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A STUDY ON THE π-DUAL RICKART MODULES

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Dedicated to Prof. O. A. S. Karamzadeh

Abstract. The right R-module M is said to be a π-dual Rickart module, if for every endomorphism $f : M \to M$ with projection invariant image, $f(M)$, in M, $f(M)$ is a direct summand of M. We show that the class of the π-dual Rickart modules contains properly the class of all π-dual Baer modules and the dual Rickart modules. We also investigate the transfering between a base ring R and $R[x]$ (and $R[[x]]$). It is shown that, in general, the class of π-dual Rickart modules is neither closed under direct summands nor closed under direct sums. We conclude the paper by giving a connection between the classes of π-dual Baer and π-lifting modules.

1. Introduction

Throughout this paper, R will be an associative ring with unity and any module M will be a unital right R-module. For a right R-module M, $S = \mathrm{End}_R(M)$ will denote the endomorphism ring of M, and $Mat_n(R)$ denotes an $n \times n$ matrix ring over the ring R. For two R-modules M and N, $\mathrm{Hom}_R(M, N)$ will indicate the set of all homomorphisms from M to N. The notations $N \leq M$ and $N \leq_d M$ mean that N is a submodule of M and N is a direct summand of M, respectively. By $\mathbb{Q}$ and $\mathbb{Z}$ we denote the ring of rational and integer numbers, respectively. $E(M)$ denotes the injective hull of a module M and $\mathbb{Z}(p^\infty)$ denotes the Prüfer p-group for any prime integer p. We also denote $r_M(I) = \{m \in M \mid Im = 0\}, r_S(I) = \{\varphi \in S \mid I\varphi = 0\}$ for $\emptyset \neq I \subseteq S$; $r_R(N) = \{r \in R \mid Nr = 0\}, l_S(N) = \{\varphi \in S \mid \varphi(N) = 0\}$ for $N \leq M$. For a subset X of S and a submodule N of M, we denote the submodule $\sum_{f\in X} f(N)$ by $X(N)$.

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Let $N \leq M$ for a module M. Then N is called a *fully invariant* submodule of M (denoted by $N \trianglelefteq M$) if $f(N) \subseteq N$ for all $f \in S$, and N is called a *projection invariant* submodule of M (denoted by $N \trianglelefteq_p M$) if $e(N) \subseteq N$ for all idempotent endomorphisms $e \in S$. Clearly, every fully invariant submodule is projection invariant. By [1, Proposition 3.1], if $N \leq_d M$, then $N \trianglelefteq M$ if and only if $N \trianglelefteq_p M$. Note that a right ideal I of a ring R is called *projection invariant* in R_R (denoted by $I \trianglelefteq_p R_R$) if $eI \subseteq I$ for all $e^2 = e \in R$. Moreover, fully invariant right ideals of R coincide with two-sided ideals of R.

In 2010, dual Baer modules were introduced by Keskin Tütüncü and Tribak. Let M be a module. M is called a *dual Baer* module if for every submodule N of M, $D_S(N) = \{f \in S \mid f(M) \subseteq N\}$ is a direct summand right ideal of S_S (see [7]). Later in 2013, Amouzegar and Talebi introduced quasi-dual Baer modules. A module M is said to be *quasi-dual Baer* if, for every fully invariant submodule N of M, there exists an idempotent $e \in S$ such that $D_S(N) = eS$ (see [2]). In 2021, Kara and in 2022, Keskin Tütüncü and Tribak defined π-dual Baer modules (according to Kara, dual π-endo Baer modules). A module M is called π-*dual Baer* if for each $N \trianglelefteq_p M$, $D_S(N) = eS$ for some $e^2 = e \in S$ (see [6] and [8]). Clearly, M is dual Baer $\Rightarrow$ M is π-dual Baer $\Rightarrow$ M is quasi-dual Baer, for any module M. Also, in 2011, Lee, Rizvi, and Roman introduced dual Rickart modules. A module M is called *dual Rickart*, if for every endomorphism $f : M \to M$, $f(M) \leq_d M$ (see [11]).

Motivated by all these works ([2, 6–8] and [11]), we introduce π-dual Rickart modules, in this paper. A module M is called a π-*dual Rickart* module if $f(M) \trianglelefteq_p M$, then $f(M) \leq_d M$, for every endomorphism $f : M \to M$. Our aim is to present some properties of these modules and investigate direct summands and direct sums of them.

Section 2 is devoted to the study of some basic properties and direct summands of π-dual Rickart modules. We construct some examples showing that π-dual Rickart modules are proper generalizations of dual Rickart modules (Example 2.5) and π-dual Baer modules (Example 2.6). We will say that R is a right π-dual Rickart ring, whenever the R-module R_R is a π-dual Rickart module, for any ring R. We investigate the transfer of the right π-dual Rickart condition between a base ring R and $R[x]$ (and $R[[x]]$). We prove that if $R[x]$ ($R[[x]]$) is a right π-dual Rickart ring, then R is a right π-dual Rickart ring (Proposition 2.12). Also, we illustrate that $R[x]$ and $R[[x]]$ may not be right π-dual Rickart rings, if R is a right π-dual Rickart ring (Example 2.13). In this section, finally, we study the direct summands of π-dual Rickart modules. We prove that if $M = M_1 \oplus M_2$ is a π-dual Rickart module with $M_1 \trianglelefteq_p M$, then M_1 and M_2 are π-dual Rickart (Corollary 2.19).

The investigations in Section 3 focus on the question of when is the direct sum of π-dual Rickart modules, π-dual Rickart? Mainly, we prove that if $M = \bigoplus_{i \in I} M_i$ with $M_i \trianglelefteq_p M$ for all $i \in I$, then M is a π-dual Rickart module if and only if M_i is a π-dual Rickart module for all $i \in I$ (Theorem 3.7).

The focus in Section 4 is on obtaining a connection between the classes of π-dual Baer and π-lifting modules. Firstly, we give the definitions of π-lifting modules, π-dual nonsingular modules, and π-dual cononsingular modules. Finally, we prove that a module M is π-dual Baer and π-dual cononsingular if and only if it is π-lifting and π-dual nonsingular (Theorem 4.10).

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2. π-Dual Rickart modules and direct summands

We start with the definition of π-dual Rickart modules.

Definition 2.1. An arbitrary module M is called a π-*dual Rickart* module, if $\mathrm{Im} f \trianglelefteq_p M$ then $\mathrm{Im} f \leq_d M$, for every endomorphism $f : M \to M$.

Lemma 2.2. *Let M be an arbitrary module, and $S = \mathrm{End}_R(M)$. M is a π-dual Rickart module if and only if for every $g \in S$ with $g(M) \trianglelefteq_p M$, $D_S(g(M))$ is a direct summand of S_S.*

Proof. ($\Rightarrow$) Suppose M is π-dual Rickart, and $g : M \to M$ an endomorphism with $g(M) \trianglelefteq_p M$. Then there exists an idempotent $e \in S$ such that $g(M) = e(M)$. Clearly, $D_S(e(M)) = eS$.

($\Leftarrow$) Let $g : M \to M$ be an endomorphism, with $g(M) \trianglelefteq_p M$. By hypothesis, $D_S(g(M)) = eS$ for some idempotent $e \in S$. Since $e \in D_S(g(M))$, $e(M) \subseteq g(M)$, and since $g \in D_S(g(M))$, $g = es$ for some $s \in S$. Therefore $g(M) \subseteq e(M)$. Hence $g(M) = e(M)$, which is a direct summand of M. □

Examples 2.3. Clearly, every dual Rickart module is π-dual Rickart. Every semisimple module is a π-dual Rickart module. Every injective module over a right hereditary ring is π-dual Rickart. Any module which has a von Neumann regular endomorphism ring is π-dual Rickart. The $\mathbb{Z}$-modules $\mathbb{Z}(p^\infty)$ (p is any prime integer), $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are π-dual Rickart modules (see [11, Example 2.3]).

Lemma 2.4. *Every π-dual Baer module is π-dual Rickart.*

Proof. Let $f : M \to M$ with $\mathrm{Im} f \trianglelefteq_p M$ be an endomorphism, and $S = \mathrm{End}(M)$. Since M is π-dual Baer, $D_S(\mathrm{Im} f) = eS$ for some $e^2 = e \in S$. By Lemma 2.2, M is π-dual Rickart. □

There exists a π-dual Rickart module which is not dual Rickart as we see in the following example.

Example 2.5. Let k be any field of characteristic 0. By [10, Corollary 3.17], the first Weyl algebra $A_1(k)$ is a simple domain, which is not a division ring. Therefore $A_1(k)$ is not a von Neumann regular ring, because over domains von Neumann regular rings and division rings are coincide. Now, let $R = \begin{bmatrix} A_1(k) & A_1(k) \\ A_1(k) & A_1(k) \end{bmatrix}$ be the 2-by-2 matrix ring over $A_1(k)$. Then, clearly, R is a simple ring (see [10, Theorem 3.1]) which is not a domain. By [9, Corollary 18.6], R is not von Neumann regular. Therefore R_R is not dual Rickart by [11, Remark 2.2]. On the other hand, R_R is π-dual Rickart by [8, Example 4.11] and Lemma 2.4. Note that ${}_RR$ is π-dual Rickart, as well.

There exists a π-dual Rickart module which is not π-dual Baer as exhibited in the next example.

Example 2.6. Let F be a field and I be an infinite index set. Let $R = \prod_{i \in I} F_i$ where $F_i = F$ for each $i \in I$. We know that $\mathrm{Soc}(R_R) = \bigoplus_{i \in I} F_i$ and it is essential in R_R. On the other hand, R_R is π-dual Rickart, because it is dual Rickart by [11, Example 5.1]. By [8, Proposition 4.18], R_R is not π-dual Baer since R is not semisimple.

Remark 2.7. Let M be an indecomposable module. Then M is dual Baer iff it is π-dual Baer iff it is dual Rickart iff it is π-dual Rickart. Because for an indecomposable module, every submodule is projection invariant.

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Recall that for an R-module M and a direct summand N of M, $N \trianglelefteq_p M$ if and only if $N \trianglelefteq M$ (see [1, Proposition 3.1]). Also, recall that a module M is said to have the *FI-strong summand sum property* (briefly, *FI-SSSP*), if the sum of any number of fully invariant direct summands is again a direct summand (see [2, page 80]). Therefore, M has the FI-SSSP if and only if the sum of any number of projection invariant direct summands is again a direct summand. In the same manner, any module M is said to have the *FI-SSP*, if the sum of any two projection invariant direct summands is again a direct summand.

Lemma 2.8. *Let M be a π-dual Baer module. Then M has the FI-SSSP.*

Proof. By [2, Lemma 2.2] and [8, Remark 2.9]. □

Now, we can give the following result similar to [2, Theorem 2.2].

Theorem 2.9. *Let M be a module with $S = \mathrm{End}(M)$ and $\mathrm{Im} f \trianglelefteq_p M$, for all $f \in S$. Then the following are equivalent:*

(i) *M is π-dual Baer;*
(ii) *M has the FI-SSSP and M is π-dual Rickart;*
(iii) *M is dual Baer.*

Proof. (i) ⇒ (ii) By Lemmas 2.4 and 2.8.
(ii) ⇒ (iii) Let $I \leq S_S$. By hypothesis, $\mathrm{Im} f \trianglelefteq_p M$ for all $f \in I$. Since M is π-dual Rickart, $\mathrm{Im} f \leq_d M$, for all $f \in I$. Then $\sum_{f \in I} \mathrm{Im} f \leq_d M$, since M has the FI-SSSP. Therefore M is dual Baer by [7, Theorem 2.1].
(iii) ⇒ (i) Clear. □

We can investigate the endomorphism rings of indecomposable π-dual Rickart modules as follows.

Theorem 2.10. *Let M be a module with $S = \mathrm{End}_R(M)$. The following are equivalent.*

(i) *M is indecomposable and dual Rickart;*
(ii) *M is indecomposable and π-dual Rickart;*
(iii) *S is a domain and $\varphi(M) = r_M(l_S(\varphi(M)))$ for all $\varphi \in S$;*
(iv) *every nonzero endomorphism $\varphi \in S$ is an epimorphism.*

Proof. It follows by Remark 2.7 and [11, Proposition 4.4]. □

Next, we characterize π-dual Rickart rings as proved in the following.

Lemma 2.11. *Let R be any ring. R_R is π-dual Baer if and only if every projection invariant cyclic right ideal xR is a direct summand of R_R.*

Proof. (⇒) Let $xR \trianglelefteq_p R_R$. Consider the R-homomorphism $f : R \to R$ defined by $f(r) = xr$. Then $\mathrm{Im} f = xR$. Since R_R is π-dual Rickart, we have $xR \leq_d R_R$.

(⇐) Let $f : R \to R$ be an R-homomorphism with $\mathrm{Im} f \trianglelefteq_p R_R$. Let $f(1) = x$. Then $\mathrm{Im} f = xR$. By hypothesis, $\mathrm{Im} f \leq_d R_R$. Hence R_R is π-dual Rickart. □

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Lemma 2.11 will be very useful to investigate the transfer of the right π-dual Rickart condition between a base ring R and $R[x]$ (and $R[[x]]$).

Proposition 2.12. *Let R be a ring satisfying one of the following conditions:*

(i) *$R[x]$ is a right π-dual Rickart ring;*

(ii) *$R[[x]]$ is a right π-dual Rickart ring.*

Then R is a right π-dual Rickart ring.

Proof. (i) Let $R[x]$ be a π-dual Rickart ring. Let $I = aR \trianglelefteq_p R_R$, where $a \in R$. By [4, Lemma 4.1(iv)], $I[x] = aR[x] \trianglelefteq_p R[x]_{R[x]}$. This implies that $I[x] = e(x)R[x]$ for some idempotent $e(x) = e_0 + e_1x + \ldots + e_nx^n \in R[x]$ by Lemma 2.11. By the same proof as in [8, Proposition 4.19], $I = e_0R$. Therefore R is a right π-dual Rickart ring by Lemma 2.11.

(ii) This is achieved by the same method as in (i). □

If R is a right π-dual Rickart ring, then $R[x]$ and $R[[x]]$ may not be right π-dual Rickart rings, as the next example illustrates.

Example 2.13. Let F be a field. Clearly, F is a right π-dual Rickart ring. By [8, Example 4.20] and Remark 2.7, neither $F[x]$ nor $F[[x]]$ is right π-dual Rickart.

The following example shows that the right π-dual Rickart property is not Morita invariant.

Example 2.14. We know that for any ring R and any positive integer n, the rings R and the full matrix ring $Mat_n(R)$ are Morita equivalent. Let R be a simple ring which is a domain but not a division ring. By [8, Example 3.5] and Remark 2.7, R_R is not π-dual Rickart, but it is quasi-dual Baer. Therefore for every positive integer $n > 1$, $Mat_n(R)$ is a right π-dual Rickart ring by [8, Proposition 4.21].

Next, we give the following definition to investigate direct summands of π-dual Rickart modules.

Definition 2.15. A module M is called *N-π-dual Rickart* if $f(M) \trianglelefteq_p N$, then $f(M) \leq_d N$, for every homomorphism $f : M \to N$.

Clearly, any module M is π-dual Rickart if and only if M is M-π-dual Rickart. The next example illustrates this definition in a similar manner to [11, Example 2.15].

Example 2.16. Let N be a semisimple module. Then M is N-π-dual Rickart for any module M. Let p be any prime integer, $M_{\mathbb{Z}} = \mathbb{Z}(p^\infty)$ and $N_{\mathbb{Z}} = \mathbb{Z}_p$. Then M is N-π-dual Rickart, but N is not M-π-dual Rickart. Note that here M and N are π-dual Rickart modules. Also $\mathbb{Z}_4$ is $\mathbb{Z}_3$-π-dual Rickart, while $\mathbb{Z}_4$ is not a π-dual Rickart $\mathbb{Z}$-module.

Theorem 2.17. *Let M and N be right R-modules. Then M is N-π-dual Rickart if and only if for any direct summand M' of M and any projection invariant submodule N' of N, M' is N'-π-dual Rickart.*

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Proof. Let $M' \leq_d M$ and $N' \trianglelefteq_p N$. Take $f : M' \to N'$ with $f(M') \trianglelefteq_p N'$. Since $M' \leq_d M$, there exists an idempotent $e : M \to M$ with $e(M) = M'$. Now we can take the homomorphism $ife : M \to N$, where $i : N' \to N$ is the inclusion map. By [5, Lemma 3.1], $(ife)(M) = f(M') \trianglelefteq_p N$. Since M is N-π-dual Rickart, $f(M') \leq_d N$, and hence $f(M') \leq_d N'$. Therefore M' is N'-π-dual Rickart. The converse is clear. □

Corollary 2.18. *The following are equivalent for a module M.*

(i) *M is π-dual Rickart;*

(ii) *for every projection invariant submodule N of M, every direct summand L of M is N-π-dual Rickart;*

(iii) *for every pair of submodules L and N of M with $L \leq_d M$ and $N \trianglelefteq_p M$ and any $f : M \to N$ with $f(M) \trianglelefteq_p N$, the image of the restricted homomorphism $f_{|L}$ with $f_{|L}(L) \trianglelefteq_p N$ is a direct summand of N.*

Proof. (i) ⇒ (ii) It is clear by Theorem 2.17.

(ii) ⇒ (iii) Let $L \leq_d M$, $N \trianglelefteq_p M$ and $f : M \to N$ be any homomorphism with $f(M) \trianglelefteq_p N$. Let $g = f_{|L} : L \to N$ and assume that $g(L) \trianglelefteq_p N$. By (ii), $g(L) \leq_d N$.

(iii) ⇒ (i) Take $M = L = N$ in (iii). □

We know that the $\mathbb{Z}$-module $\mathbb{Q}$ is π-dual Rickart. Consider the submodule $\mathbb{Z}$ of $\mathbb{Q}$. Since for every integer $n \geq 2$, $D_S(n\mathbb{Z})$ is non-zero and proper right ideal of $S = \mathrm{End}_{\mathbb{Z}}(\mathbb{Z})$, $\mathbb{Z}_{\mathbb{Z}}$ is not π-dual Rickart. Therefore π-dual Rickart property does not always transfer from a module to each of its submodules. Next, we will show that a projection invariant direct summand of a π-dual Rickart module inherits the property.

Corollary 2.19. *Let $M = M_1 \oplus M_2$ be a π-dual Rickart module for some submodules M_1 and M_2 of M. If $M_1 \trianglelefteq_p M$, then M_1 and M_2 are π-dual Rickart.*

Proof. M_1 is π-dual Rickart by Corollary 2.18.

Now let $f : M_2 \to M_2$ be a homomorphism with $f(M_2) \trianglelefteq_p M_2$. By [3, Lemma 4.13], $M_1 \oplus f(M_2) \trianglelefteq_p M$. Let $\varphi : M \to M$ be the homomorphism defined by $\varphi(m_1 + m_2) = m_1 + f(m_2) = (1_{M_1} \oplus f)(m_1 + m_2)$. Then $\varphi(M) = M_1 \oplus f(M_2)$. Since M is π-dual Rickart, $M_1 \oplus f(M_2) \leq_d M_1 \oplus M_2$ and so $f(M_2) \leq_d M_2$. Therefore M_2 is π-dual Rickart. □

The following example illustrates that projection invariant condition is necessary in Corollary 2.19.

Example 2.20. Let R be a simple ring which is a domain but not a division ring. As we mentioned in Example 2.14, R_R is not π-dual Rickart. Now, consider a free right R-module $F_R = \oplus_{i=1}^{n} R_i$ for some integer $n > 1$, where $R_i \cong R$ for all $1 \leq i \leq n$. By [8, Example 3.5], F_R is π-dual Baer, and so it is π-dual Rickart by Lemma 2.4.

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3. Direct sums of π-dual Rickart modules

As seen in the example below, the direct sum of π-dual Rickart modules is not a π-dual Rickart, necessarily.

Example 3.1. (see [11, Example 2.10]) Let $M_1 = \mathbb{Z}(p^\infty)$ and $M_2 = \langle \frac{1}{p} + Z \rangle$, for some prime integer p. Note that M_1 and M_2 are both π-dual Rickart modules. Now, we will show that $M = M_1 \oplus M_2$ is not π-dual Rickart. Let $f : M \to M$ be the endomorphism defined by $f(\frac{a}{p^n} + \mathbb{Z}, \frac{b}{p} + \mathbb{Z}) = (\frac{b}{p} + \mathbb{Z}, 0)$ where $a, b \in \mathbb{Z}$ and $n \in \mathbb{N}$. Then $f(M) = \langle \frac{1}{p} + Z \rangle \oplus 0$. We know that $\langle \frac{1}{p} + Z \rangle \oplus 0$ is an essential submodule in $\mathbb{Z}(p^\infty) \oplus 0$. Therefore $f(M)$ can not be a direct summand of M. Here, $f(M) \trianglelefteq_p \mathbb{Z}(p^\infty) \oplus 0$ because $\mathbb{Z}(p^\infty) \oplus 0$ is indecomposable. Since $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}(\mathrm{p}^\infty) \oplus 0, 0 \oplus \langle \frac{1}{\mathrm{p}} + \mathrm{Z} \rangle) = 0$, $\mathbb{Z}(p^\infty) \oplus 0 \trianglelefteq M$ by [12, Lemma 1.9]. Since $\mathbb{Z}(p^\infty) \oplus 0$ is a direct summand of M, it is projection invariant in M, as well. Now, by [5, Lemma 3.1], $f(M) \trianglelefteq_p M$. Therefore M is not π-dual Rickart.

We can generalize the above example as follows.

Example 3.2. Let L be a simple R-module such that the injective hull $E(L)$ of L has no maximal submodules. Note that $L \trianglelefteq_p E(L)$, since L is quasi-injective. On the other hand, $E(L) \trianglelefteq_p E(L) \oplus L$ since $\mathrm{Hom}_R(E(L), L) = 0$. Now let $M = E(L) \oplus L$, and $i : L \to E(L)$ be the inclusion map. Since L is not a direct summand of $E(L)$, L is not $E(L)$-π-dual Rickart. Therefore by Corollary 2.18, M is not π-dual Rickart. Now let R be a discrete valuation ring with maximal ideal I and quotient field K. It is well known that $K/R \cong E(R/I)$. Therefore the R-module $(K/R) \oplus (R/I)$ is not π-dual Rickart. On the other hand, note that K/R and R/I are π-dual Rickart modules (see [8, Exeample 3.1]).

In this section, we focus on when a direct sum of π-dual Rickart modules is also π-dual Rickart.

Let $M = \bigoplus_{i \in I} M_i$, and $M_i \trianglelefteq_p M$. From Corollary 2.18, if M is a π-dual Rickart module then M_i is M_j-π-dual Rickart, for all $i, j \in I$. Now, we give the following results.

Proposition 3.3. *Let $M = \bigoplus_{i \in I} M_i$, and let N be an indecomposable module. Then M is N-π-dual Rickart if and only if M_i is N-π-dual Rickart, for all $i \in I$.*

Proof. Since N is indecomposable, N has the SSSP and every submodule is projection invariant. Now the result follows by Theorem 2.17 and [11, Proposition 5.3(ii)]. □

Corollary 3.4. *Let $M = \bigoplus_{i \in I} M_i$ where each M_i is indecomposable. Then for each $j \in I$, M is M_j-π-dual Rickart if and only if M_i is M_j-π-dual Rickart for all $i \in I$.*

We can give the following applications of the above corollary.

Example 3.5. The $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z}$ is $\mathbb{Z}(p^\infty)$-π-dual Rickart for all prime integers p. Because $\mathbb{Q}/\mathbb{Z} = \bigoplus_{p \in \mathbb{P}} \mathbb{Z}(p^\infty)$, where $\mathbb{P}$ is the set of all prime integers and $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}(p^\infty), \mathbb{Z}(q^\infty)) = 0$ for all distinct primes p and q. The $\mathbb{Z}$-module $M = \mathbb{Z} \oplus \mathbb{Q}$ is not $\mathbb{Q}$-π-dual Rickart. Because $\mathbb{Z} \trianglelefteq_p \mathbb{Q}$, but $\mathbb{Z}$ is not a direct summand of $\mathbb{Q}$. Also M is not $\mathbb{Z}$-π-dual Rickart since $\mathbb{Z}$ is not π-dual Rickart.

Theorem 3.6. *Let $M = \bigoplus_{i \in I} M_i$ with $M_i \trianglelefteq_p M$ for all $i \in I$, and let N be an arbitrary module. Then N is M-π-dual Rickart if and only if N is M_j-π-dual Rickart for all $j \in I$.*

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Proof. (⇒)Use Theorem 2.17.

(⇐) Let $f : N \to M$ be a homomorphism with $f(N) \trianglelefteq_p M$. We will show that $f(N) \leq_d M$. By [5, Lemma 3.1], $f(N) = \bigoplus_{i\in I}(f(N) \cap M_i)$ and $f(N) \cap M_i \trianglelefteq_p M_i$ for all $i \in I$. Let $\pi_i : M \to M_i$ be the projection map for each $i \in I$. Then we have the homomorphisms $\pi_i f : N \to M_i$ with $(\pi_i f)(N) = f(N) \cap M_i$, for all $i \in I$ since $f(N) \trianglelefteq_p M$ and π_i is an idempotent endomorphism of M for each $i \in I$. Since N is M_i-π-dual Rickart, $f(N) \cap M_i \leq_d M_i$ for each $i \in I$. Therefore $f(N) \leq_d M$. □

Finally, we can give the following last theorem and its corollary describing the direct sums of π-dual Rickart modules.

Theorem 3.7. *Let $M = \bigoplus_{i\in I} M_i$ with $M_i \trianglelefteq_p M$ for all $i \in I$. Then M is a π-dual Rickart module if and only if M_i is a π-dual Rickart module for all $i \in I$.*

Proof. The necessity follows from Corollary 2.19. Conversely, assume that every M_i is π-dual Rickart for each $i \in I$. Now, we will prove that $M = \bigoplus_{i\in I} M_i$ is π-dual Rickart. Let $f : M \to M$ be a homomorphism with $f(M) \trianglelefteq_p M$. Let $j_i : M_i \to M$ be the inclusion map and $\pi_i : M \to M_i$ be the projection map for each $i \in I$. Then we have the homomorphisms $\pi_i f j_i : M_i \to M_i$ for each $i \in I$. Now, we have that $(\pi_i f j_i)(M_i) = \pi_i(f(M_i)) = f(M_i)$ for all $i \in I$, since each $M_i \trianglelefteq M$ (because $M_i \leq_d M$ and $M_i \trianglelefteq_p M$). On the other hand, $f(M) = \bigoplus_{i\in I}(f(M) \cap M_i)$ and $f(M) \cap M_i \trianglelefteq_p M_i$ for all $i \in I$, since $f(M) \trianglelefteq_p M$ (see [5, Lemma 3.1]). Note that $f(M_i) = f(M) \cap M_i$ for each $i \in I$. Therefore, $f(M) = \bigoplus_{i\in I} f(M_i) \leq_d M$, since each M_i is π-dual Rickart for each $i \in I$. □

Recall that the Jacobson radical $\text{Rad}(M)$ of any module M is the sum of all small submodules of M and $\text{Rad}(M) \trianglelefteq M$. Remember that any submodule S of any module M is called *small*, if whenever $M = S + X$ for a submodule X of M, then $M = X$.

Corollary 3.8. *Let an R-module M be a direct sum of submodules M_1 and M_2 such that* $\text{Rad}(M_1) = M_1$, *and M_2 is a semisimple module. If M is π-dual Rickart, then M_1 is π-dual Rickart. The converse holds when* $\text{Hom}_R(M_2, M_1) = 0$.

Proof. Note that $\text{Rad}(M) = \text{Rad}(M_1) \oplus \text{Rad}(M_2) = M_1 \trianglelefteq_p M$.

(⇒) Assume that M is π-dual Rickart. By Corollary 2.19, M_1 is π-dual Rickart.

(⇐) Since $\text{Hom}_R(M_2, M_1) = 0$, $M_2 \trianglelefteq_p M$. Now the result is clear by Theorem 3.7. □

4. π-Dual Baer modules and some singularity conditions

Let M be a module. We will say that M is π-*lifting*, if for any $N \trianglelefteq_p M$ there exists a direct summand K of M such that $K \subseteq N$ and $N/K \ll M/K$. This definition is given under the name of PI-lifting in [1]. Note that if M is a lifting module, then it is π-lifting. But the converse is not true, in general as we see in the following example.

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Example 4.1. Let $R = \begin{bmatrix} \mathbb{Z} & 0 \\ \mathbb{Z}_2 & \mathbb{Z}_2 \end{bmatrix}$. We know that R_R is not lifting since $\mathbb{Z}_{\mathbb{Z}}$ is not lifting. Note that projection invariant right ideals of R_R are $I_1 = \begin{bmatrix} 0 & 0 \\ \mathbb{Z}_2 & 0 \end{bmatrix}$ and $I_2 = \begin{bmatrix} 0 & 0 \\ \mathbb{Z}_2 & \mathbb{Z}_2 \end{bmatrix}$. Since I_2 is a direct summand of R_R, we can write $I_2/I_2 \ll R/I_2$. Note that Jacobson radical of R_R is $J(R_R) = I_1 = \begin{bmatrix} 0 & 0 \\ \mathbb{Z}_2 & 0 \end{bmatrix}$. So $I_1 \ll R_R$. Therefore, $I_1/0 \ll R/0$. Hence R_R is π-lifting.

In [5], the authors defined π-e.nonsingular and π-e.cononsingular modules as follows (see [5, Definition 4.9 and Proposition 4.10]):

A module M is called π-*e.nonsingular*, if for each projection invariant left ideal Y of S for which $r_M(Y)$ is essential in $e(M)$, where $e^2 = e \in S$ we have $r_M(Y) = e(M)$, and it is called π-*e.cononsingular*, if for each $N \trianglelefteq_p M$ with $r_M(l_S(N))$ a direct summand of M, we have N is essential in $r_M(l_S(N))$. Dually, we can define the following:

Definition 4.2. Let M be a module with $S = \mathrm{End}_R(M)$. M is called

(i) π-*dual nonsingular* if for each $I \trianglelefteq_p S_S$ with $e(M) \subseteq I(M)$ and $I(M)/e(M) \ll M/e(M)$, where $e^2 = e \in S$, we have $I(M) = e(M)$.

(ii) π-*dual cononsingular* if for each $N \trianglelefteq_p M$ with $D_S(N)(M)$ a direct summand of M, we have $N/D_S(N)(M) \ll M/D_S(N)(M)$.

In this section, our aim is to obtain a connection between the classes of π-dual Baer and π-lifting modules via the dual singularity conditions defined above. Firstly, we give the following characterization of π-dual nonsingular modules.

Theorem 4.3. *Let M be a module with $S = \mathrm{End}_R(M)$. The following are equivalent:*

(i) *M is π-dual nonsingular;*

(ii) *For all $I \trianglelefteq_p S_S$ with $I(M)/e(M) \ll M/e(M)$ for some $e^2 = e \in S$, we have $I \subseteq eS$;*

(iii) *For all $N \trianglelefteq_p M$ such that $N/e(M) \ll M/e(M)$ where $e^2 = e \in S$, we have$D_S(N)(M) = e(M)$ and $e \in S_l(S)$.*

Proof. (i) ⇒ (ii) Let M be π-dual nonsingular. Let $I \trianglelefteq_p S_S$ with $I(M)/e(M) \ll M/e(M)$ for some $e^2 = e \in S$. Then by π-dual nonsingularity, we have $e(M) = I(M)$. Hence $I \subseteq D_S(e(M)) = eS$.

(ii) ⇒ (iii) Let $N \trianglelefteq_p M$ such that $N/e(M) \ll M/e(M)$, where $e^2 = e \in S$. Note that $D_S(N) \trianglelefteq_p S_S$. Clearly, $e(M) \subseteq D_S(N)(M) \subseteq N$. Hence $D_S(N)(M)/e(M) \ll M/e(M)$. By (ii), $D_S(N) \subseteq eS$. On the other hand, $eS \subseteq D_S(N)$. Hence $D_S(N) = eS$, and so $D_S(N)(M) = eS(M) \subseteq e(M)$. Consequently, $D_S(N)(M) = e(M)$. Note that $D_S(N)(M) \trianglelefteq_p M$ and hence $e(M) \trianglelefteq_p M$. Then $e \in S_l(S)$ by [5, Lemma 3.1].

(iii) ⇒ (i) Let $I \trianglelefteq_p S_S$ with $I(M)/e(M) \ll M/e(M)$ where $e^2 = e \in S$. Note tat $I(M) \trianglelefteq_p M$. By (iii), $D_S(I(M))(M) = e(M)$. Therefore $I(M) = e(M)$. Consequently, M is π-dual nonsingular. □

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Corollary 4.4. *Let M be a π-dual nonsingular module with $S = \mathrm{End}_R(M)$. Let $N \trianglelefteq_p M$. If $N/e_1(M) \ll M/e_1(M)$ and $N/e_2(M) \ll M/e_2(M)$, where $e_1^2 = e_1, e_2^2 = e_2 \in S$, then $e_1(M) = e_2(M)$ and $e_1, e_2 \in S_l(S)$.*

Next, after a series of the following lemmas, we will reach the connection between the classes of π-dual Baer and π-lifting modules, which is the main aim of this section.

Lemma 4.5. *If M is π-lifting, then it is π-dual cononsingular.*

Proof. Assume that M is π-lifting. Let $N \trianglelefteq_p M$ with $D_S(N)(M) = e(M)$ for some $e^2 = e \in S$. Note that $D_S(N)(M) \subseteq N$. Since M is π-lifting, there exists $f^2 = f \in S$ such that $f(M) \subseteq N$ and $N/f(M) \ll M/f(M)$. Clearly, $f(M) \subseteq e(M)$ and hence $N/e(M) \ll M/e(M)$. Therefore M is π-dual cononsingular. □

Lemma 4.6. *Let M be a π-dual cononsingular and π-dual Baer module. Then M is π-lifting.*

Proof. Let $N \trianglelefteq_p M$. Then $D_S(N) = eS$ for some $e^2 = e \in S$ since M is π-dual Baer. Now $e(M) = D_S(N)(M)$. Since M is π-dual cononsingular, $N/e(M) \ll M/e(M)$. Hence M is π-lifting. □

Lemma 4.7. *Let M be π-dual Baer module. Then M is π-dual nonsingular.*

Proof. Assume that $I(M)/e(M) \ll M/e(M)$ for some $I \trianglelefteq_p S_S$, where $e^2 = e \in S$. Since M is π-dual Baer, $I(M) = f(M)$ for some $f^2 = f \in S$ by [8, Theorem 2.4]. On the other hand, $f(M)/e(M)$ is a direct summand of $M/e(M)$, as well. Hence $f(M) = e(M) = I(M)$. Hence M is π-dual nonsingular by Theorem 4.3. □

Lemma 4.8. *Let M be a π-dual nonsingular and π-lifting module. Then M is π-dual Baer.*

Proof. Let $I \trianglelefteq_p S_S$. Then $I(M) \trianglelefteq_p M$. Since M is π-lifting, there exists an idempotent $e \in S$ such that $I(M)/e(M) \ll M/e(M)$. By π-dual nonsingularity, $I(M) = e(M)$. Therefore M is π-dual Baer by [8, Theorem 2.4]. □

Example 4.9. The converses of Lemmas 4.5 and 4.7 are not true in general. Consider the $\mathbb{Z}$-module $M_{\mathbb{Z}} = \mathbb{Z}_{\mathbb{Z}}$. M is not π-lifting and not π-dual Baer. It is easy to see that M is π-dual cononsingular and π-dual nonsingular.

Theorem 4.10. *The following are equivalent for a module M*

(i) *M is π-dual Baer and π-dual cononsingular;*
(ii) *M is π-lifting and π-dual nonsingular.*

Proof. Combine Lemmas 4.5, 4.6, 4.7 and 4.8. □

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