ correspond to the conjugacy classes of elements $1, a, a^{9}$, and $a b$. Now, since $q \equiv 1 \bmod 2$, the order of $q$ modulo 2 is denoted as $d$, which is equal to 1 according to [3]. Consequently, there are four simple components in $\frac{\mathbb{F}_{q} G}{\mathcal{J}\left(\mathbb{F}_{q} G\right)}$ with a total dimension of 4. Therefore, we can conclude that $\frac{\mathbb{F}_{q} G}{\mathcal{J}\left(\mathbb{F}_{q} G\right)}$ is isomorphic to $\mathbb{F}_{q} \oplus \mathbb{F}_{q} \oplus \mathbb{F}_{q} \oplus \mathbb{F}_{q}$.
(2). Proof is a straightforward application of the fact that for any two ring $\mathcal{R}_{1}$ and $\mathcal{R}_{2}, \mathcal{U}\left(\mathcal{R}_{1} \oplus \mathcal{R}_{2}\right)=$
$\mathcal{U}\left(\mathcal{R}_{1}\right) \times \mathcal{U}\left(\mathcal{R}_{2}\right)$ in 1 .
(3). Since $\operatorname{Dim}_{\mathbb{F}_{q}} \mathcal{J}\left(\mathbb{F}_{q} D_{36}\right)=32$, it follows that $\left|\mathcal{V}_{1}\right|=3^{32 n}$. The non-abelian part can be deduced from [14, Chapter 3, Theorem 15.2]. Furthermore, since $G$ is solvable, it is also 3 -solvable, with a normal Sylow 3 -subgroup. By [14, Chapter 7, Proposition 2.4], the index of nilpotency of $\mathcal{J}\left(\mathbb{F}_{q} G\right)$ is 9 , implying that the exponent of the group $\mathcal{V}_{1}$ is 9 .
(4). If $O_{3}(G)=\left\{1, a^{2}, a^{4}, a^{6}, a^{8}, a^{10}, a^{12}, a^{14}, a^{16}\right\}$, then $O_{3}(G)$ is a maximal normal 3 -subgroup of $G$. Let $x \in G \cap \mathcal{V}_{1}$. Thus, $x-1 \in 1+\mathcal{J}\left(\mathbb{F}_{q} G\right)$ is a nilpotent element. Consequently, $(x-1)^{3^{2}}=$ $x^{3^{2}}-1=0$, implying $x^{3^{2}}=1$. This shows that $G \cap \mathcal{V}_{1}$ is a 3-group, and hence $G \cap \mathcal{V}_{1} \subseteq O_{3}(G)=$ $\left\{1, a^{2}, a^{4}, a^{6}, a^{8}, a^{10}, a^{12}, a^{14}, a^{16}\right\}$. Now $a^{2}-1=2+a^{2} \in \mathcal{J}\left(\mathbb{F}_{q} G\right)$, and hence $2 b+a^{2} b, 2 a^{2}+a^{4}, 2 a b+$ $a^{3} b, 2 a^{6}+a^{8}, 2 a^{12}+a^{14}, 2 a^{8}+a^{10}$ and $a^{10}+a^{12}$ are elements of $\mathcal{J}\left(\mathbb{F}_{q} G\right)$. Let $x=a^{2}, y=1+2 b+a^{2} b, z=$ $1+2 a^{2}+a^{4}, w=1+2 a b+a^{3} b, r=1+2 a^{6}+a^{8}, s=1+2 a^{12}+a^{14}, t=1+2 a^{8}+a^{10}, u=1+2 a^{10}+a^{12}$ be elements of $\mathcal{V}_{1}$. Then
$[x, y]=1+2 b+2 a^{2}+a^{4}+2 a^{6}+a^{8}+2 a^{10}+a^{12}+2 b a^{2}+2 b a^{4}+b a^{6}+2 b a^{8}+b a^{10}+2 b a^{12}+b a^{14}+2 b a^{16}=A$, $[z, A]=b+a^{4}+a^{6}+2 a^{8}+2 a^{10}+a^{14}+b a^{2}+b a^{4}+2 b a^{8}+b a^{10}+b a^{12}+2 b a^{14}=B$,
$[w, B]=a^{4}+a^{5}+a^{7}+2 a^{8}+a^{10}+2 a^{11}+a^{12}+2 a^{13}+2 a^{16}+b a+2 b a^{2}+b a^{3}+b a^{5}+b a^{6}+b a^{8}+b a^{9}+$ $2 b a^{11}+2 b a^{12}+2 b a^{13}+b a^{15}=C$,
$[r, C]=1+b+b a+2 b a^{3}+b a^{4}+b a^{5}+2 b a^{8}+b a^{9}+2 b a^{12}+2 b a^{13}+b a^{14}+2 b a^{16}+2 b a^{17}=D$,
$[s, D]=1+b a+b a^{2}+b a^{7}+b a^{8}+b a^{13}+b a^{14}=E$,
$[t, E]=1+2 b+b a+b a^{2}+2 b a^{6}+b a^{7}+b a^{8}+2 b a^{9}+2 b a^{1} 1+2 b a^{12}+b a^{13}+b a^{14}+2 b a^{17}=F$,
$[u, F]=1+b+b a+b a^{2}+b a^{3}+b a^{4}+b a^{5}+b a^{6}+b a^{7}+b a^{8}+b a^{9}+b a^{10}+b a^{11}+b a^{12}+b a^{13}+b a^{14}+$ $b a^{15}+b a^{16}+b a^{17} \neq 1$ and hence, $\mathcal{V}_{1}$ is a nilpotent group of class 8 .
(5). Using ([24], Chapter 2, Lemma 2.8), we establish that $\mathcal{J}\left(\mathbb{F}_{q} G\right)$ is nilpotent, if and only if $\mathcal{J}\left(\mathbb{F}_{q} G^{\prime}\right)$ is nilpotent. However, $\mathcal{J}\left(\mathbb{F}_{q} G\right)$ is the maximal nilpotent ideal, so we have $\omega\left(\mathbb{F}_{q} G^{\prime}\right)=$ $\mathcal{J}\left(\mathbb{F}_{q} G^{\prime}\right) \subseteq \mathcal{J}\left(\mathbb{F}_{q} G\right)$. Furthermore, we observe that $\mathbb{F}_{q}\left(\frac{D_{36}}{D_{36}^{\prime}}\right) \cong \mathbb{F}_{q}\left(\mathcal{C}_{2} \times \mathcal{C}_{2}\right) \cong \frac{\mathbb{F}_{q} G}{\omega\left(G^{\prime}\right)}$, which implies that $\operatorname{Dim}_{\mathbb{F}_{q}} \omega\left(\mathbb{F}_{q} G^{\prime}\right)=32$. Therefore, we conclude that $\mathcal{V}_{1}=\mathcal{V}_{2}$.
(6). As $\mathcal{V}_{1}=\mathcal{V}_{2}$, we can say that $\mathcal{B}=\left\{a^{i}-1 \mid i=2,4, \cdots, 16\right\} \cup\left\{a\left(a^{i}-1\right) \mid i=2,4, \cdots, 16\right\} \cup$ $\left\{a b\left(a^{i}-1\right) \mid i=2,4, \cdots, 16\right\} \cup\left\{b\left(a^{i}-1\right) \mid i=2,4, \cdots, 16\right\}$ forms a basis of $\mathcal{J}\left(\mathbb{F}_{q} G\right)$. Now, the elements $1+a+a^{17}$ commute with each basis element, thus belong to $\mathcal{Z}\left(\mathcal{J}\left(\mathbb{F}_{q} G\right)\right)$. This implies that $2+a+a^{17} \in 1+\mathcal{Z}\left(\mathcal{J}\left(\mathbb{F}_{q} G\right)\right)=\mathcal{Z}\left(\mathcal{V}_{1}\right)$. Furthermore, since $\left(2+a+a^{17}\right)^{3}=2+a^{3}+a^{15} \neq 1$, we conclude that $2+a+a^{17}$ is an element of order 9 . This proves the theorem.

Theorem 2.9. Let $q=2^{n}, G=D_{36}$. Then

$$
\mathcal{U}\left(\mathbb{F}_{q} G\right) \cong \begin{cases}\left(\mathcal{C}_{2}^{19 n}\right) \rtimes\left(\mathbb{F}_{q}^{*} \times G L_{2}\left(\mathbb{F}_{q}\right)^{4}\right) & \text { if } 2^{n} \equiv \pm 1 \quad(\bmod 9), \\ \left(\mathcal{C}_{2}^{19 n}\right) \rtimes\left(\mathbb{F}_{q}^{*} \times G L_{2}\left(\mathbb{F}_{q}\right) \times G L_{2}\left(\mathbb{F}_{q^{3}}\right)\right) & \text { otherwise }\end{cases}
$$

Proof. Combining Lemma 2.5 and Lemma 2.6, gives the proof of the theorem.

Finally, let us now discuss the semisimple case.

Theorem 2.10. Let $q=p^{n}, G=D_{36}$ and $p>3$. Then

$$
\mathcal{U}\left(\mathbb{F}_{q} G\right) \cong \begin{cases}\mathbb{F}_{q}^{* 4} \times G L_{2}\left(\mathbb{F}_{q}\right)^{8} & p^{n} \equiv \pm 1 \quad(\bmod 9) \\ \mathbb{F}_{q}^{* 4} \times G L_{2}\left(\mathbb{F}_{q}\right)^{2} \times G L_{2}\left(\mathbb{F}_{q^{3}}\right)^{2} & \text { otherwise }\end{cases}
$$

Proof. Since $p>3$ and $\mathbb{F}_{q}\left(\frac{G}{G^{\prime}}\right) \cong F_{q}\left(\mathcal{C}_{2} \times \mathcal{C}_{2}\right) \cong \mathbb{F}_{q} \oplus F_{q} \oplus F_{q} \oplus F_{q}$, using Wedderburn decomposition thorem and [23, Proposition 3.6.11], we have

$$
\mathbb{F}_{q} G \cong \mathbb{F}_{q} \oplus \mathbb{F}_{q} \oplus \mathbb{F}_{q} \oplus \mathbb{F}_{q} \oplus \sum_{i=1}^{k} \mathbb{M}_{n_{i}}\left(K_{i}\right)
$$

where $K_{i}$ being finite dimensional division algebras over field $\mathbb{F}_{q}$ and $n_{i} \geq 2$.
We divide the proof in two cases:
Case 1. Let $q=p^{n} \equiv \pm 1(\bmod 9)$.
If $q=p^{n} \equiv \pm 1(\bmod 9)$, then

$$
\begin{aligned}
d & =o(q) \quad(\bmod 9) \\
& =1 \text { or } 2 \quad(\bmod 9)
\end{aligned}
$$

and $T_{\mathbb{F}_{q}, G}=\{1\}$ or $\{ \pm 1\}$. But we know that conjugate class of an element and its inverse is the same, therefore in this case we have

$$
S_{\mathbb{F}_{q}\left(\gamma_{g}\right)}=\left\{\gamma_{g}\right\} \text { i.e. }\left|S_{\mathbb{F}_{q}\left(\gamma_{g}\right)}\right|=1
$$

for each element $g \in G$ in [3, Theorem 1.3]. Hence the Wedderburn decomposition is given by

$$
\begin{equation*}
\mathbb{F}_{q} G \cong \mathbb{F}_{q}^{4} \oplus \sum_{i=1}^{8} \mathbb{M}_{n_{i}}\left(\mathbb{F}_{q}\right) \tag{2.3}
\end{equation*}
$$

Using dimension constraints on equation (2.3), we have

$$
\begin{aligned}
& 36=4+\sum_{i=1}^{8} n_{i}^{2} \\
& \text { or } 32=\sum_{i=1}^{8} n_{i}^{2}
\end{aligned}
$$

Only possible solution of equation 2.4 is

$$
n_{i}=2, \text { for } i=1,2, \cdots, 8
$$

and hence

$$
\begin{equation*}
\mathbb{F}_{q} G \cong \mathbb{F}_{q}^{4} \oplus \sum_{i=1}^{8} \mathbb{M}_{2}\left(\mathbb{F}_{q}\right) \tag{2.5}
\end{equation*}
$$

Case 2. Let $q=p^{n} \equiv \pm 1(\bmod 9)$. Then

$$
T_{\mathbb{F}_{q}, G}=\{1,7,13\} \text { or }\{1,5,7,11,13,17\}
$$

Therefore, we simply assume

$$
T_{\mathbb{F}_{q}, G}=\{1,5,7\}
$$

If we consider cyclotomic $\mathbb{F}_{q}$-classes, we have

$$
\begin{aligned}
& S_{\mathbb{F}_{q}\left(\gamma_{e}\right)}=\left\{\gamma_{e}\right\} \quad \Longrightarrow \quad\left|S_{\mathbb{F}_{q}\left(\gamma_{e}\right)}\right|=1 \Longrightarrow K_{i} \cong \mathbb{F}_{q}, \\
& S_{\mathbb{F}_{q}\left(\gamma_{a}\right)}=\left\{\gamma_{a}, \gamma_{a^{5}}, \gamma_{a^{7}}\right\} \Longrightarrow\left|S_{\mathbb{F}_{q}\left(\gamma_{a}\right)}\right|=3 \Longrightarrow K_{i} \cong \mathbb{F}_{q^{3}} \text {, } \\
& S_{\mathbb{F}_{q}\left(\gamma_{a^{2}}\right)}=\left\{\gamma_{a^{2}}, \gamma_{a^{8}}, \gamma_{a^{4}}\right\} \quad \Longrightarrow \quad\left|S_{\mathbb{F}_{q}\left(\gamma_{a^{2}}\right)}\right|=3 \Longrightarrow K_{i} \cong \mathbb{F}_{q^{3}} \text {, } \\
& S_{\mathbb{F}_{q}\left(\gamma_{a^{3}}\right)}=\left\{\gamma_{a^{3}}\right\} \Longrightarrow\left|S_{\mathbb{F}_{q}\left(\gamma_{a^{3}}\right)}\right|=1 \Longrightarrow K_{i} \cong \mathbb{F}_{q} \text {, } \\
& S_{\mathbb{F}_{q}\left(\gamma_{a^{6}}\right)}=\left\{\gamma_{a^{6}}\right\} \Longrightarrow\left|S_{\mathbb{F}_{q}\left(\gamma_{a^{6}}\right)}\right|=1 \Longrightarrow K_{i} \cong \mathbb{F}_{q} \text {, } \\
& S_{\mathbb{F}_{q}\left(\gamma_{a} 9\right)}=\left\{\gamma_{a^{9}}\right\} \quad \Longrightarrow \quad\left|S_{\mathbb{F}_{q}\left(\gamma_{a^{9}}\right)}\right|=1 \Longrightarrow K_{i} \cong \mathbb{F}_{q} \text {, } \\
& S_{\mathbb{F}_{q}\left(\gamma_{b}\right)}=\left\{\gamma_{b}\right\} \quad \Longrightarrow \quad\left|S_{\mathbb{F}_{q}\left(\gamma_{b}\right)}\right|=1 \Longrightarrow K_{i} \cong \mathbb{F}_{q}, \\
& S_{\mathbb{F}_{q}\left(\gamma_{a} 9_{b}\right)}=\left\{\gamma_{a^{9} b}\right\} \quad \Longrightarrow \quad\left|S_{\mathbb{F}_{q}\left(\gamma_{a} 9_{b}\right)}\right|=1 \Longrightarrow K_{i} \cong \mathbb{F}_{q} \text {. }
\end{aligned}
$$

So, in this case the Wedderburn decomposition is given by

$$
\begin{equation*}
\mathbb{F}_{q} G \cong \mathbb{F}_{q}^{4} \oplus \mathbb{M}_{n_{1}}\left(\mathbb{F}_{q}\right) \oplus \mathbb{M}_{n_{2}}\left(\mathbb{F}_{q}\right) \oplus \mathbb{M}_{n_{3}}\left(\mathbb{F}_{q^{3}}\right) \oplus \mathbb{M}_{n_{4}}\left(\mathbb{F}_{q^{3}}\right) . \tag{2.6}
\end{equation*}
$$

Again using dimension constrains on (2.6), we have

$$
\begin{gather*}
36=4+n_{1}^{2}+n_{2}^{2}+3 n_{3}^{2}+3 n_{4}^{2},  \tag{2.7}\\
\text { or } 32=n_{1}^{2}+n_{2}^{2}+3 n_{3}^{2}+3 n_{4}^{2} . \tag{2.8}
\end{gather*}
$$

But equation 2.8 has only solution as

$$
n_{1}=2, n_{2}=2, n_{3}=2, n_{4}=2 .
$$

Thus, we have

$$
\mathbb{F}_{q} G \cong \mathbb{F}_{q}^{4} \oplus \mathbb{M}_{2}\left(\mathbb{F}_{q}\right)^{2} \oplus \mathbb{M}_{2}\left(\mathbb{F}_{q^{3}}\right) \oplus \mathbb{M}_{2}\left(\mathbb{F}_{q^{3}}\right)
$$

This proves the theorem.

## Acknowledgments

The authors wish to express their deep appreciation to the referee for the valuable insights provided.

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[^0]:    Communicated by Saeid Azam
    MSC(2020): Primary: 16S34; Secondary: 20C05.
    Keywords: Group algebra, Wedderburn decomposition, unit group.
    Received: 7 July 2023, Accepted: 10 January 2024.
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