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RINGS OF QUOTIENTS OF THE RING $\mathcal{R}(L)$ BY coz-FILTERS

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Dedicated to Prof. O. A. S. Karamzadeh

Abstract. In this article, we first introduce the concept of z-sets in the ring $\mathcal{R}(L)$ of real-valued continuous functions on a completely regular frame L, and give some properties of them. Let $S_{\mathcal{F}}^{-1}\mathcal{R}(L)$ denote the ring of fractions of the ring $\mathcal{R}(L)$, where $\mathcal{F}$ is a coz-filter on L and $S_{\mathcal{F}}$ is a multiplicatively closed subset related to $\mathcal{F}$. We show that $S_{\mathcal{F}}^{-1}\mathcal{R}(L)$ may be realized as the direct limits of the subrings $\mathcal{R}(A)$, where $A \in \{\mathfrak{o}_L\big(\mathrm{coz}(\alpha)\big) : \alpha \in S_{\mathcal{F}}\}$. Also, we show that $\mathrm{Q_{cl}}\mathcal{R}(L) = \mathcal{R}(L)$, if and only if $\mathcal{R}(L)$ is a special saturated ring.

1. Introduction and Preliminaries

In the point-free (localic) approach to topology, topological spaces are replaced by locales, seen as generalized spaces in which points are not explicitly mentioned. Formally, a *frame* L is defined as a special complete lattice (where we denote *top* (respectively, *bottom*) by $\top$ (respectively, $\bot$)), usually called a *locale* in which finite meets distribute over arbitrary joins; that is, $a \wedge \bigvee S = \bigvee\{a \wedge s : s \in S\}$ for all $a \in L$ and $S \subseteq L$. A frame homomorphism is a map between frames that preserves finite meets and arbitrary joins.

Throughout this paper, L will be a frame. Every frame is a complete Heyting algebra with the Heyting implication given by

$$a \to b = \bigvee\{x \in L : a \wedge x \leq b\}.$$

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For any $a \in L$, the element $a \to \bot$ is called the *pseudo complement* of a and is denoted by a^*.

For every $a, b \in L$, a is said to be *rather below* $b \in L$, written $a \prec b$, provided that $a^* \vee b = \top$. On the other hand, a is *completely below* b, written $a \prec\!\!\prec b$, if there are elements (c_q) indexed by the rational numbers $\mathbb{Q} \cap [0,1]$ such that $c_0 = a$, $c_1 = b$, and $c_p \prec c_q$ for $p < q$. A frame L is said to be *regular*, if $a = \bigvee\{x \in L \mid x \prec a\}$ for each $a \in L$, and it is *completely regular*, if $a = \bigvee\{x \in L \mid x \prec\!\!\prec a\}$ for each $a \in L$. An ideal J of a frame L is completely regular, if for every $x \in J$, there is $y \in J$ such that $x \prec\!\!\prec y$. For a completely regular frame L, the frame of its completely regular ideals is denoted by βL. The join map $\beta L \to L$ is dense onto and is referred to as the Stone-Čech compactification of L. We denote its right adjoint by r. A straightforward calculation shows that $r(a) = \{x \in L\colon x \prec\!\!\prec a\}$ for each $a \in L$. Throughout the paper, all frames will be assumed to be completely regular.

The frame $\mathcal{L}(\mathbb{R})$ of reals is obtained by taking the ordered pairs (p, q) of rational numbers as generators and imposing the following relations:

(R1) $(p, q) \wedge (r, s) = (p \vee r, q \wedge s)$,

(R2) $(p, q) \vee (r, s) = (p, s)$ whenever $p \leq r < q \leq s$,

(R3) $(p, q) = \bigvee\{(r, s) | p < r < s < q\}$,

(R4) $\top = \bigvee\{(p, q) | p, q \in \mathbb{Q}\}$.

Note that the pairs (p, q) in $\mathcal{L}(\mathbb{R})$ and the open intervals $\langle p, q\rangle = \{x \in \mathbb{R} : p < x < q\}$ in the frame $\mathfrak{O}\mathbb{R}$ have the same role. Let $\mathcal{R}(L)$ be the set of all frame maps from $\mathcal{L}(\mathbb{R})$ to a completely regular frame L, which is an f-ring. The reader can see [7] for more details of all these facts.

The properties of mapping $\operatorname{coz}\colon \mathcal{R}(L) \to L$, defined by $\operatorname{coz}(\varphi) = \varphi(-, 0) \vee \varphi(0, -)$, which are often used in this paper, are as follows:

(1) $\operatorname{coz}(\alpha\beta) = \operatorname{coz}(\alpha) \wedge \operatorname{coz}(\beta)$.

(2) $\operatorname{coz}(\alpha + \beta) \leq \operatorname{coz}(\alpha) \vee \operatorname{coz}(\beta)$.

(3) $\alpha \in \mathcal{R}(L)$ is invertible, if and only if $\operatorname{coz}(\alpha) = \top$.

(4) $\operatorname{coz}(\alpha) = \bot$, if and only if $\alpha = \mathbf{0}$.

(5) $\operatorname{coz}(\alpha^2 + \beta^2) = \operatorname{coz}(\alpha) \vee \operatorname{coz}(\beta)$.

For $A \subseteq \mathcal{R}(L)$, let $\operatorname{Coz}(A) := \{\operatorname{coz}(\alpha)\ ;\ \alpha \in A\}$, and let the cozero part of L, is denoted by $\operatorname{Coz}(L)$, be the regular sub-σ-frame consisting of all the cozero elements of L. It is known that L is completely regular, if and only if $\operatorname{Coz}(L)$ generates L. For $A \subseteq \operatorname{Coz}(L)$, we write $\operatorname{Coz}^{\leftarrow}(A)$ to designate the family of frame maps $\{\alpha \in \mathcal{R}(L) : \operatorname{coz}(\alpha) \in A\}$.

A *sublocale* of a locale L is a subset $S \subseteq L$, closed under arbitrary meets, such that $x \to s \in S$ for every $(x, s) \in L \times S$. Throughout this paper, let $\mathcal{S}\ell(L)$ be the set of all sublocales, which is a coframe with the relation of inclusion. The smallest sublocale of L is $\mathsf{O} = \{\top\}$, called the void sublocale. For any $a \in L$, sets

$$\mathfrak{o}_L(a) := \{a \to x \mid x \in L\} = \{x \mid x = a \to x\} \text{ and } \mathfrak{c}_L(a) := \{x \in L \mid x \geq a\},$$

denote the open and closed sublocales of L determined by a, respectively (see [19]). The closure and interior of every $S \in \mathcal{S}\ell(L)$, are defined, respectively, by

$$\operatorname{cl}_L S := \bigcap\{\mathfrak{c}_L(a) \in \mathcal{S}\ell(L)\colon S \subseteq \mathfrak{c}_L(a)\} = \mathfrak{c}_L\left(\bigwedge S\right) \text{ and } \operatorname{int}_L S := \bigvee\{\mathfrak{o}_L(a) \in \mathcal{S}\ell(L)\colon \mathfrak{o}_L(a) \subseteq S\}.$$

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A sublocale S of L is dense, if $cl_L(S) = L$. This is the case if and only if the bottom of S is the bottom element of L.

We shall freely use the following properties of these sublocales:

(1) $\mathfrak{o}_L(\bot) = \mathfrak{c}_L(\top) = \mathsf{O}$ and $\mathfrak{o}_L(\top) = \mathfrak{c}_L(\bot) = L$.
(2) $\mathfrak{c}_L(a) \subseteq \mathfrak{o}_L(b)$, if and only if $a \vee b = \top$, and also, $\mathfrak{o}_L(a) \subseteq \mathfrak{c}_L(b)$, if and only if $a \wedge b = \bot$.
(3) $\mathrm{int}_L \mathfrak{c}_L(a) = \mathfrak{o}_L(a^*)$ and $\mathrm{cl}_L \mathfrak{o}_L(a) = \mathfrak{c}_L(a^*)$.
(4) $\mathfrak{o}_L(a) \cap \mathfrak{o}_L(b) = \mathfrak{o}_L(a \wedge b)$ and $\mathfrak{c}_L(a) \vee \mathfrak{c}_L(b) = \mathfrak{c}_L(a \wedge b)$.
(5) $\bigvee_i \mathfrak{o}_L(a_i) = \mathfrak{o}_L\Big(\bigvee_i a_i\Big)$ and $\bigcap_i \mathfrak{c}_L(a_i) = \mathfrak{c}_L\Big(\bigvee_i a_i\Big)$.

If S is a sublocale of L, then we write $\nu_S \colon L \to S$ for the associated frame surjection. Recall from [19] that, for any $a \in L$, $\nu_S(a) = \bigwedge\{x \in S \mid a \leq x\} = \bigwedge\big(S \cap \mathfrak{c}_L(a)\big)$. Also the nuclei associated with $\mathfrak{o}_L(a)$ is $\nu_{\mathfrak{o}_L(a)}(x) = a \to x$. Recall from [16] that, a ring homomorphism $h \colon A \to R$ is an exoteric homomorphism, if for all pairs (I, J) of finitely generated ideals of A, $\mathrm{Ann}_A(I) = \mathrm{Ann}_A(J)$ implies $\mathrm{Ann}_R\big(h(I)\big) = \mathrm{Ann}_R\big(h(J)\big)$. An ideal of a ring A is said to be exoteric, if it is the kernel of an exoteric homomorphism. Dube [12] showed that $Soc\big(\mathcal{R}(L)\big)$ is an exoteric ideal of $\mathcal{R}(L)$. He also showed that if L has a finite number of atoms, then one can create a ring homomorphism with the domain $\mathcal{R}(L)$ whose core is $Soc\big(\mathcal{R}(L)\big)$.

Let A be a ring and let B be a commutative ring containing A with the same unit element. Then B is a ring of quotients or rational extension of A provided that for every $b \in B$, $b^{-1}A := \{a \in A \colon ba \in A\}$ is a dense ideal in B. That is, for every pair of elements $b, b' \in B$ with $b' \neq 0$, there is $a \in A$ such that $ba \in A$ and $b'a \neq 0$.

The concept of rational extension and quotient ring was defined by Johnson [18] and Utumi [22]. Then in [17], this concept was applied to the ring $C(X)$ of all continuous real-valued functions on a completely regular space X. The classical ring of quotients of A denoted by $Q_{cl}(A)$, is the ring of equivalence classes of formal quotients $\frac{c}{d}$ for $c \in A$ and a non-zero-divisor d of A. Let $Q(X)$ (resp., $Q_{cl}(X)$) denote the maximal ring of quotients (resp., the classical ring of quotients) of $C(X)$. These rings have been studied by Fine, Gillman, and Lambek [17] and realized as the ring of all real-valued continuous functions on the dense open sets (dense cozero sets) of X and the ring of all continuous functions on the dense cozero-sets in X. In special cases, the ring $Q(X)$ reduces to the classical ring of quotients of $C(X)$, but in general, the classical ring is a proper subring of $Q(X)$. Now let $Q(\mathcal{R}(L))$ $\big($resp., $Q_{cl}(\mathcal{R}(L))\big)$ denote the maximal ring of quotients (resp., the classical ring of quotients) of the ring $\mathcal{R}(L)$ of real-valued continuous functions on a completely regular frame L. These rings have been studied by Abedi [1] and realized as the direct limit of the subrings $\mathcal{R}(\downarrow c)$, where c is a dense element (dense cozero element) of L. Also, Abedi applied these representations of $Q(\mathcal{R}(L))$ and $Q_{cl}(\mathcal{R}(L))$ to describe the equality between different rings of quotients of $\mathcal{R}(L)$.

Let X be a topological space. To study the ring $C(X)$, zero sets and z-ideals play important roles (see [4–6] for more details). The first author [15] studied a generalization of z-ideals in the ring $C(X)$ of continuous real-valued functions on a completely regular Hausdorff space. Salehi [20] introduced multiplicative closed ξ-subsets of the ring $C(X)$ and showed that the ring of fraction $S^{-1}C(X)$ and the

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ring of direct limits of continuous functions on the members of $\mathcal{F}_S$ are isomorphic. In [10, 11, 13, 14], cozeros were used to introduce z-ideals in $\mathcal{R}(L)$ (also, see [3]).
This paper is organized as follows. Section 1 presents the basic concepts and preliminaries, which will be used in the next sections. In Section 2, we introduce the concepts of z-sets, the smallest z-set containing A, and the biggest z-set included in A, where A is a subset of $\mathcal{R}(L)$, and also, we give some properties of them. Section 3 is devoted to the study of the ring of fractions $S_{\mathcal{F}}^{-1}\mathcal{R}(L)$, where $\mathcal{F}$ is a coz-filter on L and $S_{\mathcal{F}}$ is a multiplicatively closed subset related to $\mathcal{F}$. We show that this ring is isomorphic to the direct limits of the subrings $\mathcal{R}(A)$, where $A \in \{\mathfrak{o}_L(\operatorname{coz}(\alpha)) : \alpha \in S_{\mathcal{F}}\}$. In Corollary 3.17, we show that the classical quotient ring of $\mathcal{R}(L)$ is isomorphic to the following direct limit: $\left[\biguplus_{a\in\mathcal{V}} \mathcal{R}(\mathfrak{o}_L(a))\right]$, where $\mathcal{V} = \left\{\operatorname{coz}(\alpha) : \mathfrak{o}_L(\operatorname{coz}(\alpha)) \text{ is a dense sublocal of } L\right\}$. In the last section, we show that $\mathrm{Q_{cl}}\mathcal{R}(L) = \mathcal{R}(L)$, if and only if $\mathcal{R}(L)$ is a special saturated ring.

2. z-Sets in point-free topology

Let A be a subset of $\mathcal{R}(L)$. In this section, we introduce the concepts of z-sets, the smallest z-set containing A, and the biggest z-set included in A and give some properties of them.
For any $\alpha \in \mathcal{R}(L)$, M_α denotes the intersection of all maximal ideals containing α, and $m_\alpha = \left\{\beta \in \mathcal{R}(L) \ ; \ M_\alpha = M_\beta\right\}$.

Definition 2.1. *A subset A of $\mathcal{R}(L)$ is called a z-set, if $M_\alpha = M_\beta$ and $\alpha \in A$ imply $\beta \in A$.*

Lemma 2.2 ([9])**.** *Let Q be an ideal of $\mathcal{R}(L)$. Then*

$$M_Q = \left\{\varphi \in \mathcal{R}(L) \mid r\big(\operatorname{coz}(\varphi)\big) \leq \bigvee_{\alpha\in Q} r\big(\operatorname{coz}(\alpha)\big)\right\}.$$

Hence, for any $\gamma \in \mathcal{R}(L)$, it holds that $M_\gamma = \left\{\varphi \in \mathcal{R}(L) \mid \operatorname{coz}(\varphi) \leq \operatorname{coz}(\gamma)\right\}$.

Lemma 2.3. *Let $\alpha \in \mathcal{R}(L)$. Then $m_\alpha = \left\{\beta \in \mathcal{R}(L) \ ; \ \operatorname{coz}(\alpha) = \operatorname{coz}(\beta)\right\}$.*

Proof. Let $T = \left\{\beta \in \mathcal{R}(L) \ : \ \operatorname{coz}(\alpha) = \operatorname{coz}(\beta)\right\}$, and let $\beta \in m_\alpha$. Then $M_\alpha = M_\beta$ and by Lemma 2.2,

$$\left\{\gamma \in \mathcal{R}(L) : \operatorname{coz}(\gamma) \leq \operatorname{coz}(\alpha)\right\} = \left\{\gamma \in \mathcal{R}(L) : \operatorname{coz}(\gamma) \leq \operatorname{coz}(\beta)\right\}.$$

Therefore, $\operatorname{coz}(\alpha) = \operatorname{coz}(\beta)$ and $\beta \in T$. Now, let $\beta \in T$ and M be a maximal ideal of $\mathcal{R}(L)$ such that $\alpha \in M$. Then $\operatorname{coz}(\alpha) \in \operatorname{Coz}(M)$ and $\operatorname{coz}(\beta) \in \operatorname{Coz}(M)$. Hence $\beta \in \operatorname{Coz}^{\leftarrow}\big(\operatorname{Coz}(M)\big) \subseteq M$, and so $\beta \in m_\alpha$. □

Proposition 2.4. *For any subset A of $\mathcal{R}(L)$, the following conditions are equivalent:*

1. *A is a z-set.*
2. *For any $\alpha, \beta \in \mathcal{R}(L)$, $\alpha \in A$ and $\operatorname{coz}(\alpha) = \operatorname{coz}(\beta)$ imply $\beta \in A$.*

Proof. It is straightforward. □

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Since the union and intersection of any arbitrary family of z-sets of $\mathcal{R}(L)$ are themselves z-sets, therefore for every subset A of $\mathcal{R}(L)$, two sets $\bigcap\{B : B \text{ is a } z\text{-set }, B \supseteq A\}$ and $\bigcup\{B : B \text{ is a } z\text{-set}, B \subseteq A\}$ are z-sets, which are denoted by A_z and A^z, respectively. It is observed that A_z is the *smallest z-set* containing A and that A^z is the *biggest z-set* included in A.

Proposition 2.5. *Let A be a subset of $\mathcal{R}(L)$. Then the following properties hold:*

(1) $A_z = \bigcup_{\alpha\in A} m_\alpha$.
(2) $A^z = \{\alpha \in A \ ; \ m_\alpha \subseteq A\} = \bigcup_{m_\alpha \subseteq A} m_\alpha$.

Proof. (1) Since m_α is a z-set for any $\alpha \in A$, we have $\bigcup_{\alpha\in A} m_\alpha$ is a z-set. Now, we show that $\bigcup_{\alpha\in A} m_\alpha$ is the smallest z-set containing A. It is clear that $A \subseteq \bigcup_{\alpha\in A} m_\alpha$. Now, suppose that B is a z-set such that $A \subseteq B$. Let $\beta \in \bigcup_{\alpha\in A} m_\alpha$. Then there exists an element α in A such that $\beta \in M_\alpha$, and hence $M_\alpha = M_\beta$. Since $\alpha \in B$ and B is a z-set, we have $\beta \in B$. Therefore $\bigcup_{\alpha\in A} m_\alpha \subseteq B$. Thus $\bigcup_{\alpha\in A} m_\alpha = A_z$ and the proof is complete.
(2) First, we show that $J = \{\alpha \in A \ ; \ m_\alpha \subseteq A\}$ is a z-set. To do this, suppose that $M_\beta = M_\varphi$, where $\beta \in J$ and $\varphi \in \mathcal{R}(L)$. So $\beta \in J$ implies that $m_\beta \subseteq A$ and hence $m_\varphi \subseteq A$. Therefore $\varphi \in J$. Thus J is a z-set. Now, we show that J is the biggest z-set included in A. It is clear that $J \subseteq A$, because if $\alpha \in J$, then $m_\alpha \subseteq A$. Indeed $\alpha \in m_\alpha$ implies $\alpha \in A$. Now suppose that K is a z-set such that $K \subseteq A$. Let $\beta \in K$. Since K is a z-set, $m_\beta \subseteq K$. Indeed $K \subseteq A$; therefore $m_\beta \subseteq A$, and so $\beta \in J$. Hence $K \subseteq J$. Thus $J = A^z$. □

We say that a subset S of a commutative ring R is multiplicative closed, when $1 \in S$ and whenever $s_1, s_2 \in S$, then $s_1 s_2 \in S$. Also, a multiplicative closed subset S of a commutative ring R is saturated precisely when $s_1 s_2 \in S$ implies both s_1 and s_2 belong to S. Let S be any multiplicatively closed subset of A. The smallest saturated multiplicatively closed subset containing S is denoted by $\overline{S}$.
For use in the upcoming proof, recall from [2, Lemma 3.2] that

$$M_{\alpha\beta} = M_\alpha \cap M_\beta = M_\alpha M_\beta \text{ for any } \alpha, \beta \in \mathcal{R}(L).$$

Proposition 2.6. *Let S be a multiplicatively closed subset of $\mathcal{R}(L)$. Then*

$$\overline{S}_z = \{\alpha \in \mathcal{R}(L) \ ; \ M_\gamma \subseteq M_\alpha \textit{ for some } \gamma \in S\}.$$

Proof. Put $T = \{\alpha \in \mathcal{R}(L) \ ; \ M_\gamma \subseteq M_\alpha \text{ for some } \gamma \in S\}$. It is easy to see that T is a multiplicatively closed subset of $\mathcal{R}(L)$ and that $S_z \subseteq T$. Now, we show that T is a saturated set. For doing this, suppose that $\alpha\beta \in T$. Then there exists $\gamma \in S$ such that $M_\gamma \subseteq M_{\alpha\beta}$. Therefore $M_\gamma \subseteq M_\alpha$ and $M_\gamma \subseteq M_\beta$. Since $\gamma \in S$, we conclude that $\alpha \in T$ and $\beta \in T$. Thus T is a saturated set.
Finally, we show that T is the smallest saturated multiplicatively closed subset containing S_z. Suppose that K is a saturated multiplicatively closed subset such that $S_z \subseteq K$. Let $\alpha \in T$. Then there exists an element γ in S such that $M_\gamma \subseteq M_\alpha$, and hence $M_{\alpha\gamma} = M_\alpha \cap M_\gamma = M_\gamma$. Then $\alpha\gamma \in S_z$. It follows from $\gamma \in S$ and $S_z \subseteq K$ that $\alpha\gamma \in K$. Now, since K is a saturated set, we have $\alpha \in K$. Therefore $T \subseteq K$. Thus $T = \overline{S}_z$ and the proof is complete. □

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Remark 2.7. *For every multiplicatively closed subset S of $\mathcal{R}(L)$, it holds that $S \subseteq S_z \subseteq \overline{S}_z$.*

Proposition 2.8. *Let S be a multiplicatively closed subset of $\mathcal{R}(L)$. Then*

$$\overline{S}_z = \{\alpha \in \mathcal{R}(L) \ : \ M_\alpha \cap S \neq \emptyset\}.$$

Proof. Suppose that $\alpha \in \overline{S}_z$. Then, by Proposition 2.6, there exists $\gamma \in S$ such that $M_\gamma \subseteq M_\alpha$, implying $\gamma \in M_\alpha$. Consequently, $\gamma \in M_\alpha \cap S$, proving the inclusion $\subseteq$. For The reverse inclusion, pick $\alpha \in \mathcal{R}(L)$ such that $M_\alpha \cap S \neq \emptyset$. Now, choose $\gamma \in M_\alpha \cap S$. Then $M_\gamma \subseteq M_\alpha$ and so, by Proposition 2.6, $\alpha \in \overline{S}_z$. So the reverse inclusion also holds. □

Proposition 2.9. *If S is a multiplicatively closed z-set of $\mathcal{R}(L)$, then $\overline{S}$ is a z-set of $\mathcal{R}(L)$.*

Proof. We know that S is a z-set, which implies that $S_z = S$, and hence $\overline{S} = \overline{S}_z$. Now, we show that $\overline{S}_z$ is a z-set. Suppose that $M_\alpha = M_\beta$, where $\alpha \in \overline{S}_z$ and $\beta \in \mathcal{R}(L)$. Then, there exists an element $\gamma \in S$ such that $M_\gamma \subseteq M_\alpha$. Therefore, $M_\gamma \subseteq M_\beta$ and $\gamma \in S$. Hence $\beta \in \overline{S}_z$. □

Corollary 2.10. *Let S be a multiplicatively closed subset of $\mathcal{R}(L)$. Then the following conditions are equivalent:*

(1) *S is a saturated z-set.*
(2) *If $\alpha \in \mathcal{R}(L)$ and $M_\alpha \cap S \neq \emptyset$, then $\alpha \in S$.*
(3) *If $\alpha \in \mathcal{R}(L)$, $\gamma \in S$, and $\gamma \in M_\alpha$, then $\alpha \in S$.*
(4) *If $\alpha \in \mathcal{R}(L)$, $\gamma \in S$, and $M_\gamma \subseteq M_\alpha$, then $\alpha \in S$.*

Proof. (1) $\Rightarrow$ (2) Since S is a saturated z-set, $S = \overline{S} = \overline{S}_z$. Now, suppose that $\alpha \in \mathcal{R}(L)$ and $M_\alpha \cap S \neq \emptyset$. So, by Proposition 2.8, $\alpha \in \overline{S}_z$, and hence $\alpha \in S$.
(2) $\Rightarrow$ (3) It is immediate that (2) implies (3).
(3) $\Rightarrow$ (4) Let $\alpha \in \mathcal{R}(L)$, let $\gamma \in S$, and let $M_\gamma \subseteq M_\alpha$. Then $\gamma \in M_\gamma \subseteq M_\alpha$, and so by (3), $\alpha \in S$.
(4) $\Rightarrow$ (1) First, we show that S is a z-set. Suppose that $M_\alpha = M_\beta$, where $\alpha \in S$ and $\beta \in \mathcal{R}(L)$. Then, by (4), $\beta \in S$. Thus S is a z-set. Now, we show that S is saturated. Suppose that $\alpha\beta \in S$, where $\alpha, \beta \in \mathcal{R}(L)$. Then

$$M_{\alpha\beta} = M_\alpha \cap M_\beta \subseteq M_\alpha,$$

and so by (4), $\alpha \in S$. In a similar way, $\beta \in S$. □

Definition 2.11. *If $\mathcal{F}$ is a subset of $\mathrm{Coz}(L)$ and $\mathcal{F}$ is a filter, then it is called a* coz*-filter.*

Lemma 2.12. *Let $\mathcal{F}$ be a* coz*-filter on $\mathrm{Coz}(L)$ or L. Then the following statements are true:*

(1) *$\mathrm{Coz}^{\leftarrow}(\mathcal{F})$ is a z-set.*
(2) *$\mathrm{Coz}^{\leftarrow}(\mathcal{F})$ is a multiplicatively closed subset of $\mathcal{R}(L)$.*

Proof. (1) Suppose that $\mathrm{coz}(\alpha) = \mathrm{coz}(\beta)$, where $\alpha \in \mathrm{Coz}^{\leftarrow}(\mathcal{F})$. Since $\alpha \in \mathrm{Coz}^{\leftarrow}(\mathcal{F})$, we have $\mathrm{coz}(\alpha) \in \mathcal{F}$, and hence $\mathrm{coz}(\beta) \in \mathcal{F}$. Therefore $\beta \in \mathrm{Coz}^{\leftarrow}(\mathcal{F})$. Thus by Proposition 2.4, $\mathrm{Coz}^{\leftarrow}(\mathcal{F})$ is a z-set.
(2) Suppose that $\alpha, \beta \in \mathrm{Coz}^{\leftarrow}(\mathcal{F})$. Then $\mathrm{coz}(\alpha), \mathrm{coz}(\beta) \in \mathcal{F}$. Since $\mathrm{coz}(\alpha\beta) = \mathrm{coz}(\alpha) \wedge \mathrm{coz}(\beta)$ and $\mathcal{F}$ is a coz-filter, we have $\mathrm{coz}(\alpha\beta) \in \mathcal{F}$, and hence $\alpha\beta \in \mathrm{Coz}^{\leftarrow}(\mathcal{F})$. Therefore $\mathrm{Coz}^{\leftarrow}(\mathcal{F})$ is a multiplicatively closed subset of $\mathcal{R}(L)$. □

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We put $S_{\mathcal{F}} = \mathrm{Coz}^{\leftarrow}(\mathcal{F})$. The set $S_{\mathcal{F}}$ is called *multiplicatively closed set related to* $\mathcal{F}$. Throughout the present paper,

$$Cozf[S] := \{\mathrm{coz}(\beta) \ : \ \mathrm{coz}(\alpha) \leq \mathrm{coz}(\beta), \text{for some } \alpha \in S\},$$

for every multiplicatively closed subset S of $\mathcal{R}(L)$.

Proposition 2.13. *Let S be a multiplicatively closed subset of $\mathcal{R}(L)$. Then $Cozf[S]$ is a* coz*-filter on* $\mathrm{Coz}(L)$.

Proof. To show our result, we must verify the three axioms of a filter.
(1) It is clear that for every $\alpha \in S$, $\mathrm{coz}(\alpha) \leq \top = \mathrm{coz}(\mathbf{1})$. Then $\top = \mathrm{coz}(\mathbf{1}) \in Cozf[S]$.
(2) If $\mathrm{coz}(\beta_1), \mathrm{coz}(\beta_2) \in Cozf[S]$, then $\mathrm{coz}(\alpha_1) \leq \mathrm{coz}(\beta_1)$ and $\mathrm{coz}(\alpha_2) \leq \mathrm{coz}(\beta_2)$ for some $\alpha_1, \alpha_2 \in S$. Since S is multiplicatively subset of $\mathcal{R}(L)$, $\alpha_1\alpha_2 \in S$. Then

$$\mathrm{coz}(\alpha_1\alpha_2) = \mathrm{coz}(\alpha_1) \wedge \mathrm{coz}(\alpha_2) \leq \mathrm{coz}(\beta_1) \wedge \mathrm{coz}(\beta_2),$$

and so $\mathrm{coz}(\beta_1) \wedge \mathrm{coz}(\beta_2) \in Cozf[S]$.
(3) Suppose that $\mathrm{coz}(\beta) \leq \mathrm{coz}(\gamma)$ for some $\mathrm{coz}(\beta) \in Cozf[S]$, and $\mathrm{coz}(\gamma) \in \mathrm{Coz}(L)$. Since $\mathrm{coz}(\beta) \in Cozf[S]$, there exists $\alpha \in S$ such that $\mathrm{coz}(\alpha) \leq \mathrm{coz}(\beta)$. Hence, $\mathrm{coz}(\alpha) \leq \mathrm{coz}(\gamma)$, and so $\mathrm{coz}(\gamma) \in Cozf[S]$. □

Lemma 2.14. *Let S be a multiplicatively closed subset of $\mathcal{R}(L)$. Then $Cozf[S] = Cozf[\bar{S}]$.*

Proof. By Corollary 2.10 the proof is straightforward. □

Proposition 2.15. *Let S be a multiplicatively closed subset of $\mathcal{R}(L)$ and let $\mathcal{F}$ be a filter on L. Then the following statements are true:*

(1) $S = \mathrm{Coz}^{\leftarrow}(Cozf[S])$, *if and only if S is a saturated z-set.*
(2) $\mathcal{F} = Cozf[\mathrm{Coz}^{\leftarrow}(\mathcal{F})]$ *if and only if $\mathcal{F}$ is a* coz*-filter on* $\mathrm{Coz}(L)$.

Proof. (1) *Necessity.* Suppose that $S = \mathrm{Coz}^{\leftarrow}(Cozf[S])$. By Lemma 2.12 and Proposition 2.13, S is a multiplicatively closed z-set. Now, let $\alpha\beta \in S$, where $\alpha, \beta \in \mathcal{R}(L)$. Since $\alpha\beta \in S$, we conclude that there exists $\gamma \in S$ such that $\mathrm{coz}(\gamma) \leq \mathrm{coz}(\alpha\beta)$. Therefore, $\mathrm{coz}(\gamma) \leq \mathrm{coz}(\alpha)$ and $\mathrm{coz}(\gamma) \leq \mathrm{coz}(\beta)$. Then $\mathrm{coz}(\alpha), \mathrm{coz}(\beta) \in Cozf[S]$ and so $\alpha, \beta \in \mathrm{Coz}^{\leftarrow}(Cozf[S]) = S$.
Sufficiency. Let S be a saturated z-set. It is immediate that $S \subseteq \mathrm{Coz}^{\leftarrow}(Cozf[S])$. Conversely, let $\alpha \in \mathrm{Coz}^{\leftarrow}(Cozf[S])$. Then $\mathrm{coz}(\alpha) \in Cozf[S]$, and hence there exists $\gamma \in S$ such that $\mathrm{coz}(\gamma) \leq \mathrm{coz}(\alpha)$. Since S is a saturated z-sets, we deduce from Corollary 2.10 that $\alpha \in S$. Therefore $\mathrm{Coz}^{\leftarrow}(Cozf[S]) \subseteq S$.

(2) *Necessity.* It follows from Lemma 2.12 and Proposition 2.13.
Sufficiency. Let $\mathcal{F}$ be a coz-filter on $\mathrm{Coz}(L)$. Suppose that $\alpha \in \mathcal{R}(L)$ such that $\mathrm{coz}(\alpha) \in \mathcal{F}$. Then $\alpha \in \mathrm{Coz}^{\leftarrow}(\mathcal{F})$. Cleary $\mathrm{coz}(\alpha) \in Cozf[\mathrm{Coz}^{\leftarrow}(\mathcal{F})]$, proving $\mathcal{F} \subseteq Cozf[\mathrm{Coz}^{\leftarrow}(\mathcal{F})]$. Conversely, let $\mathrm{coz}(\beta) \in Cozf[\mathrm{Coz}^{\leftarrow}(\mathcal{F})]$. Then, $\mathrm{coz}(\gamma) \leq \mathrm{coz}(\beta)$ for some $\gamma \in \mathrm{Coz}^{\leftarrow}(\mathcal{F})$. Since $\mathcal{F}$ is a coz-filter on $\mathrm{Coz}(L)$ and $\mathrm{coz}(\gamma) \leq \mathrm{coz}(\beta)$, we conclude that $\mathrm{coz}(\beta) \in \mathcal{F}$ and so $Cozf[\mathrm{Coz}^{\leftarrow}(\mathcal{F})] \subseteq \mathcal{F}$. □

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3. Rings of quotients of the ring $\mathcal{R}(L)$ by coz-filters

This section is devoted to the study of the ring of fractions $S_{\mathcal{F}}^{-1}\mathcal{R}(L)$, where $\mathcal{F}$ is a coz-filter on L and $S_{\mathcal{F}}$ is a multiplicatively closed subset related to $\mathcal{F}$. We show that this ring is isomorphic to the direct limits of the subrings $\mathcal{R}(A)$, where $A \in \{\mathfrak{o}_L\big(\mathrm{coz}(\alpha)\big) \colon \alpha \in S_{\mathcal{F}}\}$.

Let X be a topological space and let $\mathcal{S}$ be a family of nonvoid subsets of X. When $\mathcal{S}$ is a filter base (i.e., when $\mathcal{S}$ is closed under finite intersection), we are invited to consider the direct limit ring $\varinjlim_{S\in\mathcal{S}} C(S)$, with respect to the restriction homomorphisms $f \to f|_{S'}$, where $f \in C(S)$ and $S' \subseteq S$. When $\mathcal{S}$ is a family of dense sets, all these homomorphisms are one-one, and $\varinjlim_{S\in\mathcal{S}} C(S)$ may be considered as $\bigcup_{S\in\mathcal{S}} C(S)$, where we identify $f_1 \in C(S_1)$ with $f_2 \in C(S_2)$ whenever f_1 and f_2 agree on $S_1 \cap S_2$ (see [17]).

Now, let $\mathcal{S}$ be a family of sublocales of L that is closed under finite intersection. It is evident that $(\mathcal{S}, \supseteq)$ is a partially ordered set. For every $A, B \in \mathcal{S}$ with $A \leq B$ ($B \subseteq A$), we define

$$\varphi_{AB}(\alpha \to \nu_{B}\alpha) \colon \mathcal{R}(A) \to \mathcal{R}(B),$$

where

$$\nu_{B}\alpha(v) = \bigwedge\{b \in B \colon \alpha(v) \leq b\},$$

for every $v \in \mathcal{L}(\mathbb{R})$. Then $(A, \varphi_{AB})_{A,B\in\mathcal{S}}$ is a direct system over $\mathcal{S}$ in the category of rings. We define the equivalence relation $\sim$ on $\biguplus_{A\in\mathcal{S}} \mathcal{R}(A)$ by

$$\alpha \sim \beta \Leftrightarrow \nu_{A\cap B}\alpha = \nu_{A\cap B}\beta \text{ for all } (\alpha, \beta) \in \mathcal{R}(A) \times \mathcal{R}(B) \text{ and every } A, B \in \mathcal{S}.$$

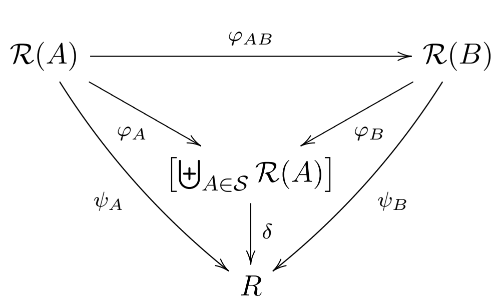
The equivalence relation $\sim$ on $\biguplus_{A\in\mathcal{S}} \mathcal{R}(A)$ induces the ring $\biguplus_{A\in\mathcal{S}} \mathcal{R}(A)/\sim$, which is denoted by $\big[\biguplus_{A\in\mathcal{S}} \mathcal{R}(A)\big]$. Let $A \in \mathcal{S}$. We define

$$\varphi_A(\alpha \to [\alpha]) \colon \mathcal{R}(A) \to \left[\biguplus_{A\in\mathcal{S}} \mathcal{R}(A)\right].$$

Then $\varphi_A = \varphi_B\varphi_{AB}$ for every $A, B \in \mathcal{S}$ with $A \leq B$. We claim that $(\big[\biguplus_{A\in\mathcal{S}} \mathcal{R}(A)\big], \varphi_A)_{A\in\mathcal{S}}$ is a direct limit of the direct system $(A, \varphi_{AB})_{A,B\in\mathcal{S}}$ in the category of rings. Let R be a ring, and let there be given morphisms $\psi_A \colon \mathcal{R}(A) \to R$ satisfying $\psi_B\varphi_{AB} = \psi_A$ for every $A, B \in \mathcal{S}$ with $A \leq B$. We define

$$\delta([\alpha] \to \psi_A(\alpha)) \colon \left[\biguplus_{A\in\mathcal{S}} \mathcal{R}(A)\right] \to R$$

for all $\alpha \in \mathcal{R}(A)$ and every $A \in \mathcal{S}$. Then the following diagram commutes for all $A, B \in \mathcal{S}$ with $A \leq B$:



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Hence, $\left([\biguplus_{A\in\mathcal{S}}\mathcal{R}(A)], \varphi_A\right)_{A\in\mathcal{S}}$ is the direct limit of the direct system $(A,\varphi_{AB})_{A,B\in\mathcal{S}}$ over $\mathcal{S}$.
By the above explanation, we have the following proposition

Proposition 3.1. *Let $\mathcal{S}$ be a family of sublocales of L that is closed under finite intersection. Then $\left([\biguplus_{A\in\mathcal{S}}\mathcal{R}(A)], \varphi_A\right)_{A\in\mathcal{S}}$ is the direct limit of the direct system $(A,\varphi_{AB})_{A,B\in\mathcal{S}}$ over $\mathcal{S}$.*

Corollary 3.2. *If $\mathcal{D}_o(L)$ is the family of all dense open sublocales in L, then $\mathcal{D}_o(L)$ is closed under finite intersection and*

$$\varinjlim_{A\in\mathcal{D}_o(L)}\mathcal{R}(A)=\left(\left[\biguplus_{A\in\mathcal{D}_o(L)}\mathcal{R}(A)\right],\varphi_A\right)_{A\in\mathcal{S}}.$$

Let D be an ideal of a commutative ring A. We recall from [17] that D is (rationally) dense in A provided that $\operatorname{Ann}(D)=(0)$.

Proposition 3.3. *An ideal D in $\mathcal{R}(L)$ (or $\mathcal{R}^*(L)$) is dense, if and only if $\bigvee_{\alpha\in D}\mathfrak{o}_L\big(\operatorname{coz}(\alpha)\big)$ is dense in L.*

Proof. Since

$$\begin{aligned}\beta\in\operatorname{Ann}(D)&\Leftrightarrow \text{ for all } \alpha\in D(\alpha\beta=\mathbf{0})\\&\Leftrightarrow \bot=\bigvee_{\alpha\in D}\operatorname{coz}(\alpha\beta)=\operatorname{coz}(\beta)\wedge\bigvee_{\alpha\in D}\operatorname{coz}(\alpha)\\&\Leftrightarrow \mathsf{O}=\mathfrak{o}_L\big(\operatorname{coz}(\beta)\big)\wedge\mathfrak{o}_L\big(\bigvee_{\alpha\in D}\operatorname{coz}(\alpha)\big)\\&\Leftrightarrow \mathsf{O}=\mathfrak{o}_L\big(\operatorname{coz}(\beta)\big)\wedge\bigvee_{\alpha\in D}\mathfrak{o}_L\big(\operatorname{coz}(\alpha)\big),\end{aligned}$$

and L is completely regular, then D in $\mathcal{R}(L)$ is dense, if and only if $\bigvee_{\alpha\in D}\mathfrak{o}_L\big(\operatorname{coz}(\alpha)\big)$ is dense in L. □

Proposition 3.4. *For every $a,b\in L$, if $b\notin\mathfrak{c}_L(a)$, then there exists $\alpha\in\mathcal{R}^*(L)$ such that $b\notin\mathfrak{c}_L(\operatorname{coz}(\alpha))$ and $\mathfrak{c}_L(a)\subseteq\operatorname{int}_L(\mathfrak{c}_L(\operatorname{coz}(\alpha)))$.*

Proof. Since L is a completely regular locale and $a\not\leq b$, we conclude that there exists $\alpha\in\mathcal{R}^*(L)$ such that $\operatorname{coz}(\alpha)\not\leq b$ and $\operatorname{coz}(\alpha)\prec\!\!\prec a$, which implies that $b\notin\mathfrak{c}_L\big(\operatorname{coz}(\alpha)\big)$ and

$$\operatorname{coz}(\alpha)\prec a\Rightarrow\operatorname{coz}(\alpha)^*\vee a=\top\Rightarrow\mathfrak{c}_L(a)\subseteq\mathfrak{o}_L\big(\operatorname{coz}(\alpha)^*\big)=\operatorname{int}_L(\mathfrak{c}_L(\operatorname{coz}(\alpha))).$$

□

Abedi [1] showed that $\mathcal{R}(L)$ is a ring of quotients of $\mathcal{R}^*(L)$ and that $\mathcal{R}(L)$ and $\mathcal{R}^*(L)$ have the same maximal ring of quotients and the same classical ring of quotients.

Proposition 3.5. *Let A be a dense sublocale of L. Then $\mathcal{R}(A)$ is a ring of quotients of $\nu_A\big(\mathcal{R}(L)\big):=\{\nu_A\alpha\colon\alpha\in\mathcal{R}(L)\}$.*

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Proof. Let $\mathbf{0} \neq \alpha \in \mathcal{R}(A)$ be given. Then there exists $\top \neq b \in \mathfrak{o}_L\big(\mathrm{coz}(\alpha)\big) \cap A$, and so, by Proposition 3.4, there exists $\beta \in \mathcal{R}^*(L)$ such that $\mathfrak{c}_L\big(\mathrm{coz}(\alpha)\big) \subseteq \mathrm{int}_L\Big(\mathfrak{c}_L\big(\mathrm{coz}(\beta)\big)\Big)$ and $b \notin \mathfrak{c}_L\big(\mathrm{coz}(\beta)\big)$. Since $b \in \mathfrak{o}_L\big(\mathrm{coz}(\alpha)\big) \setminus \mathfrak{c}_L\big(\mathrm{coz}(\beta)\big)$, we conclude that

$$\top \neq \big(\mathrm{coz}(\beta) \longrightarrow b\big) \in \mathfrak{o}_L\big(\mathrm{coz}(\alpha)\big) \cap \mathfrak{o}_L\big(\mathrm{coz}(\beta)\big) \neq \mathsf{O}.$$

Now, we claim that $\alpha\nu_A\beta \neq \mathbf{0}$. Suppose, by way of contradiction , that $\alpha\nu_A\beta = \mathbf{0}$. Then

$$\mathrm{coz}(\alpha) \wedge \mathrm{coz}(\beta) \leq \mathrm{coz}(\alpha) \wedge \mathrm{coz}(\nu_A\beta) = \bot_A = \bot_L \Rightarrow \mathfrak{o}_L\big(\mathrm{coz}(\alpha)\big) \cap \mathfrak{o}_L\big(\mathrm{coz}(\beta)\big) = \mathsf{O},$$

which is a contradiction, and so, $\alpha\nu_A\beta \neq \mathbf{0}$. To complete the proof, it suffices to show that $\alpha\nu_A\beta \in \nu_A(\mathcal{R}(L))$. For this, define $\gamma(p,q) = \alpha\nu_A\beta(p,q)$ for each $p,q \in \mathbb{Q}$. Then $\gamma \in \mathcal{R}(L)$ and $\nu_A\gamma = \alpha\nu_A\beta$. Therefore $\alpha\nu_A\beta \in \nu_A(\mathcal{R}(L))$ and so $\mathcal{R}(L)$ is a ring of quotients of $\nu_A(\mathcal{R}(L))$. □

Lemma 3.6. *If* $\alpha\beta = \mathbf{0}$*, then* $\nu_{\mathfrak{o}_L(\mathrm{coz}(\alpha))}\beta = \mathbf{0}$ *for every* $\alpha, \beta \in \mathcal{R}(L)$.

Proof. Suppose that $\alpha, \beta \in \mathcal{R}(L)$ and $\alpha\beta = \mathbf{0}$. Then $\mathrm{coz}(\alpha) \wedge \mathrm{coz}(\beta) = \bot$, which implies that $\mathfrak{o}_L\big(\mathrm{coz}(\alpha)\big) \subseteq \mathfrak{c}_L\big(\mathrm{coz}(\beta)\big)$. Hence

$$\begin{aligned}\mathrm{coz}\big(\nu_{\mathfrak{o}_L(\mathrm{coz}(\alpha))}\beta\big) &= \nu_{\mathfrak{o}_L(\mathrm{coz}(\alpha))}\big(\mathrm{coz}(\beta)\big)\\ &= \bigwedge \mathfrak{o}_L\big(\mathrm{coz}(\alpha)\big) \cap \mathfrak{c}_L\big(\mathrm{coz}(\beta)\big)\\ &= \bigwedge \mathfrak{o}_L\big(\mathrm{coz}(\alpha)\big)\\ &= \bot.\end{aligned}$$

Therefore, $\nu_{\mathfrak{o}_L(\mathrm{coz}(\alpha))}\beta = \mathbf{0}$. □

Let $\mathcal{F} \subseteq \mathrm{Coz}(L)$ be a coz-filter on L. Then by Lemma 2.12, $S_{\mathcal{F}}$ is a multiplicatively closed subset of $\mathcal{R}(L)$. Let $S_{\mathcal{F}}^{-1}\mathcal{R}(L)$ be the ring of fractions $\mathcal{R}(L)$ with respect to the multiplicatively closed subset $S_{\mathcal{F}}$.

Lemma 3.7. *If* $\mathcal{F} \subseteq \mathrm{Coz}(L)$ *is a coz-filter on* L*, then* $\frac{\alpha}{\beta} \in S_{\mathcal{F}}^{-1}\mathcal{R}(L)$*, if and only if* $\alpha \in \mathcal{R}(L)$ *and* $\mathrm{coz}(\beta) \in \mathcal{F}$.

Proof. The proof is straightforward. □

Proposition 3.8. *Let* $\mathcal{F} \subseteq \mathrm{Coz}(L)$ *be a coz-filter on* L *and let* $\frac{f}{g}, \frac{h}{k} \in S_{\mathcal{F}}^{-1}\mathcal{R}(L)$*. Then*

$$\frac{f}{g} = \frac{h}{k} \Leftrightarrow \textit{there exists } \mathrm{coz}(\alpha) \in \mathcal{F} : \nu_{\mathfrak{o}_L(\mathrm{coz}(\alpha))}(fk - gh) = \mathbf{0}.$$

Proof. Necessity. Let $\frac{f}{g}, \frac{h}{k} \in S_{\mathcal{F}}^{-1}\mathcal{R}(L)$ and $\frac{f}{g} = \frac{h}{k}$. Then, there exists $\alpha \in S_{\mathcal{F}}$ such that $\alpha(fk-gh) = \mathbf{0}$. Since $\alpha \in S_{\mathcal{F}}$, so $\mathrm{coz}(\alpha) \in \mathcal{F}$, and by Lemma 3.6, $\nu_{\mathfrak{o}_L(\mathrm{coz}(\alpha))}(fk - gh) = \mathbf{0}$.

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Sufficiency. We set $A := \mathfrak{o}_L\big(\mathrm{coz}(\alpha)\big)$. Then we obtain

$$\begin{aligned}
\nu_{_A}(fk-gh)=\mathbf{0} &\Rightarrow \mathrm{coz}\Big(\nu_{_A}(fk-gh)\Big)=\bot_A\\
&\Rightarrow \nu_{_A}\big(\mathrm{coz}(fk-gh)\big)=\bigwedge A\\
&\Rightarrow \bigwedge A\cap \mathfrak{c}_L\big(\mathrm{coz}(fk-gh)\big)=\bigwedge A\\
&\Rightarrow cl_L\Big(A\cap \mathfrak{c}_L\big(\mathrm{coz}(fk-gh)\big)\Big)=cl_L A\\
&\Rightarrow \mathfrak{c}_L\Big(\bigwedge\big(A\cap \mathfrak{c}_L(\mathrm{coz}(fk-gh))\big)\Big)=cl_L\big(\mathfrak{o}_L\big(\mathrm{coz}(\alpha)\big)\big)\\
&\Rightarrow \mathfrak{c}_L\Big(\nu_A\big(\mathrm{coz}(fk-gh)\big)\Big)=\mathfrak{c}_L\Big(\big(\mathrm{coz}(\alpha)\big)^*\Big)\\
&\Rightarrow \mathfrak{c}_L\big(\mathrm{coz}(\alpha)\to \mathrm{coz}(fk-gh)\big)=\mathfrak{c}_L\big((\mathrm{coz}(\alpha))^*\big)\\
&\Rightarrow \big(\mathrm{coz}(\alpha)\to \mathrm{coz}(fk-gh)\big)=\big(\mathrm{coz}(\alpha)\big)^*\\
&\Rightarrow \big(\mathrm{coz}(\alpha)\to \mathrm{coz}(fk-gh)\big)\wedge \mathrm{coz}(\alpha)=\big(\mathrm{coz}(\alpha)\big)^*\wedge \mathrm{coz}(\alpha)\\
&\Rightarrow \mathrm{coz}(\alpha)\wedge \mathrm{coz}(fk-gh)=\bot\\
&\Rightarrow \mathrm{coz}\big(\alpha(fk-gh)\big)=\bot\\
&\Rightarrow \alpha(fk-gh)=\mathbf{0}.
\end{aligned}$$

Therefore, $\frac{f}{g}=\frac{h}{k}$. □

Lemma 3.9. *Let $\mathcal{F}\subseteq \mathrm{Coz}(L)$ be a coz-filter on L and let $\beta\in\mathcal{R}(L)$. Then $[\beta]=[\nu_{\mathfrak{o}_L(\mathrm{coz}(\beta))}\beta]$.*

Proof. Let $\beta\in\mathcal{R}(L)$ and set $A:=\mathfrak{o}_L\big(\mathrm{coz}(\beta)\big)$. Then $\nu_{_A}\beta\in\mathcal{R}(A)$ and $\nu_{_{A\cap L}}\nu_{_A}\beta=\nu_{_{A\cap L}}\beta$, which means that $\nu_{_A}\beta\sim\beta$, and so $[\beta]=[\nu_{_A}\beta]$. □

Proposition 3.10. *Let $\mathcal{F}\subseteq\mathrm{Coz}(L)$ be a coz-filter on L, $\beta\in S_{\mathcal{F}}$ and $\mathcal{O}_L(S_{\mathcal{F}}):=\big\{\mathfrak{o}_L\big(\mathrm{coz}(\alpha)\big)\colon \alpha\in S_{\mathcal{F}}\}$. Then $[\beta]$ is invertible in $\Big[\biguplus_{A\in\mathcal{O}_L(S_{\mathcal{F}})}\mathcal{R}(A)\Big]$.*

Proof. We set $A:=\mathfrak{o}_L\big(\mathrm{coz}(\beta)\big)$. First we show that $\nu_{_A}\beta$ is invertible in $\mathcal{R}(A)$. We have

$$\begin{aligned}
\mathrm{coz}\big(\nu_{_A}\beta\big)&=\nu_{_A}\big(\mathrm{coz}(\beta)\big)\\
&=\bigwedge\Big(A\cap\mathfrak{c}_L\big(\mathrm{coz}(\beta)\big)\Big)\\
&=\nu_A\big(\mathrm{coz}(\beta)\big)\\
&=\nu_{\mathfrak{o}_L\left(\mathrm{coz}(\beta)\right)}\big(\mathrm{coz}(\beta)\big)\\
&=\mathrm{coz}(\beta)\longrightarrow\mathrm{coz}(\beta)\\
&=\top.
\end{aligned}$$

Hence $\nu_{_A}\beta$ is invertible in $\mathcal{R}(A)$. Then, there exists $\gamma\in\mathcal{R}(A)$ such that $\gamma\nu_{_A}\beta=\mathbf{1}$. Therefore by Lemma 3.9,

$$[\mathbf{1}]=[\gamma\nu_{_A}\beta]=[\gamma][\nu_{_A}\beta]=[\gamma][\beta],$$

which means that $[\beta]$ is invertible in $\Big[\biguplus_{A\in\mathcal{O}_L(S_{\mathcal{F}})}\mathcal{R}(A)\Big]$. □

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Lemma 3.11. *Let $A \in \mathcal{S}\ell(L)$ be given. Then, for*

$$\Psi(\alpha \to \nu_A \alpha) \colon \mathcal{R}(L) \to \mathcal{R}(A),$$

the following statements are true:

(1) *Ψ is an f-ring homomorphism.*

(2) *If A is a dense sublocale of L, then Ψ is an f-ring monomorphism.*

Proof. (1). This is evident that Ψ is a function. Let $\diamond \in \{+, ., \wedge, \vee\}$ be given. Suppose that $p, q \in \mathbb{Q}$ and $\alpha, \beta \in \mathcal{R}(L)$. Then

$$\begin{aligned}\nu_A(\alpha \diamond \beta)(p,q) &= \nu_A\Big(\bigvee\{\alpha(r,s) \wedge \beta(u,v) \colon < r,s > \diamond < u,v > \subseteq < p,q > \}\Big)\\ &= \bigvee\{\nu_A\big(\alpha(r,s)\big) \wedge \nu_A\big(\beta(u,v)\big) \colon < r,s > \diamond < u,v > \subseteq < p,q > \}\\ &= (\nu_A\alpha \diamond \nu_A\beta)(p,q)\end{aligned}$$

Hence, Ψ is an f-ring homomorphism.

(2). Let $\alpha \in \mathcal{R}(L)$ with $\Psi(\alpha) = \mathbf{0}$ be given. Since A is a dense sublocale of L and ν_A is a nucleus, we conclude that

$$\mathrm{coz}(\alpha) \leq \nu_A\big(\mathrm{coz}(\alpha)\big) = \mathrm{coz}(\nu_A\alpha) = \mathrm{coz}\big(\Psi(\alpha)\big) = \mathrm{coz}(\mathbf{0}) = \bot_A = \bot_L,$$

and so $\alpha = \mathbf{0}$. Therefore, Ψ is an f-ring monomorphism. □

Let A be a sublocale of L. Consider homomorphisms $\Psi \colon \mathcal{R}(L) \to \mathcal{R}(A)$ and $\varphi_A \colon \mathcal{R}(A) \to \Big[\biguplus_{A\in\mathcal{O}(S_\mathcal{F})}\mathcal{R}(A)\Big]$. We put $\theta = \varphi_A\Psi$. Then θ is a homomorphism from $\mathcal{R}(L)$ to $\Big[\biguplus_{A\in\mathcal{O}(S_\mathcal{F})}\mathcal{R}(A)\Big]$, where $\theta(\alpha) = [\nu_A\alpha]$ for every $\alpha \in \mathcal{R}(L)$. Also, Consider the canonical homomorphism $\varphi \colon \mathcal{R}(L) \to S_\mathcal{F}^{-1}\mathcal{R}(L)$ such that $\varphi(\alpha) = \dfrac{\alpha}{\mathbf{1}}$.

Remark 3.12 ([21])**.** *Let S be a multiplicatively closed subset of the commutative ring R; also, let $f \colon R \to S^{-1}R$ denote the natural ring homomorphism. Let R' be a second commutative ring, and let $g \colon R \to R'$ be a ring homomorphism with the property that $g(s)$ is a unit of R' for all $s \in S$. Then there is a unique ring homomorphism $h \colon S^{-1}R \to R'$ such that $h \circ f = g$. In fact, $h\left(\frac{a}{s}\right) = g(a)(g(s))^{-1}$ for all $a \in R$, $s \in S$.*

Proposition 3.13. *Let $\mathcal{F} \subseteq \mathrm{Coz}(L)$ be a coz-filter on L. Then, there exists a unique homomorphism π from $S_\mathcal{F}^{-1}\mathcal{R}(L)$ to $\Big[\biguplus_{A\in\mathcal{O}(S_\mathcal{F})}\mathcal{R}(A)\Big]$ such that $\pi \circ \varphi = \theta$. In fact, $\pi\left(\frac{\alpha}{\beta}\right) = \theta(\alpha)\big(\theta(\beta)\big)^{-1} = [\alpha][\beta]^{-1}$ for all $\alpha, \beta \in \mathcal{R}(L)$, where $\mathrm{coz}(\beta) \in \mathcal{F}$.*

Proof. Combine Proposition 3.10 and Remark 3.12. □

Lemma 3.14. *Let $\beta \in \mathcal{R}(L)$, $B := \mathfrak{c}_L\big(\mathrm{coz}(\beta)\big)$, $C := \mathfrak{o}_L\big(\mathrm{coz}(\beta)\big)$, and $D := \mathrm{cl}_L C = \mathfrak{c}_L\big(\mathrm{coz}(\beta)^*\big)$ be given. If $\alpha \in \mathcal{R}^*(C)$, then there exists $\gamma \in \mathcal{R}(L)$ such that for all $u \in \mathcal{L}(\mathbb{R})$, $\gamma(u) \vee \mathrm{coz}(\beta)^* = \alpha\nu_C\beta(u)$ and $\gamma(u) \vee \mathrm{coz}(\beta) = \mathbf{0}(u)$, where $\mathbf{0} \in \mathcal{R}(B)$.*

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Proof. Let $\gamma_1 := \mathbf{0} \in \mathcal{R}(B)$ be given. We put $\gamma_2 := \alpha\nu_{_C}\beta \in \mathcal{R}(D)$. We set $a := \mathrm{coz}(\beta) \vee \mathrm{coz}(\beta)^*$ and define

$$\delta_i\big((u) \mapsto \gamma_i(u) \vee a\big) \colon \mathcal{L}(\mathbb{R}) \to \mathfrak{c}_L(a),$$

for every $i = 1, 2$. We must show that $\delta_1(u) = \delta_2(u)$ for every $u \in \mathcal{L}(\mathbb{R})$. Consider the following four cases:

Case 1: If $q \leq 0$, then

$$\delta_2(-, q) = \alpha\nu_{_C}\beta(-, q) \vee a \geq a = a \vee \mathrm{coz}(\beta) = a \vee \bot_B = a \vee \gamma_1(-, q) = \delta_1(-, q).$$

Case 2: Let $q > 0$. Consider $\epsilon > 0$. Then

$$\alpha\nu_{_C}\beta(-, q) = \bigvee\Big\{\alpha(r, s) \wedge \nu_{_C}\beta(u, v) \colon\ < r, s >< u, v > \subseteq (-\infty, q)\Big\} \geq \alpha(-\epsilon, \epsilon) \wedge \nu_{_C}\beta\left(\frac{-q}{\epsilon}, \frac{q}{\epsilon}\right).$$

Now,

$$\begin{aligned} \delta_2(-, q) &\geq a \vee \left(\alpha(-\epsilon, \epsilon) \wedge \nu_{_C}\beta\left(\frac{-q}{\epsilon}, \frac{q}{\epsilon}\right)\right) \\ &= \big(a \vee \alpha(-\epsilon, \epsilon)\big) \wedge \left(a \vee \nu_{_C}\beta\left(\frac{-q}{\epsilon}, \frac{q}{\epsilon}\right)\right) \\ &\geq \big(a \vee \alpha(-\epsilon, \epsilon)\big) \wedge \left(\mathrm{coz}(\beta) \vee \mathrm{coz}(\beta)^* \vee \beta\left(\frac{-q}{\epsilon}, \frac{q}{\epsilon}\right)\right) \\ &= a \vee \alpha(-\epsilon, \epsilon). \end{aligned}$$

Then

$$\delta_2(-, q) \geq \bigvee_{\epsilon>0} a \vee \alpha(-\epsilon, \epsilon) = a \vee \bigvee_{\epsilon>0} \alpha(-\epsilon, \epsilon) = a \vee \alpha\big(\bigvee_{\epsilon>0}(-\epsilon, \epsilon)\big) = \top,$$

which implies that

$$\delta_2(-, q) = \top = \mathbf{0}(-, q) \vee a = \delta_1(-, q).$$

Case 3: If $p \geq 0$, then

$$\delta_2(p, -) \geq a = a \vee \mathrm{coz}(\beta) = a \vee \gamma_1(p, -) = \delta_1(p, -).$$

Case 4: Let $p < 0$ and let $\epsilon > 0$. Similar to Case 2, we have

$$\delta_2(p, -) \geq \bigvee_{\epsilon>0} a \vee \alpha(-\epsilon, \epsilon) = a \vee \bigvee_{\epsilon>0} \alpha(-\epsilon, \epsilon) = a \vee \alpha\big(\bigvee_{\epsilon>0}(-\epsilon, \epsilon)\big) = \top = a \vee \mathbf{0}(p, -) = \delta_1(p, -).$$

Therefore, $\delta_2(u) \geq \delta_1(u)$, for every $u \in \mathcal{L}(\mathbb{R})$. Since $\mathcal{L}(\mathbb{R})$ is a regular frame, $\delta_2 = \delta_1$. By [8, propositin 1.2], there exists $\gamma \in \mathcal{R}(L)$ such that

$$\gamma(u) \vee \mathrm{coz}(\beta)^* = \gamma_2(u) = \alpha\nu_{_C}\beta(u)$$

and

$$\gamma(u) \vee \mathrm{coz}(\beta) = \gamma_1(u) = \mathbf{0}(u),$$

where $\mathbf{0} \in \mathcal{R}(B)$. □

Theorem 3.15. *Let $\mathcal{F} \subseteq \mathrm{Coz}(L)$ be a coz-filter on L. Then the homomorphism π in Proposition 3.13 is an isomorphism; that is,* $S_{\mathcal{F}}^{-1}\mathcal{R}(L) \cong \Big[\biguplus_{A \in \mathcal{O}_L(S_{\mathcal{F}})} \mathcal{R}(A)\Big]$.

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Proof. First we show that π is a monomorphism. Let $\frac{f}{g}, \frac{h}{k} \in S_{\mathcal{F}}^{-1}\mathcal{R}(L)$ and let $\pi\left(\frac{f}{g}\right) = \pi\left(\frac{h}{k}\right)$. Then $[f][g]^{-1} = [h][k]^{-1}$ and so $[f][k] = [h][g]$. By Lemma 3.9, $\left[\nu_{\mathfrak{o}_L(\mathrm{coz}(fk))} fk\right] = \left[\nu_{\mathfrak{o}_L(\mathrm{coz}(hg))} hg\right]$. We put $B := \mathfrak{o}_L(\mathrm{coz}(fk))$, $C = \mathfrak{o}_L(\mathrm{coz}(hg))$, and $D = B \cap C$. Then

$$\begin{aligned}\nu_{_D}\nu_{_B} fk = \nu_{_D}\nu_{_C} hg &\Rightarrow \nu_{_{D\cap B}} fk = \nu_{_{D\cap C}} hg\\ &\Rightarrow \nu_{_D} fk = \nu_{_D} hg\\ &\Rightarrow \nu_{_D}(fk - hg) = \mathbf{0}.\end{aligned}$$

Then by Proposition 3.8, $\frac{f}{g} = \frac{h}{k}$, and so π is a monomorphism.

Now, let $[f] \in \left[\biguplus_{A\in\mathcal{O}(S_{\mathcal{F}})} \mathcal{R}(A)\right]$. Thus there exists $A \in \mathcal{O}_L(S_{\mathcal{F}})$ and $g \in \mathcal{R}(A)$ such that $[f] = [g]$ and so there exists $h \in S_{\mathcal{F}}$, where $g \in \mathcal{R}\big(\mathfrak{o}_L(\mathrm{coz}(h))\big)$. We put $B = \mathfrak{c}_L(\mathrm{coz}(h))$, $C = \mathfrak{o}_L(\mathrm{coz}(h))$, and $D = \mathfrak{c}_L(\mathrm{coz}(h)^*)$. Now, let $r = \dfrac{1}{1+|g|}$ and let $s = \dfrac{g}{1+|g|}$. It is easy to see that $r, s \in \mathcal{R}^*\big(\mathfrak{o}_L(\mathrm{coz}(h))\big)$. Then by Lemma 3.14, there exist $\gamma, \delta \in \mathcal{R}(L)$ such that

$$\nu_{\mathfrak{c}_L(\mathrm{coz}(h^*))}\gamma = r\nu_{_C} h,\ \nu_{_B}\gamma = \mathbf{0} \in \mathcal{R}(B),$$

and

$$\nu_{\mathfrak{c}_L(\mathrm{coz}(h^*))}\delta = s\nu_{_C} h,\ \nu_{_B}\delta = \mathbf{0} \in \mathcal{R}(B).$$

First we show that $\mathrm{coz}(\gamma) = \mathrm{coz}(h)$. We have

$$\begin{aligned}\mathrm{coz}(\gamma) &= \mathrm{coz}(\gamma) \vee \perp\\ &= \mathrm{coz}(\gamma) \vee \big(\mathrm{coz}(h) \wedge (\mathrm{coz}(h))^*\big)\\ &= \big(\mathrm{coz}(\gamma) \vee \mathrm{coz}(h)\big) \wedge \big(\mathrm{coz}(\gamma) \vee (\mathrm{coz}(h))^*\big)\\ &= \nu_{\mathfrak{c}_L(\mathrm{coz}(h))}\mathrm{coz}(\gamma) \wedge \nu_{_{\mathfrak{c}_L\left((\mathrm{coz}(h))^*\right)}}\mathrm{coz}(\gamma)\\ &= \mathrm{coz}(\mathbf{0}) \wedge \mathrm{coz}(r\nu_{_C} h)\\ &= \perp_B \wedge \mathrm{coz}(r) \wedge \mathrm{coz}(\nu_{_C} h)\\ &= \mathrm{coz}(h) \wedge \top \wedge \nu_{_C}(\mathrm{coz}(h))\\ &= \mathrm{coz}(h) \wedge \big(\mathrm{coz}(h) \longrightarrow \mathrm{coz}(h)\big)\\ &= \mathrm{coz}(h).\end{aligned}$$

Hence $\mathrm{coz}(\gamma) \in \mathcal{F}$, and so $\dfrac{\delta}{\gamma} \in S_{\mathcal{F}}^{-1}\mathcal{R}(L)$. Therefore

$$[\gamma]^{-1} = [s\nu_{_C} h][r\nu_{_C} h]^{-1} = [s][r]^{-1} = [\frac{g}{1+|g|}][\frac{1}{1+|g|}]^{-1} = [g][\frac{1}{1+|g|}][\frac{1}{1+|g|}]^{-1} = [g] = [f].$$

Therefore $\pi\left(\dfrac{\delta}{\gamma}\right) = [f]$, which means that π is an epimorphism. $\square$

Theorem 3.16. *The following statements are true for any completely regular locale L:*

(1) *If S is a multiplicatively closed z-set of $\mathcal{R}(L)$, then $S^{-1}\mathcal{R}(L) \cong \left[\biguplus_{A\in\mathcal{O}_L(S)} \mathcal{R}(A)\right]$.*

(2) *If $\mathcal{F}$ is a filter on L, then $S^{-1}_{Cozf[\mathrm{Coz}^{\leftarrow}(\mathcal{F})]}\mathcal{R}(L) \cong \left[\biguplus_{A\in\mathcal{O}(S_{Cozf[\mathrm{Coz}^{\leftarrow}(\mathcal{F})]})} \mathcal{R}(A)\right]$.*

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Proof. (1) By Lemma 2.14, $Cozf[S] = Cozf[\bar{S}]$. Moreover, by [21, Exercise 5.12], $S^{-1}\mathcal{R}(L) \cong \bar{S}^{-1}\mathcal{R}(L)$. Therefore without loss of generality, we may assume that S is a saturated multiplicatively closed z-set of $\mathcal{R}(L)$. Then, by Proposition 2.15, we have $S_{\mathcal{F}} = S_{Cozf[S]} = \mathrm{Coz}^{\leftarrow}\big(Cozf[S]\big) = S$, and so by (1), $S^{-1}\mathcal{R}(L) \cong \Big[\biguplus_{A\in\mathcal{O}_L(S)} \mathcal{R}(A)\Big]$.
(2) It is clear. □

Corollary 3.17. *If* $\mathcal{V} = \big\{\mathrm{coz}(\alpha)\colon \mathfrak{o}_L\big(\mathrm{coz}(\alpha)\big)$ *is a dense sublocal of* $L\big\}$, *then* $\mathcal{V}$ *is a coz-filter on* L *and*

$$Q_{cl}(\mathcal{R}(L)) \cong \left[\biguplus_{a\in\mathcal{V}} \mathcal{R}(\mathfrak{o}_L(a))\right].$$

4. Special saturated ring

A commutative ring R is said to be special saturated if it contains no finitely generated dense ideals. In this section, we consider the ring $\mathcal{R}(L)$ and give equivalent conditions for the ring $\mathcal{R}(L)$ such that it is special saturated. If the Jacobson radical, $J(R)$, of a commutative reduced ring R is zero, then every exoteric ideal of R is a z-ideal. As an immediate consequence from [16, Theorem 5.4.], we now have the following result.

Proposition 4.1. *If* $\mathcal{R}(L)$ *is a special saturated ring, then the classes of* z*-ideals and exoteric ideals coincide.*

Proposition 4.2. *For* $\mathcal{R}(L)$, *the following conditions are equivalent:*

(1) $\mathcal{R}(L)$ *is a special saturated ring.*
(2) *For every* $\alpha \in \mathcal{R}(L)$ *with* $\mathrm{coz}(\alpha) \neq \top$, *there exists* $\mathbf{0} \neq \beta \in \mathcal{R}(L)$ *such that* $\mathrm{coz}(\alpha)\wedge\mathrm{coz}(\beta) = \bot$.
(3) *For the classical quotient ring of* $\mathcal{R}(L)$, *we have* $\mathrm{Q_{cl}}\mathcal{R}(L) = \mathcal{R}(L)$.

Proof. (1) ⇒ (2). Let $\alpha \in \mathcal{R}(L)$ with $\mathrm{coz}(\alpha) \neq \top$ be given. Then $I := \alpha\mathcal{R}(L) \neq \mathcal{R}(L)$ and by the hypothesis, $\mathrm{Ann}_{\mathcal{R}(L)}(I) \neq (0)$. Hence, there exists $\mathbf{0} \neq \beta \in \mathcal{R}(L)$ such that $\alpha\beta = \mathbf{0}$, which implies that $\mathrm{coz}(\alpha) \wedge \mathrm{coz}(\beta) = \bot$.

(2) ⇒ (1). Let $n \in \mathbb{N}$ and let $\alpha_1, \ldots, \alpha_n \in \mathcal{R}(L)$ with $I := (\alpha_1, \ldots, \alpha_n) \subsetneq \mathcal{R}(L)$ be given. Then $\mathrm{Ann}_{\mathcal{R}(L)}(I) = \mathrm{Ann}_{\mathcal{R}(L)}(\sum_{i=1}^{n} \alpha_i^2)$ and $\mathrm{coz}(\sum_{i=1}^{n} \alpha_i^2) \neq \top$. By the hypothesis, there exists $\mathbf{0} \neq \beta \in \mathcal{R}(L)$ such that $\mathrm{coz}(\sum_{i=1}^{n} \alpha_i^2) \wedge \mathrm{coz}(\beta) = \bot$, which implies that $\beta \sum_{i=1}^{n} \alpha_i^2 = \mathbf{0}$, and so, $\beta \in \mathrm{Ann}_{\mathcal{R}(L)}(I)$; that is, I is not dense. Therefore, $\mathcal{R}(L)$ is a special saturated ring.

(1) ⇔ (3). It is evident. □

Abedi [1, Proposition 4.5] showed that $\mathrm{Q_{cl}}\mathcal{R}(L) = \mathcal{R}(L)$, if and only if L is an almost P-frame. Then by Proposition 4.2, we can conclude the next corollary.

Corollary 4.3. *Frame* L *is an almost* P*-frame, if and only if* $\mathcal{R}(L)$ *is a special saturated ring.*

Proposition 4.4. *For a frame* L, *the following conditions are equivalent:*

(1) *Every map* $\varphi\colon \mathcal{R}(L) \to S$, *where* S *is a regular ring, is an exoteric homomorphism.*
(2) *Every prime ideal of* $\mathcal{R}(L)$ *is exoteric.*

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(3) $\mathrm{Ann}_{\mathcal{R}(L)}\mathrm{Ann}_{\mathcal{R}(L)}(I) = \sqrt{I}$ *for every finitely generated right ideal* I *of* $\mathcal{R}(L)$.

(4) L *is a* P*-frame.*

Proof. By [16, Proposition 5.8], (1), (2), and (3) are equivalent. It is clear that if $\mathcal{R}(L)$ satisfies the equivalent condition (1), (2), and (3), then it is a special saturated ring. Hence, (1) and (4) are equivalent. □

Let I be an ideal of a commutative ring R. The ideal I is called a torsion ideal, if

$$\text{for all } i \in I, \text{ for all } r \in R (r \notin I \Rightarrow \text{there exists } r' \in R \text{ with } ir' = 0 \text{ and } rr' \notin I).$$

Proposition 4.5. *Every torsion ideal of* $\mathcal{R}(L)$ *is a* z*-ideal.*

Proof. Let I be a torsion ideal of $\mathcal{R}(L)$ and let $(\alpha, \beta) \in I \times \mathcal{R}(L)$ with $\mathrm{coz}(\alpha) = \mathrm{coz}(\beta)$ be given. If $\beta \notin I$, then there exists $\gamma \in \mathcal{R}(L)$ such that $\alpha\gamma = \mathbf{0}$ and $\beta\gamma \notin I$. hence

$$\bot = \mathrm{coz}(\alpha\gamma) = \mathrm{coz}(\alpha) \wedge \mathrm{coz}(\gamma) = \mathrm{coz}(\beta) \wedge \mathrm{coz}(\gamma) = \mathrm{coz}(\beta\gamma) \Rightarrow \beta\gamma = \mathbf{0} \in I,$$

a contradiction, which implies that $\beta \in I$. Therefore, I is a z-ideal of $\mathcal{R}(L)$. □

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