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A SURVEY OF RECENT RESULTS CONNECTED WITH SUBNORMAL SUBGROUPS

M. R. DIXON*, M. FERRARA AND M. TROMBETTI

Dedicated to Derek J. S. Robinson, our friend and colleague, on the occasion of his 85th birthday.

Abstract. In this paper, we give a brief survey of some highlights from the theory of subnormal subgroups and then reveal some recent extensions of this theory due to the current authors and others.

1. Introduction

The theory of subnormal subgroups originates from the paper [69] of Wielandt. Since then numerous authors have contributed results on this topic, many of which are included in the excellent book [42] of Lennox and Stonehewer and in the excellent survey of Casolo [10]. The aim of the current survey paper is to give a very brief snapshot of some of the highlights of the theory of subnormal subgroups and use this as a springboard to discuss some generalizations and recent results. Our notation is generally standard and can be found in the wonderful books of Robinson [57,59], when not explained.

2. Some topics in subnormal subgroups

A subgroup H of a group G is called *subnormal* in G, if there is a natural number n and a finite series

$$H = H_0 \triangleleft H_1 \triangleleft \ldots \triangleleft H_n = G.$$

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∗Corresponding author

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The length of the shortest such series is called the *defect* of H in G and it is well-known that the *normal closure series* defined by $K_1 = H^G$ (the normal closure of H in G) and $K_{i+1} = H^{K_i}$ for each $i \geq 1$, constitutes such a shortest series for a subnormal subgroup H of G.

More generally, the subgroup H is called

- *ascendant* in G, if there is an ordinal γ and a series
$$H = H_0 \triangleleft H_1 \triangleleft \dots H_\alpha \triangleleft H_{\alpha+1} \triangleleft \dots H_\gamma = G;$$
- *descendant* in G, if there is an ordinal γ and a series
$$H = H_\gamma \dots \triangleleft H_{\alpha+1} \triangleleft H_\alpha \dots \triangleleft H_1 \triangleleft H_0 = G;$$
- *serial* in G, if there is a chain of subgroups between X and G such that, whenever $H < K$ are consecutive terms of such chain, then $H \triangleleft K$ (a more descriptive definition of serial subgroups has been given by Hall and can be for instance be found in [57, Section 1.2])

2.1. **The join problem.** Since the join of two normal subgroups is again normal, a natural question arises as to whether the join of two subnormal subgroups is also subnormal and it is this question that provided the initial impetus for much of the subsequent research. The most basic result on the join problem concerns *permutable subgroups*, that is subgroups H and K of a group such that $HK = KH$. It is easy to prove, using the properties of the normal closure series, that the join of two permuting subnormal subgroups H and K of an arbitrary group is itself subnormal. In particular, this is true if one of these subgroups normalizes the other. The following result, which is due to Wielandt [69], is not so easy to prove.

Theorem 2.1. *Let G be a finite group, and let H, K be subnormal subgroups of G. Then $\langle H, K\rangle$ is also subnormal in G.*

Wielandt's result is much more general than this. In fact, it deals with arbitrary groups and the so-called *composition subgroups*, that is subgroups X having a series
$$X = X_0 \triangleleft X_1 \triangleleft \dots \triangleleft X_n = G$$
such that X_{i+1}/X_i is simple for every i. Of course, every composition subgroup is subnormal, and the converse also holds for finite groups.

Theorem 2.2. *Let G be a group, and let H, K be composition subgroups of G. Then $\langle H, K\rangle$ is also a composition subgroup of G. In particular, $\langle H, K\rangle$ is subnormal in G.*

It follows from Theorem 2.2 that if G satisfies both the maximal and minimal conditions on subgroups, then the join of every pair of subnormal subgroups of G is still subnormal. Recall that a group G is said to satisfy the *maximal* (respectively*, minimal*) *condition on subnormal subgroups, max-sn (resp. min-sn)* if every ascending (resp. descending) chain of subnormal subgroups terminates in finitely many steps, or equivalently, every non-empty collection of subnormal subgroups has a maximal (resp. minimal) element— the maximal and minimal conditions on other sets of subgroups are similarly defined. Evidently, Wielandt's method is sufficient to prove substantially more as follows.

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Theorem 2.3. *Let G be a group.*

(i) *If G satisfies the maximal condition on subnormal subgroups and if H, K are subnormal subgroups of G, then $\langle H, K\rangle$ is subnormal in G;*

(ii) *If H, K are finite subnormal subgroups of G, then $\langle H, K\rangle$ is finite and subnormal in G;*

Note that we can replace the maximal condition on subnormal subgroups with the minimal one in point (i) of Theorem 2.3, and that actually max-sn can be replaced by some more general condition, implying both max-sn and min-sn (see [42, Theorem 3.5.1]).

The first example of a group in which the join of two subnormal subgroups is not subnormal was given by Zassenhaus in his book [74], and since then numerous examples have been constructed showing that this or that property is not sufficient for the join of two subnormal subgroups to be subnormal. Roseblade and Stonehewer gave an infinite family of examples in [64] as follows.

Proposition 2.4. *Let S be a non-zero commutative ring with identity 1, A a free S-module of infinite dimension, and let C be the additive subgroup of S generated by 1. Then there is a group G with subnormal subgroups $H \cong A$ and $K \cong C$, such that $J = \langle H, K\rangle \neq G$ and J is self-normalizing in G.*

At this point, it is appropriate to mention the contributions of Robinson to this theory. In [60] he proved the following results which give a number of cases, that show the join of subnormal subgroups is subnormal.

Theorem 2.5. *Let G be a group and let H, K be subnormal subgroups of G.*

(i) *If G is nilpotent-by-abelian, then $\langle H, K\rangle$ is subnormal in G;*

(ii) *If $[H, K]$ has the maximal condition, then $\langle H, K\rangle$ is subnormal in G;*

(iii) *If G' satisfies the maximal condition on subnormal subgroups, then $\langle H, K\rangle$ is subnormal in G;*

(iv) *If $\langle H, K\rangle'$ satisfies the maximal condition on subnormal subgroups, then $\langle H, K\rangle$ is subnormal in G;*

(v) *If G contains a normal subgroup N such that G/N has the maximal condition and if the join of every pair of subnormal subgroups of N is subnormal in N, then $\langle H, K\rangle$ is subnormal in G.*

We note that part (v) of Theorem 2.5 has been extended in [66] to include the minimal condition on subnormal subgroups.

In this paper, Robinson also discussed groups where the join of an arbitrary collection of subnormal subgroups is subnormal, but we omit the details (some of these results have been extended by Smith [66]).

This problem concerning the join of two subnormal subgroups leads naturally to the idea of a subnormal coalition (or coalescence) class. A class of groups $\mathfrak{X}$ is a *subnormal coalition* (or *coalescence*) *class* if the join of two (so finitely many) subnormal $\mathfrak{X}$-subgroups of any group G is always a subnormal $\mathfrak{X}$-subgroup of G. For example, it follows from Theorem 2.3 that the class of finite groups is a subnormal coalition class. A result of Baer [2, Satz 3] shows that the class of finitely generated nilpotent groups is a subnormal coalition class, although an example of Hall (see [54]) shows that the class of all nilpotent groups is not a subnormal coalition class.

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In the papers [54, 61] these results are complemented as follows.

Theorem 2.6. *Let $\mathfrak{X}$ be a subclass of the class of groups satisfying the minimal condition on subnormal subgroups and suppose that every product of two normal $\mathfrak{X}$-subgroups is an $\mathfrak{X}$-group. Then in any group, the join of two subnormal $\mathfrak{X}$-subgroups is a subnormal $\mathfrak{X}$-subgroup.*

Robinson also proves a similar result for the join of two ascendant $\mathfrak{X}$-subgroups, with suitable conditions imposed on $\mathfrak{X}$.

Roseblade and Stonehewer gave many examples of such subnormal coalition classes, see [64]. The highlights of this theory of subnormal coalescence perhaps lie in a theorem due to Lennox and Stonehewer [42, Theorem 3.4.1], which is general enough to handle many of the known results on coalescence. For the sake of simplicity, we do not give the full result, but we only state a generalization of the "join part" due to Williams [72], whose proof uses properties of group rings. We recall that a group G has *finite rank* r, if r is the least natural number such that every finitely generated subgroup of G is at most r-generator; we say that G has infinite rank, if there is no such number r. We denote the ith term of the lower central series of G by $\gamma_i(G)$. The result mentioned above is as follows.

Theorem 2.7. *Let $n, d_1, d_2, \ldots, d_n, r$ be natural numbers. Then there is a function $f = f(d_1, \ldots, d_n, r)$ satisfying the following property: if G is a group such that $J'/\gamma_3(J)$ has finite rank r, and $H_1, H_2, \ldots, H_n$ are subnormal subgroups of G of defects $d_1, d_2, \ldots, d_n$, respectively, then $J = \langle H_1, H_2, \ldots, H_n\rangle$ is subnormal of defect at most f.*

Recall that a group is *f.r.d.* if it can be expressed as the direct product of a group of finite rank and a periodic divisible abelian group. Also, if A and B are abelian groups, then $A \otimes B$ denotes the tensor product of A and B over the integers.

We pay attention to the following necessary and sufficient conditions given by Williams [71]

Theorem 2.8. *Let H and K be groups.*

(i) *If $H/H' \otimes K/K'$ is f.r.d. and H, K are subnormal subgroups of a group G, then $\langle H, K\rangle$ is also subnormal in G;*

(ii) *Conversely, if $H/H' \otimes K/K'$ is not f.r.d, then there is a group G containing H, K as subnormal subgroups, but in which $\langle H, K\rangle$ is not subnormal.*

To close this subsection, we remark that although the join of subnormal subgroups may not be subnormal, it will always be serial (see [37]). In this context, note also that the join of serial subgroups may not be serial in general, but this is the case if we restrict our attention to the universe of locally finite groups (see [32]).

2.2. **The Wielandt subgroup and T-groups.** In [70], Wielandt introduced what is now known as the Wielandt subgroup of a group and obtained many of its properties.

The *Wielandt subgroup* of a group G is defined as :

$$\omega(G) = \bigcap\{N_G(H) \mid H \text{ is a subnormal subgroup of } G\}.$$

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It is an important result of Wielandt, [70], that if H is a non-abelian simple subnormal subgroup of a group G and if K is any subnormal subgroup for which $H \cap K = 1$, then $[H, K] = 1$. In particular, this implies that H normalizes every subnormal subgroup of G since if $H \cap K \neq 1$, then $H \leq K$. Thus, $\omega(G)$ contains every non-abelian simple subnormal subgroup of G; in particular, $\omega(G) \neq 1$ for every finite group G having a non-abelian minimal subnormal subgroup. Furthermore, if $\zeta(G)$ denotes the center of a group G, then $\zeta(G) \leq \omega(G)$; hence the Wielandt subgroup of a nontrivial nilpotent group is always nontrivial.

Certainly, the Wielandt subgroup can be trivial, as the example of the infinite dihedral group shows, or it can be the whole group, as the consideration of the symmetric group S_3 on 3 elements shows.

Wielandt also proved the following result.

Theorem 2.9. *Let N be a minimal normal subgroup of a group G and suppose that N satisfies the minimal condition on normal subgroups. Then $N \leq \omega(G)$.*

Besides the initial paper of Wielandt, the Wielandt subgroup has been studied in various papers including [5, 8] and [15]. The following result was established by Robinson in [54] and Roseblade in [61].

Theorem 2.10. *Let G be a group satisfying the minimal condition on subnormal subgroups. Then $|G : \omega(G)|$ is finite.*

It is evident that if M is a normal subgroup of K and K is a normal subgroup of $\omega(G)$, then M is subnormal in G and hence M is normal in $\omega(G)$. Thus in $\omega(G)$ normality is a transitive relation. A group G in which normality is a transitive relation is called a T-*group*. Much of the theory of finite T-groups was noted in [4, 31] and [73]. The influence of Derek Robinson in this area is amply illustrated in [53, 55, 56] and [58]. The following result is fundamental in the theory of T-groups.

Proposition 2.11. *Let G be a* T*-group.*

(i) *If G is soluble, then G is metabelian.*

(ii) *If G is a finitely generated soluble group, then G is finite or abelian.*

In fact, the structure of soluble T-groups can be much more precisely formulated and we refer the reader to [53] for more detailed information than we give here.

It was shown by Gaschütz [31, Satz 4] that finite soluble T-groups are precisely the finite T-groups in which the subgroups also inherit the property T, the class of so-called $\bar{\mathrm{T}}$-*groups*. This is no longer true in the infinite case, since Robinson [53] gives an example of an infinite soluble T-group which is not a $\bar{\mathrm{T}}$-group. However, he does give a precise idea of the structure of soluble $\bar{\mathrm{T}}$-groups.

In [18], groups all of whose infinite subnormal subgroups are normal (so-called *IT-groups*) were studied. Subsequent research on this topic was carried out by Heineken in [33] and the following result was obtained.

Proposition 2.12. *Let G be an infinite* IT*-group.*

(i) *If G is an infinite soluble group, then G has a normal* T*-subgroup of finite index;*

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(ii) *If G is an infinite soluble group, then G has a finite normal subgroup U such that G/U is a* T-*group.*

The authors of [3] determined, by studying [18] and [33] that an interesting subgroup to study would be the *Generalized Wielandt subgroup* $\omega_i(G)$ of a group G defined by

$$\omega_i(G) = \bigcap\{N_G(H) \,|\, H \text{ is an infinite subnormal subgroup of } G\}.$$

It is clear that $\omega(G) \leq \omega_i(G)$ and that $\omega_i(G) = G$ if G has no infinite subnormal subgroups. Furthermore, if H is an infinite subnormal subgroup of a group G, then $H \triangleleft \omega_i(G)$, so that $\omega_i(G)$ is an IT-group. Thus, G is an IT-group if and only if $G = \omega_i(G)$.

The main results concerning $\omega_i(G)$ were obtained in [3]. An important role is played by the subgroup

$$S(G) = \langle T : T \text{ is a finite soluble subnormal subgroup of } G\rangle.$$

Theorem 2.13. *Let G be a group. Then*

(i) *$\omega_i(G)/\omega(G)$ is residually finite;*
(ii) *If G is infinite, then $\omega_i(G)/\omega(G)$ is a* T-*group;*
(iii) *If $S(G)$ is not Prüfer-by-finite, then $\omega_i(G) = \omega(G)$;*
(iv) *If $S(G)$ is an infinite Prüfer-by-finite group, then $\omega_i(G)/\omega(G)$ is metabelian;*
(v) *If $S(G)$ is finite, then $\omega_i(G)/\omega(G)$ is finite.*

It follows for example from the above result that $\omega_i(G) = \omega(G)$ whenever G is an infinite residually finite group.

2.3. **Groups with all subgroups subnormal.** The structure of all groups whose subgroups are normal was obtained by Dedekind in [20] and Baer in [1]. The much more difficult problem of determining the structure of groups all of whose subgroups are subnormal takes more effort. A natural place to start is the class of finite groups, but here it is an easy graduate exercise to prove that if a group is finite and all subgroups are subnormal, then the group is nilpotent. The next phase in this program was achieved by Roseblade [62] where the following result was obtained, stated in its simplest form.

Theorem 2.14. *Let d be a fixed natural number. Then there is a natural number $c = c(d)$ depending only on d such that every group whose subgroups are subnormal of defect at most d is nilpotent of nilpotency class at most c.*

The hopes (or fears) of the group-theoretical community that perhaps every group with all subgroups subnormalmight be close to being nilpotent, were dashed to an extent by the publication of [36] where Heineken and Mohamed gave an example of a group G in which $\zeta(G)$ is trivial, but every subgroup of G is (nilpotent and) subnormal. Subsequently, Meldrum in [43] and Heineken and Mohamed in [35] showed how to construct uncountably many non-hypercentral groups with every subgroup (nilpotent and) subnormal and further examples were given by Menegazzo in [44], that showed the existence of Heineken–Mohamed type groups in which the derived subgroup has infinite exponent or in which

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the derived length is an arbitrary integer. By this time, the brilliant papers of Möhres [45–49] had appeared, the culmination of this research being the following important result.

Theorem 2.15. *Let G be a group and suppose that every subgroup of G is subnormal in G. Then G is soluble.*

At this point, it seems appropriate to quote some of the results obtained by Smith and Casolo in the context of groups with all subgroups subnormal. In particular, Smith [67] (and independently Casolo [9]) dealt with the torsion-free case, proving the following result.

Theorem 2.16. *Let G be a torsion-free group and suppose that every subgroup of G is subnormal in G. Then G is nilpotent.*

Note that it seems to be an open question if in the torsion-free case the bound on the nilpotency class given in Theorem 2.14 can be assumed to be d. The conjecture has been verified up to the case $d = 4$ (see [68]).

Robinson [57] asked if every group with all subgroups subnormal is a *Fitting group*, that is a group generated by nilpotent normal subgroups. Also, Smith [66] asked if a group with all subgroups subnormal is nilpotent provided that it is generated by finitely many nilpotent subgroups. Casolo (see [10]) positively answered these questions by proving much more than one could have expected.

Theorem 2.17. *Let G be a group all of whose subgroups are subnormal. If X is a nilpotent subgroup of G, then its normal closure X^G is nilpotent as well.*

Furthermore, Casolo described the structure of periodic groups in which all subgroups are subnormal (see [10]) by proving that every such group is an extension of a nilpotent group by a divisible abelian group of finite rank. It seems to be an open question if the periodicity assumption can be dropped — Casolo claimed this is possible if one replaces nilpotency with hypercentrality, but this result seems to be unpublished.

We end this subsection by briefly describing some collateral issues. First, groups whose subgroups are ascendant (respectively, descendant or serial) have been somewhat investigated and we refer to [10] and [57] for an account of the known results in these areas.

Secondly, it turns out that if you have a (soluble) group which is large in some sense, then requiring that all ‘‘large’’ subgroups are subnormal may have a huge impact on the structure of the group itself, sometimes even implying that all subgroups are subnormal. For example, if large means ‘‘infinite rank’’, then we have the following result of Evans and Kim [25], which we state in a weaker form.

Theorem 2.18. *Let G be a soluble group of infinite rank and let d be a positive integer. Suppose that every subgroup of infinite rank is subnormal of defect at most d. Then G is nilpotent of class bounded by a function of d.*

Moreover, if $d = 1$, then G is actually Dedekind.

Other results on the subjects can be found for example in [39] and [40]. Large can also mean ‘‘uncountable cardinality’’, and in this case we refer to [29] and its reference list for results similar to that of Evans and Kim.

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3. The theory of f-subnormal subgroups

The previous section has been concerned with subnormal subgroups and applications, together with very brief forays into notions of ascendancy, descendancy, and seriality. Such work is quite classical. In this and the following sections, we generalize this concept of subnormality in several ways, describing work over the past few years, although the concept itself is quite old and has been flirted with in the past.

A subgroup H of a group G is *f-ascendant*, if there is a (complete) series of subgroups

$$H = G_0 \leq G_1 \leq \dots G_\alpha \leq G_{\alpha+1} \leq \dots G_\lambda = G \tag{3.1}$$

with $G_\alpha \triangleleft G_{\alpha+1}$ or $|G_{\alpha+1} : G_\alpha| < \infty$, a definition due to Phillips [52]. Phillips showed, for example, that if H, K are f-ascendant subgroups of a group G and if H, K are locally (nilpotent-by-finite), then $\langle H, K\rangle$ is also locally (nilpotent-by-finite).

In equation (3.1) when λ is a finite ordinal we say that H is *f-subnormal* in G. In this case we call the chain of subgroups (3.1) an *f-subnormal series of length λ for H in G*. When $\lambda = 1$ we say that H is *f-normal* in G. Clearly subnormal subgroups and subgroups of finite index in G are f-subnormal in G. In particular, all subgroups of a finite group are f-subnormal in the group, so f-subnormal subgroups need not be subnormal, although it was shown in [12] that for locally nilpotent groups f-subnormality is equivalent to subnormality.

3.1. Chain conditions and f-subnormal subgroups.

In Section 2.1 we introduced the maximal and minimal conditions on sets of subgroups of a group, and in particular on the set of subnormal subgroups of a group. Here, besides max-sn and min-sn, we will also need to deal with *max-$f$sn* and *min-$f$sn*, the maximal and minimal condition on f-subnormal subgroups, respectively.

The maximal and minimal conditions (on subnormal subgroups) have been generalized in many ways. For example, a group G is said to have the *weak minimal condition on subnormal subgroups* (denoted by *min-∞-sn*) if every descending chain of subnormal subgroups of G

$$H_0 \geq H_1 \geq H_2 \geq \dots$$

has the property that only finitely many of the indices $|H_i : H_{i+1}|$ are infinite; one similarly defines the *weak maximal condition on subnormal subgroups* (denoted by *max-∞-sn*). Some interesting papers devoted to the conditions min-∞-sn and max-∞-sn include [38, 51, 63].

The double chain conditions are also interesting here. A group G satisfies the *double chain condition on subnormal subgroups*, if for every chain of subnormal subgroups of the form

$$\dots \leq G_{-n} \leq \dots \leq G_{-1} \leq G_0 \leq G_1 \leq \dots \leq G_n \leq \dots \tag{3.2}$$

there exists an integer k such that $G_i = G_k$ for all $i \geq k$ or $G_i = G_k$ for all $i \leq k$. Furthermore, we say that G satisfies the *weak double chain condition on subnormal subgroups*, if in every chain of the form (3.2) where G_i is subnormal in G at most finitely many of the indices $|G_{i+1} : G_i|$ are infinite.

Clearly, all the above described chain conditions can be adapted to f-subnormal subgroups; in particular, the weak minimal (resp. maximal) condition on f-subnormal subgroups will be denoted by *min-∞-$f$sn* (resp. *max-∞-$f$sn*).

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In this section we first relate some of the results of [23] where interest was focused on the relationship between chain conditions on subnormal subgroups and corresponding chain conditions on f-subnormal subgroups.

Theorem 3.1. *Let G be a group. Then*

(i) *G satisfies min-fsn if and only if G satisfies min-sn;*
(ii) *G satisfies max-fsn if and only if G satisfies max-sn;*
(iii) *G satisfies min-∞-fsn if and only if G satisfies min-∞-sn;*
(iv) *G satisfies max-∞-fsn if and only if G satisfies max-∞-sn;*
(v) *G satisfies the double chain condition on subnormal subgroups if and only if G satisfies the double chain condition on f-subnormal subgroups;*
(vi) *G satisfies the weak double chain condition on subnormal subgroups if and only if G satisfies the weak double chain condition on f-subnormal subgroups.*

Various authors have also considered groups with the weak minimal or weak maximal condition on non-subnormal subgroups. It is easy to see that every chain condition on (non-subnormal)-subgroups implies the analogous condition for (non-f-subnormal)-subgroups but an easy example shows that in contrast to Theorem 3.1 the converse is always false. Indeed, let S_3 be the symmetric group of degree 3. Consider the direct sum $G = S_3 \times \operatorname*{Dr}_{i\in I} G_i$ where $I = \{5, 7, 11, 13, \dots\}$ and each G_i is the cyclic group of order i. Every subgroup of G is f-subnormal. If we now take a non-subnormal subgroup of S_3, which we denote by H, then $H \times K$ is not subnormal in G, whenever K is any subgroup of $\operatorname*{Dr}_{i\in I} G_i$. Therefore G has an infinite double chain of non-subnormal subgroups, which guarantees that G satisfies none of the chain conditions on non-subnormal subgroups of relevance to this discussion.

3.2. **Groups with all subgroups f-subnormal.** The notion of f-subnormal subgroup encompasses many other notions of subgroups, such as those of almost subnormal subgroup and nearly normal subgroup. A subgroup H of a group G is *almost subnormal* if

$$H \le H_n \le H_{n-1} \le \dots \le H_1 \le H_0 = G \tag{3.3}$$

where $H_1 = H^G, H_i = H^{G,i} = H^{H_{i-1}}$ and $|H_n : H| < \infty$, a definition originally due to Lennox [41]. More precisely, perhaps, let n, m be two fixed non-negative integers. If there is a subgroup H_0 containing H such that $|H_0 : H| \le n$ and H_0 is subnormal in G with subnormal defect at most m, then we say that H is (n, m)-*subnormal* in G. The pairs (n, m) are ordered lexicographically. With this ordering, if H is (n, m)-subnormal in G for some pair (n, m), then the least such pair is called the *near defect* of H in G (see [41]). Clearly if H is subnormal in G of defect at most d, then H is $(1, d)$-subnormal in G and every subgroup of a finite group G is $(|G|, 0)$-subnormal in G. This terminology is also used in [12, 13, 21]. In [26], $(n, 1)$-subnormal subgroups are called *nearly normal.* Some motivation comes from the paper [50] where Neumann proves the following result.

Theorem 3.2. *Let G be a group. All subgroups of G are nearly normal if and only if G is finite-by-abelian.*

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A subgroup H is *subnormal-by-finite* in G if H contains a subnormal subgroup S of G such that $|H : S| < \infty$. Clearly we may assume $S \triangleleft H$. If S is normal in G, then we say H is *normal-by-finite*. Infinite groups all of whose subgroups are normal-by-finite were studied in the papers [6, 16, 17] and groups all of whose subgroups are subnormal-by-finite were discussed in [11, 34]. Groups with all subgroups ascendant-by-finite (in an obvious sense) were studied in [7].

It is clear that every finite subgroup of a group is normal-by-finite, and it is not difficult to see that every f-subnormal subgroup is actually subnormal-by-finite (see [12]). However, as the example of the infinite dihedral group shows, not every subnormal-by-finite subgroup need be f-subnormal.

Seeking a common origin to Roseblade's theorem (Theorem 2.14), B. H. Neumann's theorem (Theorem 3.2) and some of the other results discussed above, Lennox [41] established the following theorem. Note that if G is a finite-by-nilpotent group, with N a finite normal subgroup such that G/N is nilpotent, then every subgroup of G is almost subnormal of near defect at most (n, m), where $n = |N|$ and m is the nilpotency class of G/N.

Theorem 3.3. *Let n, m be non-negative integers. Then there is a function μ depending only on n and m satisfying the following property: if G is a group whose finitely generated subgroups are almost subnormal of near defect at most (n, m), then $|\gamma_{\mu(n,m)}(G)| \leq n!$*

We refer the reader to [42] for a fuller discussion of this theorem. The bounds here cannot be omitted. The Heineken–Mohamed groups (see [36]) are not finite-by-nilpotent, even though every subgroup is subnormal. Furthermore, FC-*groups* (that is groups in which every element has finitely many conjugates) have the property that each finitely generated subgroup is of finite index in its normal closure, but there are FC-groups that are not finite-by-nilpotent (for example the direct product of finite nilpotent groups with increasing nilpotency classes). However, in some cases these bounds can be relaxed a bit. For example, combining the results from [12] and [21], we can state the following.

Theorem 3.4. *Let n be a non-negative integer. Then there is a function μ depending only on n satisfying the following property: if G is a periodic* (*resp. torsion-free*) *group in which every subgroup H is almost subnormal of near defect (n, m) for some m depending on H, then $\gamma_{\mu(n)}(G)$ is finite* (*resp. G is nilpotent of class at most $\mu(n)$*)*.*

Lennox also obtained the following result.

Theorem 3.5. *Let G be a finitely generated group. The following are equivalent.*

(i) *Every finitely generated subgroup of G is almost subnormal of bounded near defect;*
(ii) *G is finite-by-nilpotent;*
(iii) *Every finitely generated subgroup of G is f-subnormal.*

In the above theorem, the finite generation assumption is an essential one. In fact, let $G = \operatorname*{Dr}_{n\in\mathbb{N}} S_n$ be the direct product of the symmetric groups S_n of degree n; all finitely generated subgroups of G are almost subnormal, but G is not finite-by-nilpotent.

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Note also that one cannot add "subnormal-by-finite" to the above list, not even if we require the group to satisfy the maximal condition on subgroups. In fact, the infinite dihedral group has the maximal condition and every subgroup is subnormal-by-finite but not all subgroups are almost subnormal.

In the absence of the bounds in Theorem 3.3, Casolo and Mainardis [12] have shown that if G is a group in which every subgroup has finite index in a subnormal subgroup, then G is finite-by-soluble, thus generalizing Theorem 2.15. They obtain the following result first.

Theorem 3.6. *Let G be a group. The following are equivalent.*

(i) *Every subgroup of G is f-subnormal;*
(ii) *Every subgroup of G is almost subnormal;*
(iii) *Every subgroup H of G is contained in a subgroup K such that H is subnormal in K and $|G:K|<\infty$.*

Furthermore every subgroup is subnormal-by-finite.

In order to prove their main theorem the authors consider the subgroup

$$D(G)=\langle H^{\mathfrak{N}}\,|\,H \text{ is finitely generated subgroup of } G\rangle;$$

recall that if X is any group, then $X^{\mathfrak{N}}$ is the *nilpotent residual* of X, that is the intersection of all normal subgroups N of X such that X/N is nilpotent — the *finite residual* $X^{\mathfrak{F}}$ is similarly defined. The key to unlocking information concerning groups with all subgroups f-subnormal is the following result.

Theorem 3.7. *Let G be a group in which every subgroup is f-subnormal. Then:*

(i) *Every subgroup of $G/D(G)$ is subnormal;*
(ii) *$D(G)$ is periodic and is finite-by-nilpotent;*
(iii) *G is finite-by-soluble;*
(iv) *$D(G)\cap G^{\mathfrak{F}}\leq \zeta_\omega(G)=\bigcup_i\zeta_i(G)$;*
(v) *Every element of $G^{\mathfrak{F}}$ is right Engel in G.*

In particular, if G is torsion-free, then G is nilpotent.

Casolo and Mainardis stated the last part of their theorem by saying that in the torsion-free case, the group is hypercentral. They obtained this special case by using an old result of Möhres, but now we have Theorem 2.16 and we can deduce the group is nilpotent. It should also be noted that they provided an example of a group G in which every subgroup has finite index in a subnormal subgroup of defect at most 2, and such that $D(G)=G^{\mathfrak{N}}$ has not finite exponent.

As we noted at the end of Section 2.3, a common theme recently has concerned the study of groups which have restrictions on the subgroups of infinite rank (see for example [25,39]) and we next discuss how this relates to almost subnormal subgroups. We begin with the following result of De Falco, de Giovanni and Musella [26, Theorem B] which is an analogue of Theorem 3.2. Recall that a group is said to be *radical*, if it has an ascending normal series whose factors are locally nilpotent.

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Theorem 3.8. *Let G be a radical group in which every subgroup of infinite rank is nearly normal. Then either G has finite rank or G is finite-by-abelian.*

Of course for proving these kinds of result, a large class of groups would naturally be that of *locally graded* groups, that is groups in which every non-trivial finitely generated subgroup has a proper subgroup of finite index, so as to avoid *Tarski monsters* (i.e. infinite groups whose proper subgroups are cyclic of prime order) and similar pathologies. However, the structure of locally graded groups of finite rank is not known so we often have to restrict ourselves to some large well-behaved subclass of the class of locally graded groups. Let $\mathfrak{Y}_0$ denote the class of periodic locally graded groups. Let $\mathbf{L}, \mathbf{R}, \acute{\mathbf{P}}, \grave{\mathbf{P}}$ denote the usual closure operations (see the first chapter of [57] for the precise definitions) and for each ordinal α let

$$\mathfrak{Y}_{\alpha+1} = \mathbf{L}\mathfrak{Y}_\alpha \bigcup \mathbf{R}\mathfrak{Y}_\alpha \bigcup \acute{\mathbf{P}}\mathfrak{Y}_\alpha \bigcup \grave{\mathbf{P}}\mathfrak{Y}_\alpha,$$

and as usual let $\mathfrak{Y}_\gamma = \bigcup_{\beta<\gamma} \mathfrak{Y}_\beta$, for limit ordinals γ. Set $\mathfrak{X} = \bigcup_\gamma \mathfrak{Y}_\gamma$. The class $\mathfrak{X}$ was defined by Chernikov [14] who proved that every $\mathfrak{X}$-group of finite rank is almost locally soluble and we note that, apart from this result, little appears to be known concerning locally graded groups of finite rank in general. The main result of [22] exploits the properties of the class $\mathfrak{X}$ and generalizes Theorem 3.8.

Theorem 3.9. *Let $G \in \mathfrak{X}$ have infinite rank. Suppose all subgroups of G of infinite rank are almost subnormal of bounded near defect at most (n, m). Then G is finite-by-nilpotent.*

It seems to be an open problem if one can bound the class of the nilpotent piece in the conclusion of Theorem 3.9 in terms of (n, m), but the authors proved that this is the case at least when the group is periodic or locally nilpotent.

3.3. **The f-Wielandt subgroup.** We return again to the Wielandt subgroup and now define the *f-Wielandt subgroup* of G by

$$\overline{\omega}(G) = \bigcap\{N_G(H) \,|\, H \text{ is an } f\text{-subnormal subgroup of } G\}.$$

It is always the case that $\overline{\omega}(G) \leq \omega(G)$, but easy examples show that in general the two subgroups are different. Some basic information about the behaviour of $\overline{\omega}(G)$ is given by the following lemma (see [27]).

Lemma 3.10. *Let G be a group. Then:*

(i) $\bar{\omega}(G) = \omega(G)$, *if G is locally nilpotent;*
(ii) $\bar{\omega}(G) \leq \zeta_2(G)$, *if G is finite or G is nilpotent;*
(iii) $\bar{\omega}(G)$ *is a* T*-group;*
(iv) $H \cap \bar{\omega}(G) \leq \bar{\omega}(H)$ *for all f-subnormal subgroups H of G;*
(v) $\bar{\omega}(G)N/N \leq \bar{\omega}(G/N)$ *for all normal subgroups N of G.*

The following more interesting property has been proved in [23]. It should be compared with Theorem 2.10 and with Theorem 3.1.

Theorem 3.11. *Let G be a group satisfying the minimal condition on subnormal subgroups. Then $\bar{\omega}(G)$ has finite index in G.*

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An interesting consequence of Theorem 3.11 is that in a group satisfying min-sn every f-subnormal subgroup has finitely many conjugates.

Next we define the *generalized f-Wielandt subgroup*

$$\overline{\omega}_i(G) = \bigcap \{N_G(H) \mid H \text{ is an infinite } f\text{-subnormal subgroup of } G\},$$

as a natural generalization of $\omega_i(G)$ for f-subnormal subgroups. Clearly $\overline{\omega}(G) \leq \overline{\omega}_i(G) \leq \omega_i(G)$ and if G has no infinite f-subnormal subgroups, then $\overline{\omega}_i(G) = G$. The interaction between $\overline{\omega}(G)$ and $\omega_i(G)$ has been studied in [24], where the following analogue of Theorem 2.13 (iii) is proved. Recall that the *Baer radical* of a group is the the subgroup generated by all its cyclic subnormal subgroups.

Proposition 3.12. *Let G be a group satisfying $\overline{\omega}_i(G) \neq \overline{\omega}(G)$. Then the Baer radical of G is Prüfer-by-finite and hence is nilpotent.*

It turns out that the structure of $\overline{\omega}_i(G)/\overline{\omega}(G)$ is much more tightly controlled than with the corresponding factor $\omega_i(G)/\omega(G)$. We exhibit below [24, Lemma 3.1] which is an analogue of [3, Theorem 6]; here $V_f(G)$ denotes the subgroup generated by all the finite f-subnormal subgroups of G.

Lemma 3.13. *Let G be a group in which $V_f(G)$ is Baer-by-finite. Then*

(i) *$\overline{\omega}_i(G)/\overline{\omega}(G)$ is finite;*
(ii) *There exists a finite normal subgroup N of G such that every f-subnormal subgroup of $\overline{\omega}_i(G)/N$ is a normal subgroup.*

This together with a further auxiliary result enables us to obtain the following result from [24].

Theorem 3.14. *For all groups G the quotient group $\overline{\omega}_i(G)/\overline{\omega}(G)$ is residually finite. Furthermore, $\overline{\omega}_i(G)$ is either finite or $\overline{\omega}_i(G)/\overline{\omega}(G)$ is a Dedekind group.*

An adaptation of a construction due to Heineken [33] shows that all finite Dedekind groups can appear as the quotient $\overline{\omega}_i(G)/\overline{\omega}(G)$, but no example so far is known of a group in which $\overline{\omega}_i(G)/\overline{\omega}(G)$ is infinite.

When some form of generalized solubility is used the results become stronger. We recall that a group is *subsoluble*, if it has an ascending subnormal series whose factors are abelian. We then have:

Theorem 3.15. *Let G be an infinite subsoluble group such that $\overline{\omega}_i(G) \neq \overline{\omega}(G)$. Then:*

(i) *G is a soluble group with a normal Prüfer p-subgroup P such that G/P is finite-by-(torsion-free abelian);*
(ii) *$G/\overline{\omega}(G)$ has finite exponent;*
(iii) *$\overline{\omega}_i(G)/\overline{\omega}(G)$ is a finite abelian $\{p, p-1\}$-group.*

In particular we also have:

Corollary 3.16. *Let G be an infinite subsoluble group.*

(i) *If G is finitely generated, then $\overline{\omega}_i(G) = \overline{\omega}(G)$.*
(ii) *If G contains no Prüfer subgroups, then $\overline{\omega}_i(G) = \overline{\omega}(G)$.*

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When $\overline{\omega}_i(G)$ is finite we have:

Theorem 3.17. (i) *If G is an infinite group in which $\overline{\omega}_i(G)$ is finite, then $\overline{\omega}_i(G)/\overline{\omega}(G)$ is abelian.*

(ii) *Let G be a group. Then $\overline{\omega}_i(G)$ is finite if and only if $\overline{\omega}(G)$ is finite.*

4. σ-Subnormality

In [65], the concept of σ-subnormal subgroup was introduced for finite groups and in [30] this idea is extended to locally finite ones.

Let $\mathbb{P}$ denote the set of all prime numbers and let $\sigma = \{\sigma_j \mid j \in J\}$ be a partition of $\mathbb{P}$. A subgroup X of a group G is called *σ-subnormal* in G, if there is a chain of subgroups

$$X = X_0 \leq X_1 \leq X_2 \leq \ldots \leq X_n = G \tag{4.1}$$

such that, for $1 \leq i \leq n-1$, either $X_{i-1} \triangleleft X_i$ or $X_i/(X_{i-1})_{X_i}$ is a σ_{j_i}-group for some $j_i \in J$. Clearly, every subnormal subgroup is σ-subnormal, but the converse is not true because every subgroup of a p-group is σ-subnormal for any choice of the partition σ; moreover, if we restrict attention to finite groups and we let $\overline{\sigma}$ be the *trivial partition* (that is, every set in the partition just contains a single prime), then every $\overline{\sigma}$-subnormal subgroup is obviously subnormal.

Skiba obtained many properties of σ-subnormal subgroups of finite groups; from the point of view of this survey it is interesting to note that Skiba obtained a characterisation of finite groups with all subgroups σ-subnormal and he showed that at least in finite groups a join of two σ-subnormal subgroups is again σ-subnormal (see [65, Proposition 2.3, Lemma 2.6]). The characterisation of finite groups with all subgroups σ-subnormal can immediately be transposed to the locally finite case. It is in fact proved in [30] that a σ-subnormal subgroup which is a σ_i-group for some $i \in I$ must be contained in the largest normal σ_i-subgroup. The same cannot be said for the join of two σ-subnormal subgroups. For each prime p, in [42, Theorem 1.5.2, Theorem 1.5.5], an example of a locally finite p-group of derived length 5, is given, H_p say, which has two subnormal subgroups A_p, B_p of defect at most 5, such that $\langle A_p, B_p\rangle$ is self-normalizing in H_p. Now let σ be the trivial partition of $\mathbb{P}$ and let $G = \operatorname*{Dr}_{p\in\mathbb{P}} H_p$. Then $A = \operatorname*{Dr}_{p\in\mathbb{P}} A_p, B = \operatorname*{Dr}_{p\in\mathbb{P}} B_p$ are σ-subnormal in G, but $\langle A, B\rangle$ is not σ-subnormal.

The difficulties in dealing with σ-subnormal subgroups arise from the fact that there is no canonical series that takes the place of the normal closure series for a subnormal subgroup. These difficulties can be overcome by using the concept of σ-seriality. Informally, a subgroup X is *σ-serial* in G, if there is a chain $\mathcal{S}$ of subgroups of G such that either $K \triangleleft H$ or $\pi(H/K_H) \subseteq \sigma_i$ for some $\sigma_i \in \sigma$, whenever $H < K$ are elements of $\mathcal{S}$ for which there is no element L of $\mathcal{S}$ between H and K. If the chain $\mathcal{S}$ is indexed by the linearly ordered set $(\mathcal{I}, <)$, then X is σ-subnormal precisely when $|\mathcal{I}|$ is finite. Of course, one can also define analogues of ascendant and descendant subgroups as follows: if $\mathcal{I}$ is well-ordered (inversely well-ordered) then X is said to be *σ-ascendant* (respectively *σ-descendant*) in G. In [30] the authors proved the following generalization of the main result of [32].

Theorem 4.1. *Let σ be a partition of $\mathbb{P}$ and let G be a locally finite group. Let $\mathcal{L}$ be a family of σ-serial subgroups of G. Then $\langle X \mid X \in \mathcal{L}\rangle$ is also σ-serial in G.*

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The concept of σ-serial subgroup is an invaluable instrument in studying the σ-subnormality of the join, and in fact it turns out that if a σ-serial subgroup X contains a subnormal subgroup which is nicely embedded in X, then X is σ-subnormal in the group. This has been employed in [30] to prove the following subnormality criterion.

Theorem 4.2. *Let σ be a partition of $\mathbb{P}$ and let G be a locally finite group. If H and K are σ-subnormal subgroups of G with $HK = KH$, then HK is σ-subnormal in G.*

Corollary 4.3. *Let σ be a partition of $\mathbb{P}$ and let G be a locally finite group. If H, K are σ-subnormal subgroups of G, then $\langle H, K\rangle$ is σ-subnormal in G if and only if H^K is σ-subnormal in G.*

The join of σ-subnormal subgroups is investigated in more detail in the paper [28]. There we can find analogues of many results from Section 2.1. For example, the following analogue of Theorem 2.8 is given. If σ is any partition of $\mathbb{P}$, we say that an abelian group A has *finite σ-join rank* if the following two conditions are satisfied:

(i) there is no infinite subset τ of σ such that $A_{\tau_i}/D(A_{\tau_i})$ has infinite rank for every $\tau_i \in \tau$;

(ii) there is no infinite subset τ of σ such that $A_{\tau_i}/D(A_{\tau_i})$ has finite rank for every $\tau_i \in \tau$, but $A_\rho/D(A_\rho)$ has infinite rank, where $\rho = \bigcup_{\tau_i\in\tau} \tau_i$.

(Here, if B is an abelian group, $D(B)$ denotes the largest divisible subgroup of B).

Theorem 4.4. *Let $\sigma = \{\sigma_i : i \in I\}$ be a partition of $\mathbb{P}$. Let H and K be locally finite groups.*

(1) *If $H/H' \otimes K/K'$ has finite σ-join rank, then $\langle H, K\rangle$ is σ-subnormal in any locally finite group G in which H and K can be subnormally embedded.*

(2) *If $H/H' \otimes K/K'$ has not finite σ-join rank, then there is a locally finite group G containing H and K as subnormal subgroups, such that $\langle H, K\rangle$ is not σ-subnormal in G.*

The correlation between the join of subnormal subgroups and σ-subnormal subgroups is also extensively examined in [30] and [28].

Theorem 4.5. *Let σ be a partition of $\mathbb{P}$ and let G be a locally finite group. Let H, K be σ-subnormal in G. If the join of any pair of subnormal subgroups of G contained in $\langle H, K\rangle$ is σ-subnormal in G, then $\langle H, K\rangle$ is σ-subnormal in G.*

It also turns out that if σ is a finite partition of $\mathbb{P}$, then the join of every pair of σ-subnormal subgroup is σ-subnormal. However, it seems to be an open problem if in this case the join of arbitrarily many σ-subnormal subgroups is still σ-subnormal.

5. Subnormal subgroups in linear groups

In this final short section, we give an overview of the behaviour of subnormal subgroups (and their generalizations) in the universe of linear groups. It has recently been shown, in [19], that in a periodic linear group serial subgroups are always ascendant while descendant subgroups are subnormal. A further result given in [19, Theorem 4.1] shows that if G is a soluble-by-periodic linear group and if X, Y are subnormal subgroups of G, then $\langle X, Y\rangle$ is also subnormal in G. It is also interesting to note the following result.

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Theorem 5.1. *There is a natural number n such that following property holds: if G is a periodic linear group of degree 2 over a field of characteristic 2, then every subnormal subgroup of G has defect at most n.*

This is quite surprising because if you change either the degree or the characteristic the result is not true anymore. Also, in [19], it is proved that a linear group in which all subgroups are subnormal must be nilpotent, while linear groups in which every subgroup is serial must be locally nilpotent (and so hypercentral).

Linear groups are also a great environment for dealing with σ-subnormal subgroups. If fact the following result has been proved in [30].

Theorem 5.2. *Let σ be a partition of $\mathbb{P}$ and let G be a periodic linear group. If X is a σ-serial subgroup of G, then X is σ-subnormal in G.*

It follows from Theorem 5.2 and Theorem 4.1 that the join of any family of σ-subnormal subgroups is still σ-subnormal in the universe of linear groups.

Concluding remark

It is interesting that even though the join problem and the theory of subnormal subgroups itself is a rather old one, there are many interesting results which can be obtained to generalize and complement many of the classical theorems. We hope this survey will serve as a small tribute to some of Derek Robinson's many contributions to group theory and that the results given here will stimulate further research on subnormal subgroups and their generalizations.

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Martyn R. Dixon

Department of Mathematics, University of Alabama, Tuscaloosa, AL 35487-0350, U. S. A.

Email: mdixon@ua.edu

Maria Ferrrara

Dipartimento di Matematica e Fisica, Università degli Studi della Campania "Luigi Vanvitelli", viale Lincoln 5 Caserta Italy.

Email: maria.ferrara1@unicampania.it

Marco Trombetti

Dipartimento di Matematica e Applicazioni, Università degli Studi di Napoli Federico II, Complesso Universitario Monte S. Angelo, Via Cintia, I-80126 Napoli, Italy.

Email: marco.trombetti@unina.it

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