# A NOTE ON ARITHMETIC-GEOMETRIC-HARMONIC MEAN INEQUALITY OF SEVERAL POSITIVE OPERATORS 

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Abstract. Suppose that $B_{1}, \cdots, B_{m}$ are positive operators on a Hilbert space $\mathcal{H}$. In this paper we generalize the weighted arithmetic, geometric, and harmonic means as follows:

$$
\begin{aligned}
& \mathbf{a}_{m}(\boldsymbol{\kappa} ; \mathbf{B})=\mathbf{a}_{2}\left(k_{1}, N^{\prime} ; B_{1}, \mathbf{a}_{m-1}\left(\boldsymbol{\kappa}^{\prime} ; \mathbf{B}^{\prime}\right)\right)=\frac{k_{1} B_{1}+\cdots+k_{m} B_{m}}{N}, \\
& \mathbf{h}_{m}(\boldsymbol{\kappa} ; \mathbf{B})=\mathbf{h}_{2}\left(k_{1}, N^{\prime} ; B_{1}, \mathbf{h}_{m-1}\left(\boldsymbol{\kappa}^{\prime} ; \mathbf{B}^{\prime}\right)\right)=\left(\frac{k_{1} B_{1}^{-1}+\cdots+k_{m} B_{m}^{-1}}{N}\right)^{-1}, \\
& \mathbf{g}_{m}(\boldsymbol{\kappa} ; \mathbf{B})=\mathbf{g}_{2}\left(k_{1}, N^{\prime} ; B_{1}, \mathbf{g}_{m-1}\left(\boldsymbol{\kappa}^{\prime} ; \mathbf{B}^{\prime}\right)\right),
\end{aligned}
$$

where $\boldsymbol{\kappa}=\left(k_{1}, \cdots, k_{m}\right), N=k_{1}+\cdots+k_{m}, \boldsymbol{\kappa}^{\prime}=\left(k_{2}, \cdots, k_{m}\right)$, and $N^{\prime}=k_{2}+\cdots+k_{m}$. We show that the arithmetic-geometric-harmonic mean inequality holds. Also we investigate nine properties of the geometric mean.

## 1. Introduction

The simplest and the most classical mean values are the arithmetic, the geometric, and the harmonic mean values. For a positive sequence $b=\left(b_{1}, \cdots, b_{m}\right)$ this mean values are defined respectively by

$$
\begin{equation*}
A_{m}(b)=\frac{1}{m} \sum_{i=1}^{m} b_{i}, \quad G_{m}(b)=\sqrt[m]{\prod_{i=1}^{m} b_{i}}, \quad H_{m}(b)=\frac{m}{\sum_{i=1}^{m} \frac{1}{b_{i}}} \tag{1.1}
\end{equation*}
$$

It is well known that

$$
\begin{equation*}
A_{m}(b) \geq G_{m}(b) \geq H_{m}(b) \tag{1.2}
\end{equation*}
$$

[^0]are called arithmetic mean-geometric mean-harmonic mean inequalities. The left hand of inequality (1.2) is one of the most important inequalities in mathematics and it has many applications in different sciences. In terms of the importance of this inequality, more than fifty arguments from different mathematicians are presented in chronological order [5].

The study of the arithmetic-geometric mean inequality has rich literature (for details refer to $[1,6,9,12])$. Some generalizations of this inequality are present by Pečarić, Qi, Šimić and Xue [11] and Qi [13].

Assume that $b=\left(b_{1}, \cdots, b_{m}\right)$ is a sequence of positive numbers and $\boldsymbol{\kappa}=\left(k_{1}, \cdots, k_{m}\right)$ is a sequence of nonnegative integers, such that $k_{1}+\cdots+k_{m}=N$. We define

$$
\begin{align*}
& \mathbf{a}_{m}\left(\boldsymbol{\kappa} ; b_{1}, \cdots, b_{m}\right)=\frac{k_{1} b_{1}+\cdots+k_{m} b_{m}}{N}  \tag{1.3}\\
& \mathbf{g}_{m}\left(\boldsymbol{\kappa} ; b_{1}, \cdots, b_{m}\right)=\sqrt[N]{b_{1}^{k_{1}} \cdots b_{m}^{k_{m}}}  \tag{1.4}\\
& \mathbf{h}_{m}\left(\boldsymbol{\kappa} ; b_{1}, \cdots, b_{m}\right)=\left(\frac{k_{1} b_{1}^{-1}+\cdots+k_{m} b_{m}^{-1}}{N}\right)^{-1} \tag{1.5}
\end{align*}
$$

that are the weighted arithmetic, weighted geometric, and weighted harmonic means respectively. Then the arithmetic-geometric-harmonic mean inequality is as follows:

$$
\mathbf{a}_{m}\left(\boldsymbol{\kappa} ; b_{1}, \cdots, b_{m}\right) \geq \mathbf{g}_{m}\left(\boldsymbol{\kappa} ; b_{1}, \cdots, b_{m}\right) \geq \mathbf{h}_{m}\left(\boldsymbol{\kappa} ; b_{1}, \cdots, b_{m}\right) .
$$

Let $\mathbb{B}(\mathcal{H})$ denote the $C^{*}$-algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$ with the identity $I_{\mathcal{H}}$. In the case when $\operatorname{dim} \mathcal{H}=n$, we identify $\mathbb{B}(\mathcal{H})$ with the full matrix algebra $\mathcal{M}_{n}(\mathbb{C})$ of all $n \times n$ matrices with entries in the complex field. An operator $A \in \mathbb{B}(\mathcal{H})$ is called positive if $\langle A x, x\rangle \geq 0$ for all $x \in \mathcal{H}$ and in this case we write $A \geq 0$. We write $A>0$ if $A$ is a positive invertible operator. For self-adjoint operators $A, B \in \mathbb{B}(\mathcal{H})$, we say that $A \leq B$ if $B-A \geq 0$. The Gelfand map $f(t) \mapsto f(A)$ is an isometrical *-isomorphism between the $C^{*}$-algebra $C(\sigma(A))$ of continuous functions on the spectrum $\sigma(A)$ of a self-adjoint operator $A$ and the $C^{*}$-algebra generated by $A$ and $I_{\mathcal{H}}$. If $f, g \in C(\sigma(A))$, then $f(t) \geq g(t)(t \in \sigma(A))$ implies that $f(A) \geq g(A)$. A linear map $\Phi$ on $\mathbb{B}(\mathcal{H})$ is positive if $\Phi(A) \geq 0$ whenever $A \geq 0$. It is said to be unital if $\Phi\left(I_{\mathcal{H}}\right)=I_{\mathcal{H}}$.

Let $A, B \in \mathbb{B}(\mathcal{H})$ are two positive invertible operators and $\nu \in[0,1]$. The operator-weighted arithmetic, geometric, and harmonic means are defined by

$$
\begin{aligned}
A \nabla_{\nu} B & =(1-\nu) A+\nu B, \\
A \not \sharp_{\nu} B & =A^{\frac{1}{2}}\left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right)^{\nu} A^{\frac{1}{2}}, \\
A!_{\nu} B & =\left((1-\nu) A^{-1}+\nu B^{-1}\right)^{-1},
\end{aligned}
$$

respectively. The operator version of the A-G-H mean inequality is $A!_{\nu} B \leq A \not \sharp_{\nu} B \leq A \nabla_{\nu} B$. In this regard, famous mathematicians such as T. Ando have worked, a summary of this issue can be seen in $[2,7,8,10]$. Also, for instance, in [3], Bakherad et. al. stated some extensions of interpolation between the arithmetic-geometric means inequality. In 2021, Bedrani et. al. [4], studied the geometric mean $A \not \sharp_{\nu} B$ for two accretive matrices $A, B$, when $\nu \in(1,2)$ and $\nu \in(-1,0)$. Seddik in [15] presented
the characterizations of some distinguished classes of $\mathbb{B}(\mathcal{H})$, namely, the self-adjoint operators, the normal operators, and the unitary operators in terms of operator inequalities.

In this paper, we generalize all statements of [14] for the non-equal weights, respectively. Also, in continuation, we present an example that shows the convergence of the recursive sequence defined for geometric mean.

## 2. Arithmetic, geometric and harmonic mean

Assume that $B_{1}, \cdots, B_{m}$ are positive operators in $\mathbb{B}(\mathcal{H})$. If we replace $b_{1}, \cdots, b_{m}$ by $B_{1}, \cdots, B_{m}$, respectively, in (1.3) and (1.5), and consider $B^{-1}=\lim _{\varepsilon} \searrow 0\left(B+\varepsilon I_{\mathcal{H}}\right)^{-1}$, then we get the extensions of the arithmetic and the harmonic operator means. But, we use the induction for the definition of the geometric mean $\mathbf{g}_{m}\left(B_{1}, \cdots, B_{m}\right)$ as follows:

Let $A$ and $B$ be two positive operators, we define the recursive sequence $\left\{T_{n}\right\}$,

$$
\begin{cases}T_{0}=\frac{k}{m} A+\frac{m-k}{m} B & ; 0<k<m  \tag{2.1}\\ T_{n+1}=\frac{m-k}{m} T_{n}+\frac{k}{m} A \sqrt[k]{\left(T_{n}^{-1} B\right)^{m-k}} & ;(n \geq 0)\end{cases}
$$

The following main result gives the convergence of the operator sequence $\left\{T_{n}\right\}$.
It is a well-known fact that, the product of two positive operators is not necessarily positive, in fact not necessarily self-adjoint. In this concept, for every $n=1,2,3, \cdots$, the operators $T_{n}^{-1} B$ is not self-adjoint but the operators $\left(T_{n}^{-1} B\right)^{\frac{m-k}{k}}$ exist and $A\left(T_{n}^{-1} B\right)^{\frac{m-k}{k}}$ is positive. As an example, we can consider the positive operators $A$ and $B$ given by

$$
A=\left(\begin{array}{ll}
3 & 1 \\
1 & 3
\end{array}\right) \quad \text { and } \quad B=\left(\begin{array}{ll}
6 & 1 \\
1 & 3
\end{array}\right)
$$

For the sake of simplicity, choose $m=3$ and $k=2$ in the Eq. (2.1), ie.

$$
T_{0}=\frac{2}{3} A+\frac{1}{3} B=\left(\begin{array}{ll}
4 & 1 \\
1 & 3
\end{array}\right)
$$

Its inverse is

$$
T_{0}^{-1}=\frac{1}{11}\left(\begin{array}{cc}
3 & -1 \\
-1 & 4
\end{array}\right)
$$

The recursion recommends for the next step

$$
T_{1}=\frac{1}{3} T_{0}+\frac{2}{3} A \sqrt{T_{0}^{-1} B}
$$

but a simple calculation shows

$$
T_{0}^{-1} B=\left(\begin{array}{rr}
\frac{17}{11} & 0 \\
-\frac{2}{11} & 1
\end{array}\right)
$$

This matrix has a square root but is not self-adjoint.
Theorem 2.1. With the above assumptions, the sequence $\left\{T_{n}\right\}=\left\{T_{n}(k, m-k ; A, B)\right\}$ converges decreasingly in $\mathbb{B}(\mathcal{H})$ to

$$
\begin{equation*}
A \sharp_{k / m} B=B^{1 / 2}\left(B^{-1 / 2} A B^{-1 / 2}\right)^{k / m} B^{1 / 2} . \tag{2.2}
\end{equation*}
$$

Also, the following estimation holds

$$
\begin{equation*}
0 \leq T_{n}-A \sharp_{k / m} B \leq\left(1-\frac{k}{m}\right)^{n}\left(T_{0}-\left(B^{-1 / 2} A B^{-1 / 2}\right)^{k / m}\right) \quad \forall n \geq 0 . \tag{2.3}
\end{equation*}
$$

Proof. To prove, we consider three steps as in [14, Theorem 2.1]:
Step 1. Assume that $a>0$ is a real number and consider the recurrent sequence

$$
\begin{cases}x_{0}=\frac{k}{m} a+\frac{m-k}{m} ; & 0<k<m  \tag{2.4}\\ x_{n+1}=\frac{m-k}{m} x_{n}+\frac{k}{m}\left(\frac{a}{\sqrt[k]{x_{n}^{m-k}}}\right) ; & (n \geq 0)\end{cases}
$$

by using the idea of Step 1 of [14, Theorem 2.1], we conclude that

$$
0 \leq x_{n}-\sqrt[m]{a^{k}} \leq\left(\frac{m-k}{m}\right)^{n}\left(x_{0}-\sqrt[m]{a^{k}}\right), \quad \forall n \geq 0
$$

And hence the real sequence $\left\{x_{n}\right\}$ converges to $\sqrt[m]{a^{k}}$.
Step 2. Let $A \in \mathbb{B}(\mathcal{H})$ be a positive operator and define the following sequence

$$
\begin{cases}X_{0}=\frac{k}{m} A+\frac{m-k}{m} I_{\mathcal{H}} ; & 0<k<m  \tag{2.5}\\ X_{n+1}=\frac{m-k}{m} X_{n}+\frac{k}{m} A X_{n}^{1-\frac{m}{k}} ; & (n \geq 0)\end{cases}
$$

Since that $A$ commutes with $X_{n}$ for $n \geq 0$, so, by using Gelfand's representation and the previous step, we get $\left\{X_{n}\right\}$ converges in $\mathbb{B}(\mathcal{H})$ to $A^{k / m}$ and we have

$$
0 \leq X_{n}-A^{k / m} \leq\left(\frac{m-k}{m}\right)^{n}\left(X_{0}-A^{k / m}\right), \quad \forall n \geq 0
$$

Step 3. With the same way of the second step, the sequence $\left\{Y_{n}\right\}$,

$$
\begin{cases}Y_{0}=\frac{k}{m} B^{-1 / 2} A B^{-1 / 2}+\frac{m-k}{m} I_{\mathcal{H}} ; & 0<k<m  \tag{2.6}\\ Y_{n+1}=\frac{m-k}{m} Y_{n}+\frac{k}{m} B^{-1 / 2} A B^{-1 / 2} Y_{n}^{1-\frac{m}{k}} ; & (n \geq 0)\end{cases}
$$

converges in $\mathbb{B}(\mathcal{H})$ to $\left(B^{-1 / 2} A B^{-1 / 2}\right)^{k / m}$ and

$$
0 \leq Y_{n}-\left(B^{-1 / 2} A B^{-1 / 2}\right)^{k / m} \leq\left(\frac{m-k}{m}\right)^{n}\left(Y_{0}-\left(B^{-1 / 2} A B^{-1 / 2}\right)^{k / m}\right), \quad(n \geq 0)
$$

Thus, by using (2.6), we have

$$
\left\{\begin{array}{l}
B^{1 / 2} Y_{0} B^{1 / 2}=\frac{k}{m} A+\frac{m-k}{m} B  \tag{2.7}\\
B^{1 / 2} Y_{n+1} B^{1 / 2}=\frac{m-k}{m} B^{1 / 2} Y_{n} B^{1 / 2}+\frac{k}{m} A B^{-1 / 2} Y_{n}^{1-\frac{m}{k}} B^{1 / 2} \quad(n \geq 0),
\end{array}\right.
$$

and so,

$$
B^{-1 / 2} Y_{n}^{\frac{k-m}{k}} B^{1 / 2}=\left(B^{-1 / 2} Y_{n}^{k-m} B^{1 / 2}\right)^{1 / k}
$$

Now we have

$$
\begin{aligned}
& B^{-1 / 2} Y_{n}^{k-m} B^{1 / 2}=\left(B^{-1 / 2} Y_{n}^{-1} B^{1 / 2}\right)^{m-k} \\
& =\left(B^{-1 / 2} Y_{n}^{-1} B^{-1 / 2}\right) B\left(B^{-1 / 2} Y_{n}^{-1} B^{-1 / 2}\right) B \cdots\left(B^{-1 / 2} Y_{n}^{-1} B^{-1 / 2}\right) B, \\
& =\left(T_{n}^{-1} B\right)^{m-k},
\end{aligned}
$$

where $T_{n}=B^{1 / 2} Y_{n} B^{1 / 2}$. This proves the theorem.

Remark 2.2. The $\operatorname{map}(A, B) \longmapsto A \sharp_{k / m} B$ has the conjugate symmetry relation, i.e

$$
\begin{equation*}
A \not \sharp_{k / m} B=A^{1 / 2}\left(A^{-1 / 2} B A^{-1 / 2}\right)^{(m-k) / m} A^{1 / 2}=B \sharp(m-k) / m \text {. } \tag{2.8}
\end{equation*}
$$

Corollary 2.3. With the above assumptions, the following statements hold:
(1) For a fixed positive operator $B$, the maps $X \longmapsto X \sharp_{k / m} B$, and $X \longmapsto B \sharp_{k / m} X$ are operator increasing and concave;
(2) If $L \in \mathbb{B}(\mathcal{H})$ be a invertible operator, then

$$
\left(L^{*} A L\right) \sharp_{k / m}\left(L^{*} B L\right)=L^{*}\left(A \sharp_{k / m} B\right) L .
$$

Definition 2.4. With the above notations, if $\mathbf{B}=\left(B_{1}, \cdots, B_{m}\right)$ be a m-tuple of positive operators and $\boldsymbol{\kappa}=\left(k_{1}, \cdots, k_{m}\right)$ be a m-tuple of nonnegative integers such that $k_{1}+\cdots+k_{m}=N$. We define the arithmetic, harmonic, and geometric mean of several operators as follows:

$$
\begin{align*}
& \mathbf{a}_{m}(\boldsymbol{\kappa} ; \mathbf{B})=\mathbf{a}_{2}\left(k_{1}, N^{\prime} ; B_{1}, \mathbf{a}_{m-1}\left(\boldsymbol{\kappa}^{\prime} ; \mathbf{B}^{\prime}\right)\right)=\frac{k_{1} B_{1}+\cdots+k_{m} B_{m}}{N}  \tag{2.9}\\
& \mathbf{h}_{m}(\boldsymbol{\kappa} ; \mathbf{B})=\mathbf{h}_{2}\left(k_{1}, N^{\prime} ; B_{1}, \mathbf{h}_{m-1}\left(\boldsymbol{\kappa}^{\prime} ; \mathbf{B}^{\prime}\right)\right)=\left(\frac{k_{1} B_{1}^{-1}+\cdots+k_{m} B_{m}^{-1}}{N}\right)^{-1}  \tag{2.10}\\
& \mathbf{g}_{m}(\boldsymbol{\kappa} ; \mathbf{B})=\mathbf{g}_{2}\left(k_{1}, N^{\prime} ; B_{1}, \mathbf{g}_{m-1}\left(\boldsymbol{\kappa}^{\prime} ; \mathbf{B}^{\prime}\right)\right) \tag{2.11}
\end{align*}
$$

where $\boldsymbol{\kappa}^{\prime}=\left(k_{2}, \cdots, k_{m}\right), N^{\prime}=k_{2}+\cdots+k_{m}$ and $\mathbf{B}^{\prime}=\left(B_{2}, \cdots, B_{m}\right)$.
In the next theorem, we study the properties of the operator mean $\mathbf{g}_{m}(\boldsymbol{\kappa} ; \mathbf{B})$.
Proposition 2.5. The operator mean $\mathbf{g}_{m}(\boldsymbol{\kappa} ; \mathbf{B})$ satisfies the following properties:
(1) If $B_{1}, \cdots, B_{m}$ are commuting, and $r=\left(r_{1}, \cdots, r_{m}\right)$ is a sequence of nonnegative rational numbers with $r_{1}+\cdots+r_{m}=1$ then

$$
\mathbf{g}_{m}\left(r ; B_{1}, \cdots, B_{m}\right)=B_{1}^{r_{1}} \cdots B_{m}^{r_{m}}
$$

(2) Self-duality relation, i.e

$$
\left(\mathbf{g}_{m}(\boldsymbol{\kappa} ; \mathbf{B})\right)^{-1}=\mathbf{g}_{m}\left(\boldsymbol{\kappa} ; \mathbf{B}^{-1}\right)
$$

where $\mathbf{B}^{-1}=\left(B_{1}^{-1}, \cdots, B_{m}^{-1}\right)$.
(3) The arithmetic-geometric-harmonic mean inequality, i.e

$$
\mathbf{h}_{m}(\boldsymbol{\kappa} ; \mathbf{B}) \leq \mathbf{g}_{m}(\boldsymbol{\kappa} ; \mathbf{B}) \leq \mathbf{a}_{m}(\boldsymbol{\kappa} ; \mathbf{B})
$$

(4) The algebraic equation, i.e. the equation $X(B X)^{N-1}=A(B A)^{k_{1}-1}$ has one and only one solution, $X=\mathbf{g}_{m}\left(\boldsymbol{\kappa} ; A, B^{-1}, \cdots, B^{-1}\right)$, where $k_{1}+\cdots+k_{m}=N$.

Proof. (1) Follows immediately from the definition of $\mathbf{g}_{m}$.
(2) Follows by a simple induction on $m \geq 2$ with the duality relation:

$$
\left(A \sharp_{k / m} B\right)^{-1}=A^{-1} \sharp_{k / m} B^{-1} .
$$

(3) By induction on $m \geq 2$, it is well known for $m=2$. Assume that it holds for $m-1$, and show that it holds for $m$. According to (2.3) with $n=0$, we obtain

$$
A \sharp_{k / m} B \leq \frac{k}{m} A+\frac{m-k}{m} B .
$$

Now, by using the definition of $\mathbf{g}_{m}(\boldsymbol{\kappa} ; \mathbf{B})$, we have

$$
\begin{align*}
\mathbf{g}_{m}(\boldsymbol{\kappa} ; \mathbf{B}) & =\mathbf{g}_{2}\left(k_{1}, N^{\prime} ; B_{1}, \mathbf{g}_{m-1}\left(\boldsymbol{\kappa}^{\prime} ; \mathbf{B}^{\prime}\right)\right) \leq \mathbf{g}_{2}\left(k_{1}, N^{\prime} ; B_{1}, \mathbf{a}_{m-1}\left(\boldsymbol{\kappa}^{\prime} ; \mathbf{B}^{\prime}\right)\right)  \tag{2.12}\\
& \leq \mathbf{a}_{2}\left(k_{1}, N^{\prime} ; B_{1}, \mathbf{a}_{m-1}\left(\boldsymbol{\kappa}^{\prime} ; \mathbf{B}^{\prime}\right)\right)=\mathbf{a}_{m}(\boldsymbol{\kappa} ; \mathbf{B})
\end{align*}
$$

where $\mathbf{B}=\left(B_{1}, B_{2}, \cdots, B_{m}\right)$ is positive operator hence

$$
\mathbf{g}_{m}\left(\boldsymbol{\kappa} ; \mathbf{B}^{-1}\right) \leq \mathbf{a}_{m}\left(\boldsymbol{\kappa} ; \mathbf{B}^{-1}\right)
$$

and by (1.3) and the fact that the map $X \mapsto X^{-1}$ is operator decreasing, we obtain the geometric-harmonic mean inequality.
(4) We have

$$
\begin{aligned}
X=\mathbf{g}_{m}\left(\boldsymbol{\kappa} ; A, B^{-1}, \cdots, B^{-1}\right) & =\mathbf{g}_{2}\left(k_{1}, N^{\prime} ; A, \mathbf{g}_{m-1}\left(\boldsymbol{\kappa}^{\prime} ; B^{-1}, \cdots, B^{-1}\right)\right) \\
& =\mathbf{g}_{2}\left(k_{1}, N^{\prime} ; A, B^{-1}\right) \\
& =B^{-1 / 2}\left(B^{1 / 2} A B^{1 / 2}\right)^{k_{1} / N} B^{-1 / 2},
\end{aligned}
$$

thus $B^{1 / 2} X B^{1 / 2}=\left(B^{1 / 2} A B^{1 / 2}\right)^{k_{1} / N}$, and so

$$
\left(B^{1 / 2} X B^{1 / 2}\right)^{N}=\left(B^{1 / 2} A B^{1 / 2}\right)^{k_{1}}
$$

or equivalently

$$
B^{1 / 2} X B^{1 / 2} B^{1 / 2} X B^{1 / 2} \cdots B^{1 / 2} X B^{1 / 2}=B^{1 / 2} A B^{1 / 2} B^{1 / 2} A B^{1 / 2} \cdots B^{1 / 2} A B^{1 / 2}
$$

Therefore $X(B X)^{N-1}=A(B A)^{k_{1}-1}$.

Proposition 2.6. Assume that $B_{1}, B_{2}, \cdots, B_{m} \in \mathbb{B}(\mathcal{H})$ are positive operators. Then the following statements hold:
(1) For every $\alpha_{1}, \alpha_{2}, \cdots, \alpha_{m} \in \mathbb{R}^{+}$

$$
\mathbf{g}_{m}\left(\boldsymbol{\kappa} ; \alpha_{1} B_{1}, \cdots, \alpha_{m} B_{m}\right)=\mathbf{g}_{m}\left(\boldsymbol{\kappa} ; \alpha_{1}, \cdots, \alpha_{m}\right) \mathbf{g}_{m}(\boldsymbol{\kappa} ; \mathbf{B}),
$$

where

$$
\mathbf{g}_{m}\left(\boldsymbol{\kappa} ; \alpha_{1}, \cdots, \alpha_{m}\right)=\sqrt[N]{\alpha_{1}^{k_{1}} \cdots \alpha_{m}^{k_{m}}}
$$

(2) The map $X \mapsto \mathbf{g}_{m}\left(\boldsymbol{\kappa} ; X, B_{2}, \cdots, B_{m}\right)$ is operator increasing and concave, i.e.

$$
X \leq Y \Longrightarrow \mathbf{g}_{m}\left(\boldsymbol{\kappa} ; X, B_{2}, \cdots, B_{m}\right) \leq \mathbf{g}_{m}\left(\boldsymbol{\kappa} ; Y, B_{2}, \cdots, B_{m}\right)
$$

and

$$
\begin{aligned}
& \mathbf{g}_{m}\left(\boldsymbol{\kappa} ; \lambda X+(1-\lambda) Y, B_{2}, \cdots, B_{m}\right) \\
& \geq \lambda \mathbf{g}_{m}\left(\boldsymbol{\kappa} ; X, B_{2}, \cdots, B_{m}\right)+(1-\lambda) \mathbf{g}_{m}\left(\boldsymbol{\kappa} ; Y, B_{2}, \cdots, B_{m}\right), \\
& \quad \text { DOI: https://dx.doi.org/10.30504/JIMS. 2023.415016.1144 }
\end{aligned}
$$

for all positive operators $X, Y \in \mathbb{B}(\mathcal{H})$ and $\lambda \in[0,1]$.
(3) For every invertible operator $L \in \mathbb{B}(\mathcal{H})$, the following equality holds

$$
\mathbf{g}_{m}\left(\boldsymbol{\kappa} ; L^{*} B_{1} L, \cdots, L^{*} B_{m} L\right)=L^{*} \mathbf{g}_{m}\left(\boldsymbol{\kappa} ; B_{1}, \cdots, B_{m}\right) L
$$

(4) If $\operatorname{dim} \mathcal{H}<\infty$, then

$$
\operatorname{det} \mathbf{g}_{m}(\boldsymbol{\kappa} ; \mathbf{B})=\mathbf{g}_{m}\left(\boldsymbol{\kappa} ; \operatorname{det} B_{1}, \cdots, \operatorname{det} B_{m}\right)
$$

Proof. (1) This is clear, by the definition of $\mathbf{g}_{m}$.
(2) By using induction and Corollary 2.3 (1), we get the result.
(3) This follows from the definition and Corollary 2.3 (2).
(4) By the properties of the determinant, it is easy to see that, for all positive operators $A$ and $B$,

$$
\operatorname{det}\left(A \not \sharp_{k / m} B\right)=(\operatorname{det} A) \sharp_{k / m}(\operatorname{det} B) .
$$

The above equality, the definition of $\mathbf{g}_{m}(\boldsymbol{\kappa} ; \mathbf{B})$ and, a simple induction on $m \geq 2$, implies the desired result.

Corollary 2.7. The map $X \mapsto \mathbf{g}_{m}\left(\boldsymbol{\kappa} ; B_{1}, \cdots, X, \cdots, B_{m}\right)$ is operator increasing and concave.
Proof. For $m=2$, the desired result holds. For the map

$$
X \mapsto \mathbf{g}_{m}\left(\boldsymbol{\kappa} ; X, B_{2}, \cdots, B_{m}\right)
$$

it is the statement of Proposition 2.6 (2). Now, by Remark 2.2 it is easy to see that if $X \mapsto \mathbf{g}_{m-1}\left(\boldsymbol{\kappa}^{\prime} ; B_{2}, \cdots, X, \cdots, B_{m}\right)$ is an operator increasing and concave map, then so is $X \mapsto$ $\mathbf{g}_{2}\left(k_{1}, N^{\prime} ; B_{1}, \mathbf{g}_{m-1}\left(\boldsymbol{\kappa}^{\prime} ; B_{2}, \cdots, X, \cdots, B_{m}\right)\right)$.

Corollary 2.8. For all $p \in(0,1]$, there exists a rational sequence $\left\{r_{n}\right\}$ in $[0,1]$ such that $r_{n} \nearrow p$, thus $\lim \left(A \nVdash_{n} B\right)=A \not \sharp_{p} B$. Hence we can say:
If $\mathbf{A}=\left(A_{1}, \cdots, A_{m}\right)$ is a sequence of positive operators and $\boldsymbol{\omega}=\left(p_{1}, \cdots, p_{m}\right)$ is a probability vector. Then for each $i=1,2, \cdots, m-1$, there are sequences of rational numbers $\left\{r_{n, i}\right\}$ in $[0,1]$, such that $r_{n, i} \nearrow p_{i}(i=1,2, \cdots, m-1)$. Then $r_{n, m}=1-\left(r_{n, 1}+\cdots+r_{n, m-1}\right)$ is convergent to $p_{m}$. By the continuity of $\mathbf{g}_{m}$ on $\mathbb{B}(\mathcal{H})$, we get

$$
\begin{aligned}
\mathbf{g}_{m}(\boldsymbol{\omega} ; \mathbf{A}) & =\mathbf{g}_{m}\left(\left(p_{1}, \cdots, p_{m}\right) ;\left(A_{1}, \cdots, A_{m}\right)\right) \\
& =\lim _{n \rightarrow \infty} \mathbf{g}_{m}\left(\left(r_{n, 1}, \cdots, r_{n, m}\right) ;\left(A_{1}, \cdots, A_{m}\right)\right) \\
& =\lim _{n \rightarrow \infty} \mathbf{g}_{m}\left(\mathbf{r}_{n} ; \mathbf{A}\right)
\end{aligned}
$$

where $\mathbf{r}_{n}=\left(r_{n, 1}, \cdots, r_{n, m}\right)$. According to the above propositions, $\mathbf{g}_{m}(\boldsymbol{\omega} ; \mathbf{A})$ has the following properties:
(P1) If $A_{1}, \cdots, A_{m}$ commute with each other, then

$$
\mathbf{g}_{m}(\boldsymbol{\omega} ; \mathbf{A})=A_{1}^{p_{1}} \cdots A_{m}^{p_{m}}
$$

(P2) Joint homogeneity.

$$
\mathbf{g}_{m}\left(\boldsymbol{\omega} ; a_{1} A_{1}, \cdots, a_{m} A_{m}\right)=a_{1}^{p_{1}} \cdots a_{m}^{p_{m}} \mathbf{g}_{m}(\boldsymbol{\omega}, \mathbf{A})
$$

for positive numbers $a_{i}>0(i=1, \cdots, m)$.
(P3) Monotonicity. For each $i=1,2, \cdots, m$, if $B_{i} \leq A_{i}$, then

$$
\mathbf{g}_{m}(\boldsymbol{\omega} ; \mathbf{B}) \leq \mathbf{g}_{m}(\boldsymbol{\omega} ; \mathbf{A})
$$

(P4) Continuity. For each $i=1,2, \cdots, m$, let $\left\{A_{i}^{(k)}\right\}_{k=1}^{\infty}$ be sequence of positive operators such that $A_{i}^{(k)} \rightarrow A_{i}$ as $k \rightarrow \infty$. Then

$$
\mathbf{g}_{m}\left(\boldsymbol{\omega} ; A_{1}^{(k)}, \cdots, A_{m}^{(k)}\right) \rightarrow \mathbf{g}_{m}\left(\boldsymbol{\omega} ; A_{1}, \cdots, A_{m}\right) .
$$

(P5) Congruence invariance. For any invertible operator L,

$$
\mathbf{g}_{m}\left(\boldsymbol{\omega} ; L^{*} A_{1} L, \cdots, L^{*} A_{m} L\right)=L^{*} \mathbf{g}_{m}\left(\boldsymbol{\omega} ; A_{1}, \cdots, A_{m}\right) L
$$

(P6) Joint concavity.

$$
\begin{aligned}
& \mathbf{g}_{m}\left(\boldsymbol{\omega} ; \lambda A_{1}+(1-\lambda) B_{1}, \cdots, \lambda A_{m}+(1-\lambda) B_{m}\right) \\
& \geq \lambda \mathbf{g}_{m}\left(\boldsymbol{\omega} ; A_{1}, \cdots, A_{m}\right)+(1-\lambda) \mathbf{g}_{m}\left(\boldsymbol{\omega} ; B_{1}, \cdots, B_{m}\right) \quad \text { for all } 0 \leq \lambda \leq 1
\end{aligned}
$$

(P7) Self-duality.

$$
\mathbf{g}_{m}\left(\boldsymbol{\omega} ; A_{1}^{-1}, \cdots, A_{m}^{-1}\right)^{-1}=\mathbf{g}_{m}\left(\boldsymbol{\omega} ; A_{1}, \cdots, A_{m}\right)
$$

(P8) Determinantial identity.

$$
\operatorname{det} \mathbf{g}_{m}\left(\boldsymbol{\omega} ; A_{1}, \cdots, A_{m}\right)=\prod_{i=1}^{m}\left(\operatorname{det} A_{i}\right)^{p_{i}}
$$

(P9) Arithmetic-geometric-harmonic mean inequalities.

$$
\left(\sum_{i=1}^{n} p_{i} A_{i}^{-1}\right)^{-1} \leq \mathbf{g}_{m}(\boldsymbol{\omega} ; \mathbf{A}) \leq \sum_{i=1}^{n} p_{i} A_{i} .
$$

Remark 2.9. It is well known that $(A, B) \mapsto \mathbf{g}_{2}(\boldsymbol{\omega} ; A, B)$ is symmetric. However $\mathbf{g}_{m}$ is not symmetric for $m \geq 3$, [14].

In the following example, we provide a numerical example to illustrate the above theoretical results with two numerical matrices. We denote by $\|A\|$ the Schur's norm of $A$ defined by

$$
\|A\|=\sqrt{\operatorname{Trace}\left(A^{*} A\right)} .
$$

Example 2.10. Assume that $A$ and $B$ are two matrices as follows:

$$
A=\left(\begin{array}{lll}
9 & 3 & 1 \\
3 & 8 & 2 \\
1 & 2 & 6
\end{array}\right), \quad B=\left(\begin{array}{ccc}
5 & -1 & 2 \\
-1 & 3 & 1 \\
2 & 1 & 5
\end{array}\right)
$$

In order to compute some iterations of the sequence $\left\{T_{n}\right\}$, we compute $\mathbf{g}_{2}(A, B)$ by algorithm (2.1). By using MATLAB software R2023a, we obtain numerical iterations $T_{2}, T_{3}, \cdots, T_{8}$ satisfying the following
estimations in the tables 1, 2, 3 and 4 with $m=1000$ and $k=1, \cdots, m-1$, and good approximations are obtained from the first iterations. This computation shows that the sequence $\left\{T_{n}\right\}$ converges.

| $\left\\|T_{1}-T_{0}\right\\|$ | 0.0016154728057480257676 |
| :--- | :--- |
| $\left\\|T_{2}-T_{1}\right\\|$ | 0.0011634280912704566754 |
| $\left\\|T_{3}-T_{2}\right\\|$ | 0.00051474649329987193312 |
| $\left\\|T_{4}-T_{3}\right\\|$ | 0.00008032143632017041564 |
| $\left\\|T_{5}-T_{4}\right\\|$ | 0.0000016852354079912081024 |
| $\left\\|T_{6}-T_{5}\right\\|$ | 0.00000000072232699894026335025 |
| $\left\\|T_{7}-T_{6}\right\\|$ | 0.000000000000021791503230382269578 |
| $\left\\|T_{8}-T_{7}\right\\|$ | 0 |

Table 1. estimation for $k=1$

| $\left\\|T_{1}-T_{0}\right\\|$ | 0.16541828864772079033 |
| :--- | :--- |
| $\left\\|T_{2}-T_{1}\right\\|$ | 0.073685097004985281033 |
| $\left\\|T_{3}-T_{2}\right\\|$ | 0.012398848987408535652 |
| $\left\\|T_{4}-T_{3}\right\\|$ | 0.000306904176498107729 |
| $\left\\|T_{5}-T_{4}\right\\|$ | 0.00000018290682899704290181 |
| $\left\\|T_{6}-T_{5}\right\\|$ | 0.00000000000007029675847066042085 |
| $\left\\|T_{7}-T_{6}\right\\|$ | 0.00000000000001788010648157496553 |
| $\left\\|T_{8}-T_{7}\right\\|$ | 0 |

TABLE 2. estimation for $k=100$

| $\left\\|T_{1}-T_{0}\right\\|$ | 0.22161277342124655054 |
| :--- | :--- |
| $\left\\|T_{2}-T_{1}\right\\|$ | 0.0054140138376658356392 |
| $\left\\|T_{3}-T_{2}\right\\|$ | 0.0000032366912930827457344 |
| $\left\\|T_{4}-T_{3}\right\\|$ | 0.0000000000011569560685390664217 |
| $\left\\|T_{5}-T_{4}\right\\|$ | 0.0000000000000025646512328328951085 |
| $\left\\|T_{6}-T_{5}\right\\|$ | 0.000000000000002122893325126378536 |
| $\left\\|T_{7}-T_{6}\right\\|$ | 0.0000000000000011647424834813782837 |
| $\left\\|T_{8}-T_{7}\right\\|$ | 0 |

TABLE 3. estimation for $k=500$

| $\left\\|T_{1}-T_{0}\right\\|$ | 0.0000000038666396223820110835 |
| :--- | :--- |
| $\left\\|T_{2}-T_{1}\right\\|$ | 0.0000000000000035388086795394242018 |
| $\left\\|T_{3}-T_{2}\right\\|$ | 0.000000000000005193186131580818277 |
| $\left\\|T_{4}-T_{3}\right\\|$ | 0.0000000000000023288234633381844451 |
| $\left\\|T_{5}-T_{4}\right\\|$ | 0.0000000000000012755491433176288343 |
| $\left\\|T_{6}-T_{5}\right\\|$ | 0.0000000000000028261664256307950536 |
| $\left\\|T_{7}-T_{6}\right\\|$ | 0.0000000000000029373740229761031035 |
| $\left\\|T_{8}-T_{7}\right\\|$ | 0 |
| TABLE 4. estimation for $k=999$ |  |

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