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ON INFINITE DIRECT PRODUCTS OF RINGS MODULO THEIR DIRECT SUMS

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Dedicated to Professor O. A. S. Karamzadeh

Abstract. In this article, inspiring with a result due to Karamzadeh, we examine the $\prod_{i\in I} R_i/\oplus_{i\in I} R_i$, where $\{R_i\}_{i\in I}$ is an infinite family of rings. We observe that they are not self-injective on either side. However, in some important cases, they are $\aleph_0$-self-injective. Along this line, we study the interconnection between regularity (in the sense of von Neumann), injectivity , and $\aleph_0$-injectivity.

It is not knowledge, but the act of learning, not possession but the act of getting there, which grants the greatest enjoyment. When I have clarified and exhausted a subject, then I turn away from it, in order to go into darkness again. The never-satisfied man is so strange; if he has completed a structure, then it is not in order to dwell in it peacefully, but in order to begin another. I imagine the world conqueror must feel thus, who, after one kingdom is scarcely conquered, stretches out his arms for others. (*Carl Friedrich Guass*)

Our story begins with a paper of Matlis, in which, among other things, he has studied commutative reduced rings, their minimal primes, their injective hulls and their classical rings of quotients (see [18]). As Matlis remarks in the introduction of the article, in writing the paper, he was confronted with some serious difficulties:

> Some of this information is already known. Thus in order to present more detailed results, a good deal of background information has to be used, imposing a severe strain on the general reader unfamiliar with the subject. Further compounding the

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> problem is that much of the information is scattered wholesale about the literature. An even deeper difficulty is that this information, while relatively elementary in character, is usually thrown off as pieces of debris from general construction in the theory of noncommutative rings, or category and sheaf theory, so that no easy route to the subject is available. In order to overcome these problems we shall present statements and proofs of most relevant facts about a reduced ring and its minimal prime spectrum including folklore and elementary exercises, as well as the work of other authors, giving attributions only for the deeper results.

I think the result of his effort was quite successful and the outcome is a brilliant example of research-expository articles. In the last section of his paper, he provides some interesting examples, and the first one has the central figure of the discussion.

Example 1: Let I be an infinite index set; for each $i \in I$, let $\mathbb{F}_i$ be a field, $\mathbb{K} = \prod_{i\in I} \mathbb{F}_i$ and $\mathbb{J} = \oplus_{i\in I}\mathbb{F}_i$. In this example, he studied prime ideals of $\mathbb{K}$ and $\frac{\mathbb{K}}{\mathbb{J}}$. After showing that all prime ideals of $\frac{\mathbb{K}}{\mathbb{J}}$ are essential, Matlis posed his questions:

Matlis's Questions

> (1) It is an open question whether or not the ring $R = \frac{\mathbb{K}}{\mathbb{J}}$ of Example 1 is self-injective.
>
> (2) Let R be any self-injective VNR that is not a finite direct sum of fields; and let J be the sum of all of the simple submodules of R. (J could be 0.) Then R/J is a VNR with an infinite number of prime ideals and they are all essential in R. For by Proposition 3.13 the proof can easily be reduced to the case of Example 1. The question of whether or not R/J is selfinjective is a generalization of the open question posed by Example 1.

He posed the second question, as if, he was expecting that the answer of his first question was positive (which is not). Karamzadeh had given the reference [18] to one of his MS.c students (named Mehrdad Namdari), to write a research thesis on it. It was in this way that he came across these questions. After a period of creative thinking which all mathematicians are familiar with, he finally solved the problems and published his own solution in an article entitled "on a Question of Matlis" (see [14]). I think this article, like any other good articles, has some unsaid points that here I will try to address a corner of it. As we review his answers, we try to highlight the main points of his solution and at the same time to generalize some of his results. In this way, we will reveal those "unsaid points". I can even claim that this topic has shaped a considerable part of his own intellectual and research life and also some of his students, including myself.

This article has several goals, some of them pedagogical. One of these goals is to show readers how a question can be an unending source of inspiration for the next and abundant research. And I will explain this issue not only by telling that in a general phrase but by showing with a concrete example, that is, Kazemzadeh's answers. I'd be delighted to stimulate a few readers in this way. Another issue that this article intends to highlight is that we should look at mathematics as a two-way gateway.

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Historically, it was usually non-commutative algebraists who studied commutative algebra in the hope of generalizing its theorems to non-commutative settings. This is, of course, an efficient method for research - as the Goldie school at University of Leeds, in the forties to the eighties of the twentieth century - was exactly the representative of this way of thinking. But as it can be seen from our story, it is harmless that experts in commutative algebra occasionally visit the articles of their colleagues in non-commutative algebra. Last but not least, the present writing is a tribute to Professor O.A.S. Karamzadeh in a mathematical way, the way he prefers to all other ways.

All rings are assumed to be associative with an identity and modules are unitary modules. A ring R is called *regular*(in the sense of Von Neumann), if for every $a \in R$ there exists $b \in R$ such that $a = aba$. A ring R is said to be reduced, if R has no non-zero nilpotent element. The direct sums of all minimal left ideals of a ring R, is called the *socle* of R and is denoted by $\mathrm{Soc}_l(R)$. It is well known that $\mathrm{Soc}_l(R)$ is the intersection of all essential left ideals of R. A ring R is said to be left*self-injective* (*left* $\aleph_0$*-self-injective*) (*left f-self-injective*), if every module homomorphism $\phi : I \longrightarrow R$ can be extended to a module homomorphism $\bar{\phi} : R \longrightarrow R$, whenever I is a left ideal ((a left countably generated ideal), (a left finitely generaated)) of R. It is evident that every left self-injective ring is left $\aleph_0$-self-injective. The reader is referred to [17] for undefined terms and notations.

Let R be a ring and $x = a_i(modA_i)$, $i \in I$, be a (countable) system of congruences where $a_i \in R$, for all $i \in I$ and each A_i is a principal left ideal of R. If every such system that is finitely solvable has a simultaneous solution in R we say that R is $(\aleph_0-)$linearly compact. A subset S of R is said to be orthogonal provided $xy = 0$ for all $x, y \in S$ and $x \neq y$. If $S \cap T = \emptyset$ and $S \cup T$ is an orthogonal set in R, then S is said to have a left separation from T if there exists an element $a \in R$ with $as^2 = s$, for all $s \in S$, and $a \in \mathrm{ann}_l(T)$. By a countable system of linear equations over a ring R, we mean countably many linear equations with coefficients from R involving a countable set of indeterminates where in each equation there exists only a finite set of indeterminates with nonzero coefficients. A ring R is called $\aleph_0$-algebraically compact, if every countable system of linear equations over a ring R which is finitely solvable has a simultaneous solution in R.

1. The first answer to the question of Matlis

As a matter of fact, Karamzadeh, has given two independent answers to the Matlis problems and especially the second answer is more sophisticated than the first answer and quite independent of Osofsky's result. The second proof was also more fertile than the first one and I will explain it later in Section 3, but for now, we focus on his first answer and the possibilities it opens to us for further developments.

As we said earlier, the first answer is based on a deep observation of Osofsky. We present it exactly as it has been presented in [23].

Theorem 1.1. *Let $\{e_i\}_{i \in I}$ be an infinite set of orthogonal idempotents of R. Assume for each $A \subseteq I$, there exists $m_A \in R$ with $m_A e_i = e_i$, for all $i \in A$, and $e_j m_A = 0$, for all $j \in I \setminus A$. Then for all $M_R \supseteq R_R$, $\frac{M}{\oplus e_i R + \ker \pi}$ is not injective, where $\pi : R \longrightarrow \prod_{i \in I} e_i R$, $\pi(x) = (e_i x)$.*

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In [22, Lemma 5], Osofsky has shown that if R is a right self-injective regular ring that contains an infinite set of orthogonal idempotents $\{e_n\}_{n\in\mathbb{N}}$, then $R/\oplus e_nR$ is not an injective right R-module; this immediately gives a negative answer to the second question of Matlis. Having this lemma in her arsenal, she was able to prove her well known that theorem; R is semisimple Artinian if and only if every (cyclic) finitely generated right R-module is injective (see [22, Theorem]).

Though in [14], Theorem 1.1 has been used only to show that $\frac{\mathbb{K}}{\mathbb{J}}$ is not self-injective, but actually the very same proof works well to prove the following theorem. Before we go through the result further, we remind some necessary concepts. Let R be a ring, we say that the ideal $I \subset R$ is *pure* if the quotient ring R/I is flat over R. It is well known that $\oplus R_i$ is a pure ideal of $\prod R_i$. We also recall that if K is a two-sided ideal of the ring R, then ${}_R(R/K)$ is flat if and only if every injective right R/K-module is injective as an R-module. A more formal generalization of this observation can be seen in the next result.

Proposition 1.2. ([8, Proposition 6.17] *Let $\phi : R \longrightarrow S$ be a ring map, and let A be an injective right S-module. If ${}_RS$ is flat, then A is also injective as a right R-module.*

Based on the above results and discussion, we are ready to prove the next theorem.

Theorem 1.3. *Let $\{R_i\}_{i\in I}$ be an infinite family of rings (with identity). Then*

$$\frac{\prod R_i}{\oplus R_i},$$

is never left or right self-injective.

Proof. Let $R = \prod_{i\in I} R_i$, $J = \bigoplus R_i$, and $S = R/J$. For each $i \in I$, let $e_i = (a_i) \in R$, where $a_j = 1$, if $j = i$ and $a_j = 0$, if $j \neq i$. Then clearly $R_i = e_iR$, for every $i \in I$, $e_ie_j = 0$, when $i \neq j$ and $\oplus R_i = \oplus e_iR$. For each $A \subseteq I$, we put χ_A, then clearly $e_i\chi_A = e_i$, for all $i \in A$ and $e_i\chi_A = 0$, for every $i \in I \setminus A$. We also note that if $\pi : R \longrightarrow \prod e_iR$ is the natural map, whith $\pi(x) = (e_ix)$, then $\ker \pi = (0)$. Now in view of Theorem 1.1, S is not injective as an R-module. On the other hand, S is a flat R-module (on either side), so by Proposition 1.2, any injective S-module is also injective as an R-module. But we have already observed that S is not injective as an R-module, so S cannot be self-injective on either side. □

Corollary 1.4. *Let $\{R_i\}_{i\in I}$ be an infinite family of regular rings, then $\frac{\prod R_i}{\oplus R_i}$ is neither a left nor a right self-injective ring.*

Ironically, Handelman (see [8, pp. 385–386]) has observed that, if R is a regular ring, then the ring $R^\omega/R^{(\omega)}$ is left and right $\aleph_0$-self-injective, where $\omega = \{0, 1, 2, \cdots\}$. Even a more general version of this observation can be stated.

Proposition 1.5. *Let $\{R_i\}_{i\in I}$ be an infinite family of regular rings, then*

$$\frac{\prod R_i}{\oplus R_i}$$

is $\aleph_0$-self-injective on either side.

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A generalization of Handelman's observation has been given by the author as follows.

Proposition 1.6. *Let R_i be an infinite family of left coherent, left f-injective rings. Then $\prod R_i/\oplus R_i$ is a left $\aleph_0$-self-injective ring.*

Proof. See [20, Theorem 2.7]. □

In the light of Theorem 1.3 or Corollary 1.4, we know that $R^\omega/R^{(\omega)}$ is neither right nor left self-injective ring. Now, using a result in [8], perhaps, even a more interesting point can be stated. We need a proposition from [8].

Proposition 1.7. ([8, Proposition 9.31]) *Let R be a regular, right $\aleph_0$-self-injective ring, and let J be a two-sided ideal of R. If $\frac{R}{J}$ contains no uncountable direct sums of nonzero right ideals, then $\frac{R}{J}$ is a right self-injective ring.*

We also need a classic result due to Tarski [25] and Sierpinski [24], though, unfortunately, here, just like the case of Zorn and Kuratowski on the so-called Zorn's lemma, in the literature, only the name of Tarski is mentioned. Let X be an infinite set. The family $\mathcal{A} \subset P(X)$, is called an *almost disjoint* family, if every element of $\mathcal{A}$ is infinite and the intersection of any two distinct members is finite. If $|X| = \aleph_0$, then X has an almost disjoint family $\mathcal{A}$ with $|\mathcal{A}| = 2^{\aleph_0}$.

Corollary 1.8. *Let R be a regular ring. Then $R^\omega/R^{(\omega)}$ contains an uncountable direct sums of non-zero right (and also left) ideals. In particular, its Goldie dimension is equal to, or greater than $2^{\aleph_0}$.*

Proof. Let $\mathcal{A}$ be an almost disjoint set on ω. Now the set $\{\chi_A + R^{(\omega)}\}_{A\in\mathcal{A}}$ is a set of orthogonal idempotents which has $2^{\aleph_0}$ elements. This shows that the Goldie dimension of $R^\omega/R^{(\omega)}$ is equal to, or greater than $2^{\aleph_0}$. □

Example 1.9. *This is an outcome of a result due to Tyukavkin in [26]: let $\mathbb{F}$ be a countable field, $R_i = \mathrm{M}_i(\mathbb{F})$, and $R = \prod_{i\in\mathbb{N}} R_i$, then R/M where M denotes a maximal twosided ideal containing $J = \oplus_{i\in\mathbb{N}} R_i$, is a directly finite self-injective ring that is not a V-ring (see also the introduction of [12]). Comparing Tyukavkin's result with Corollary 1.4, we observe that though R/J is not self-injective, so is R/M .*

2. The second answer to the question of Matlis

The following Theorem was proved in [14, Theorem 2.2]. The countable version has been proved in [15, Proposition 1.2]. What follows is a mixture of these two. Based on this theorem, the second answer to the question of Matlis has been provided. In results that will be proved in the sequel, the following theorem will be used freely.

Theorem 2.1. ([14, Theorem 2.2] and [15, Proposition 1.2]) *Let R be a reduced ring, then the following statements are equivalent.*

(1) *R is a regular ring which is ($\aleph_0$-)linearly compact on principal right ideals;*

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(2) *R is a regular ring and whenever $S \cup T$ is (a countable orthogonal) an orthogonal set in R with $S \cap T = \emptyset$, then R has an element which separates S from T;*

(3) *R is (an $\aleph_0$-) a self-injective ring.*

In [13, Exercise 11.45], it has been asked: Let R be the ring $\mathbb{Z}_2^{\omega}$, and let I denote the ideal $\mathbb{Z}_2^{(\omega)}$. Then (i) R is self-injective, and hence algebraically compact; (ii) R/I is not algebraically compact. In the light of the above theorem, we know that the self-injectivity and algebraically compactness of R/I are the same.

As a first application of this theorem, we can give an independent proof for Handelman's observation. The proof is exactly the same as the proof Karamzadeh has already given for $\mathbb{K}/\mathbb{J}$. Hence we do not repeat it.

Proposition 2.2. *Let $\{R_i\}_{i \in I}$ be an infinite family of abelian regular rings. Then*

(1) *$\prod R_i / \oplus R_i$ is not self-injective;*

(2) *$\prod R_i / \oplus R_i$ is $\aleph_0$-self-injective.*

Proof. (1) See [14, pages 2724-2725].
(2) See [15, pages 1510-1511]. □

Let X be a set, and $\mathcal{A}$ a family of subsets of X which is closed under complementation and formation of countable unions, that is a σ-algebra. Let $\mathcal{M}(X, \mathcal{A})$ be the collection of all $\mathcal{A}$-measurable real-valued functions on X, i.e., those $f : X \longrightarrow \mathbb{R}$ such that for an interval I of any type, or just each open interval, $\{x \in X \mid f(x) \in I\}$ belongs to $\mathcal{A}$. If f and g are members of $\mathcal{M}(X, \mathcal{A})$, then $f + g$ and fg which are defined pointwisely, belong to $\mathcal{M}(X, \mathcal{A})$ (see [10], Page 81, Theorem C), hence $\mathcal{M}(X, \mathcal{A})$, with pointwise addition and multiplication is a commutative (reduced) ring. Now, Let X be a Hasudorff space, if $\mathcal{A}$ is the σ-algebra of all Borel measurable sets, then we call $\mathcal{M}(X, \mathcal{A})$, the ring of all Borel measurable functions. If $X = \mathbb{R}^n$, then we may speak of Lebesgue measurable sets. And in this case, each Borel set is a Lebesgue measurable set. When $X = \mathbb{R}^n$, that is, when X is a finite-dimensional Euclidean space, the cardinal number of all Borel measurable functions is $c = 2^{\aleph_0}$ (see [4], Page 96, Theorem 6. 2. 8]) and the cardinal number of all Lebesgue measurable functions is 2^c. By $C(X)$, we mean the ring of all real valued continuous functions, see [7], for undefined terms and definitions, also by $\mathbb{R}^X$ we mean the ring (again by pointwise addition and multiplication) of all real-valued functions on X. Since every Borel set is generated by open sets of X, and the inverse image of any open set under a continuous function remains open, hence every continuous function is a Borel measurable function. In the special case, $X = \mathbb{R}^n$, since the class of all Borel measurable sets is a subclass of Lebesgue measurable sets, every continuous function is a Lebesgue measurable function as well. Therefore the following inclusions hold whenever $\mathcal{A}$ is either the σ-ring of all Borel measurable sets (when X is an arbitrary topological space) or the σ-ring of all Lebesgue measurable functions (when X is a finite-dimensional Euclidean space):

$$C(X) \subset \mathcal{M}(X, \mathcal{A}) \subset \mathbb{R}^X$$

But in general , i.e., when X is a topological space and $\mathcal{A}$ is an arbitrary σ-algebra, we know that both of $C(X)$ and $\mathcal{M}(X,\mathcal{A})$ are subrings of $\mathbb{R}^X$. In [9], it has been shown that $\mathcal{M}(X,\mathcal{A})$ is von Neumann regular, but we give a proof for completeness (see Lemma 3.1). By $Z(f)$ we mean $\{x \in X \mid f(x) = 0\}$.

Let X be an infinite discrete topological space, then every function in $\mathbb{R}^X$ is continuous, i.e., $C(X)$, is the whole $\mathbb{R}^X$. On the other hand, $C_F(X)$, i.e., the socle of $C(X)$ is nothing but $\mathbb{R}^{(X)}$. After giving answer to the question of Matlis, Karamzadeh got interested in the following generalization of the question: For a completely regular space X, when is $C(X)/C_F(X)$ self-injective or $\aleph_0$-self-injective? In the following lines, we first explain necessary concepts and symbols in the rings of functions (e.g., rings of continuous functions, $C(X)$, rings of measurable functions, $\mathcal{M}(X,\mathcal{A})$, and their factors).

3. When regularity and $\aleph_0$-self-injectivity come together

The fact that $\frac{\mathbb{K}}{\mathbb{J}}$, is $\aleph_0$-self-injective or Handelman's observation, gives us a strong motivation to scrutinize the interconnection between regularity, $\aleph_0$-injectivity and injectivity. In this section, we provide some "natural" sources of $\aleph_0$-self-injective regular rings. Those who are familiar with rings of (real-valued) functions know that in some important examples, regularity and $\aleph_0$-injectivity are two inseparable friends.

3.1. **Source 1: Rings of measurable functions.** Rings of measurable functions, $\mathcal{M}(X,\mathcal{A})$, are a natural source for regular $\aleph_0$-self-injective rings. We will see soon that the other source, i.e., $D(X)$ is in fact a factor of $\mathcal{M}(X,\mathcal{A})$. Since these sources are less well-known than rings of continuous rings we present them in full detail. The next lemma is well-known (see [9]) but we give proof for the sake of completeness.

Lemma 3.1. *$\mathcal{M}(X,\mathcal{A})$ is a von Neumnn regular ring.*

Proof. Let f be a measurable function. We find another measurable function g such that $fgf = f$. We define g as follows: on $Z(f)$ we define g to be zero and on $X \setminus Z(f)$ we define $g = \frac{1}{f}$. It is not difficult to see that g is measurable function and has the appropriate property. □

Now we show that $\mathcal{M}(X,\mathcal{A})$ is $\aleph_0$-self-injective. We need the following Lemma for proving our theorem:

Lemma 3.2 (Pasting Lemma for measurable functions)**.** *Let $\{A_i\}_{i=1}^{\infty}$ be a family of mutually disjoint measurable sets and $f : \bigcup A_i \longrightarrow \mathbb{R}$ be a function. If $f|_{A_i} = f_i$ is a measurable function, then f is measurable function.*

Proof. See [3]. □

Theorem 3.3. *$\mathcal{M}(X,\mathcal{A})$ is an $\aleph_0$-self-injective ring.*

Proof. Let $T \bigcup S \subseteq \mathcal{M}(X,\mathcal{A})$ be a countable orthogonal subset of $\mathcal{M}(X,\mathcal{A})$, with $T \bigcap S = \emptyset$, in the view of Theorem 2.1, we must separate T from S, that is, we must find a measurable function f such

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that $ft = t^2$ and $fs = 0$, where t and s are arbitrary elements of T and S respectively. Let $T = \{f_i\}_{i=1}^{\infty}$ and $S = \{g_i\}_{i=1}^{\infty}$ and

$$L = \bigcup_{i=1}^{\infty} \mathrm{Coz}(f_i),\ K = \bigcup_{i=1}^{\infty} \mathrm{Coz}(g_i)$$

Both L and K are measurable sets. Since $T \bigcup S$ is an orthogonal set, hence for every two distinct elements h_1, h_2 of $S \bigcup T$ we have $\mathrm{Coz}(h_1) \bigcap \mathrm{Coz}(h_2) = \emptyset$. Let $Y = L \bigcap K$, we define $f : X \longrightarrow \mathbb{R}$ by $f|_{Coz(f_i)} = f_i$ and $f|_{Coz(g_i)} = 0$ and in addition $f|_{X \setminus Y} = 0$, then f is a well-defined map. By Pasting Lemma (the above lemma) f is a measurable function, i.e., $f \in \mathcal{M}(X, \mathcal{A})$. And f has the desirable property, i.e, $ff_i = f_i^2$ and $fg_i = 0$. □

3.2. **Source 2:** $D(X)$**.** In [9], $D(X)$ has been introduced as the set of continuous functions f on topological space X to the two point compactification of the reals, $\mathbb{R} \bigcup \{+\infty, -\infty\}$, which are real valued on a dense subset X_f of X.

By a representation theorem due to Henriksen and Johnson, every Archimedean reduced F-ring is embedded in a $D(X)$. In general, under pointwise addition and multiplication, $D(X)$ is not a ring. However, $D(X)$ is an algebra, if and only if X is a quasi F-space. A completely regular space X is called a quasi F-space, if every dense co-zero set Y in X is C^*-embedded. Since $D(X) \cong D(\beta X)$, without loss of generality, we may suppose that X is a compact space. So when X is a compact space, then $D(X)$ becomes a ring with pointwise addition and multiplication. Let $\mathcal{N}$ be a subfamily of $\mathcal{A}$, with $E \in \mathcal{A}$, $F \in \mathcal{N}$, and $E \subseteq F$, implying $E \in \mathcal{N}$ and closed under countable unions - a σ-ideal. The members of $\mathcal{N}$ is called the *null sets*, and accordingly, $N = \{f \in \mathcal{M}(X, \mathcal{A}) \mid \mathrm{Coz}(f) \in \mathcal{N}\}$ is said to be the *ideal of null functions*. In [9, Theorem 2.1], it has been shown that when X is a basically disconnected compact space, then $D(X)$ is isomorphic to a ring of measurable functions modulo N.

In [8, Proposition 9.31], it has been shown that every factor ring of an $\aleph_0$-self-injective regular ring is also $\aleph_0$-self-injective regular. Therefore we have:

Corollary 3.4. *Let I be an ideal of $\mathcal{M}(X, \mathcal{A})$, then $\frac{\mathcal{M}(X,\mathcal{A})}{I}$ is also an $\aleph_0$-self-injective regular ring.*

Proof. By Theorem 3.3, we have already shown that $\mathcal{M}(X, \mathcal{A})$ is an $\aleph_0$-self-injective regular ring. Now by [8, Proposition 9.31], we are thorough. □

Based on the above concepts and results we will have:

Corollary 3.5. *Let X be a basically disconnected compact space. Then $D(X)$ is an $\aleph_0$-self-injective ring.*

Proof. By [9, Theorem 2.1], $D(X) \cong \frac{\mathcal{M}(X,\mathcal{A})}{N}$, where N is the ideal of null functions. By Corollary 3.4, $D(X)$ is an $\aleph_0$-self-injective regular ring. □

3.3. **Source 3:** $C(X)$**.** To answer the question, when is $C(X)/C_F(X)$ $\aleph_0$-self-injective, Karamzadeh (joint with Estaji) in [6], proved the following nice observation.

Theorem 3.6. ([6, Theorem 1] *The following are equivalent:*

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(1) *X is a P-space;*
(2) *$C(X)$ is $\aleph_0$-selfinjective;*
(3) *$C(X)/C_F(X)$ is $\aleph_0$-selfinjective.*

Again here, like $\mathcal{M}(X,\mathcal{A})$ and $D(X)$, we are confronted with the same phenomena, that regularity and $\aleph_0$-self-injectivity come together.

4. Where regularity and $\aleph_0$-self-injectivity do not come together

Though, in Section 4, we have already observed that, there are instances in which regularity and $\aleph_0$-self-injectivity come together, but there are instances that they are quite independent concepts. We have already dealt with rings $\prod_{p\in\mathbb{P}}\mathbb{Z}_p$ and $\frac{\prod_{p\in\mathbb{P}}\mathbb{Z}_p}{\oplus_{p\in\mathbb{P}}\mathbb{Z}_p}$. It is interesting that our source of examples comes from subrings of $\prod_{p\in\mathbb{P}}\mathbb{Z}_p$ containing $\oplus_{p\in\mathbb{P}}\mathbb{Z}_p$. Before giving our example, we need a lemma whose verification is immediate. In this section by $\mathbb{P}$, we always mean the set of all prime (natural) numbers.

Lemma 4.1. *Let n be a natural number, then $\mathbb{Z}^n$ is contained in $\prod_{p\in\mathbb{P}}\mathbb{Z}_p$ as a subring.*

The next definition and next proposition are a very special case of a more general concept and theorem, introduced in [1] and [2]. Since it has not been published yet, we give proof here. In the following, we say that R is a pure subring of S if the additive group of R is a pure subgroup of the additive subgroup of S. We say that a subgroup H of a group G is pure if for every $n\in\mathbb{N}$, $nH = H\cap nG$.

Definition 4.2. *Let $\mathbb{M}_n := \{x\in\prod_{p\in\mathbb{P}}\mathbb{Z}_p \mid \exists k\in\mathbb{N}$ such that $kx\in\mathbb{Z}^n\}$.*

Proposition 4.3. *For every $n\in\mathbb{N}$, the following holds:*

(1) *$\mathbb{M}_n$ is a subring of $\prod_{p\in\mathbb{P}}\mathbb{Z}_p$, containing $\oplus_{p\in\mathbb{P}}\mathbb{Z}_p\oplus\mathbb{Z}^n$;*
(2) *$\frac{\mathbb{M}_n}{\oplus_{p\in\mathbb{P}}\mathbb{Z}_p}$ is (ring) isomorphic with $\mathbb{Q}^n$;*
(3) *$\mathbb{M}_n$ is a regular ring;*
(4) *$\mathbb{M}_n$ is not $\aleph_0$-self-injective ring.*

Proof. (1): Let $x,y\in\mathbb{M}_n$, we know that there exist $k,l\in\mathbb{N}$ such that $kx=s$ and $ly=t$, where $s,t\in\mathbb{Z}^n$. We observe that $kl(xy)=(kx)(ly)=st\in\mathbb{Z}^n$. On the other hand $kl(x+y)=l(kx)+k(ly)=ls+kt$ which belong to $\mathbb{Z}^n$ as well. This shows that $\mathbb{M}_n$ is a ring.
(2) Define $\phi:\mathbb{M}_n\longrightarrow\mathbb{Z}^n\otimes\mathbb{Q}$ with $\phi(x)=t\otimes\frac{1}{k}$, where $kx=a+t\in(\oplus_{p\in\mathbb{P}}\mathbb{Z}_p)+\mathbb{Z}^n$. First of all, we show that ϕ is well-defined. Suppose that for $x\in\mathbb{M}_n$, there are $k,k'\in\mathbb{N}$ such $kx=a+t$ and $k'x=a'+t'$. Note that $k'kx=k'a+k't=ka'+kt'$, which implies that $(k'a+k't)\otimes\frac{1}{kk'}=(ka'+kt')\otimes\frac{1}{kk'}$. That is $k'a\otimes\frac{1}{kk'}+k't\otimes\frac{1}{kk'}=ka'\otimes\frac{1}{kk'}+kt'\otimes\frac{1}{kk'}$. But $ka\otimes\frac{1}{kk'}=0=ka'\otimes\frac{1}{kk'}$. Hence $t\otimes\frac{1}{k}=t'\otimes\frac{1}{k'}$, i.e., ϕ is well-defined.
Now we show that ϕ is a ring homomorphism. To show that $\phi(x+y)=\phi(x)+\phi(y)$, suppose that for $x,y\in\mathbb{M}_n$, there are $k,l\in\mathbb{N}$ such that $kx=a+s$ and $ly=b+t$. Now consider $lkx=la+ls$ and $kly=kb+kt$, we have $kl(x+y)=la+ls+kb+kt$, this implies that $\phi(x+y)=ls+kt\otimes\frac{1}{kl}$, but

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$ls\otimes\frac{1}{kl}+kt\otimes\frac{1}{kl}=s\otimes\frac{1}{k}+t\otimes\frac{1}{l}=\phi(x)+\phi(y)$. Now, since $klxy=(kx)(ly)=(a+s)(b+t)=ab+at+sb+st$, where $ab+at+sb\in\oplus_{p\in\mathbb{P}}\mathbb{Z}_p$, we have $\phi(xy)=st\otimes\frac{1}{kl}=(s\otimes\frac{1}{k})(t\otimes\frac{1}{l})=\phi(x)\phi(y)$. Furthermore, we show that ϕ is onto. Without loss of generality, we may suppose that $s\otimes\frac{1}{k}\in\mathbb{Z}^n\otimes\mathbb{Q}$, now the equation $k(x+\oplus_{p\in\mathbb{P}}\mathbb{Z}_p)=s+\oplus_{p\in\mathbb{P}}\mathbb{Z}_p$ is solvable, due to $\prod_{p\in\mathbb{P}}\mathbb{Z}_p/\oplus_{p\in\mathbb{P}}\mathbb{Z}_p$ being divisible. Hence, there exists $x\in\prod_{p\in\mathbb{P}}\mathbb{Z}_p$ such that $kx=s+a$, where $a\in\oplus_{p\in\mathbb{P}}\mathbb{Z}_p$, therefore $x\in\mathbb{M}_n$ and $\phi(x)=s\otimes\frac{1}{k}$. Now we show that $\ker\phi=\oplus_{p\in\mathbb{P}}\mathbb{Z}_p$. To observe this, recall first that $\mathbb{Z}^n\otimes\mathbb{Q}\cong S^{-1}\mathbb{Z}^n$, where $S=\mathbb{Z}\setminus\{0\}$. Now if $t\otimes\frac{1}{k}=0$ if and only if there exists $n\in\mathbb{Z}\setminus\{0\}\}$ such that $nt=0$, i.e., $t\in\oplus_{p\in\mathbb{P}}\mathbb{Z}_p$.
(3) Since $\oplus_{p\in\mathbb{P}}\mathbb{Z}_p$ is regular, and also $\mathbb{Z}^n\otimes\mathbb{Q}\cong\mathbb{Q}^n$ is a regular ring, we conclude that $\mathbb{M}_n$ is regular (see [8]).
(4) Since $\mathbb{M}_n$ is regular and $\mathrm{Soc}(\mathbb{M}_n)=\oplus_{p\in\mathbb{P}}\mathbb{Z}_p$ is countably generated, if it were $\aleph_0$-self-injective, it would be self-injective, due to [20, Corollary 4.4], but this is not the case because the maximal quotient ring of $\mathbb{M}_n$ is $\prod_{p\in\mathbb{P}}\mathbb{Z}_p$. □

These $\mathbb{M}_n$, are regular but they are not $\aleph_0$-self-injective. It seems that to have an $\aleph_0$-self-injective ring which is not regular is a rarer occurrence. Of course, one can examine $\prod_{n=2}^{\infty}\mathbb{Z}_n$ or the product of any family of non-regular self-injective rings. If one is looking for an $\aleph_0$-self-injective ring which is neither regular nor self-injective, one may see [21]. Let $R=\mathbb{Z}(+)\mathbb{Q}/\mathbb{Z}$ be a ring with addition $(m,n)+(p,q)=(m+p,n+q)$ and multiplication $(m,n)(p,q)=(mp,mq+np)$. The ring R is called the (trivial) idealization of $\mathbb{Z}$ with respect to the $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z}$. The ring R is a commutative ring and the ideal $0(+)\mathbb{Q}/\mathbb{Z}$ is the Jacobson radical of R. Now $S=R^{\omega}/R^{(\omega)}$ is an $\aleph_0$-self-injective such that $S/Jac(S)\cong\mathbb{Z}^{\omega}/\mathbb{Z}^{(\omega)}$. Hence S is neither regular nor self-injective.

Proposition 4.4. *There is a commutative $\aleph_0$-self-injective ring S, such that $S/Jac(S)\cong\frac{\mathbb{Z}^{\omega}}{\mathbb{Z}^{(\omega)}}$. In particular, S is neither self-injective nor regular.*

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