Journal of the Iranian Mathematical Society
ISSN (on-line): 2717-1612
J. Iran. Math. Soc. 4 (2023), no. 2, 121-129

CERTAIN MODULES WITH THE NOETHERIAN DIMENSION

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Dedicated to Prof. O. A. S. Karamzadeh

Abstract. An R-Module M with a small submodule S, such that $\frac{M}{S}$ is Noetherian, is called a SN-module. In this paper, we introduce the concept of α-SN-modules, for any ordinal $\alpha \geq 0$ (SN-modules are just 0-SN-modules). Some of the basic results of SN-modules extended to α-SN-modules. It is shown that an fs-module M, which is α-SN, has Noetherian dimension $\leq \alpha$. In particular, if M is quotient finite-dimensional and all of its submodules are α-SN, then M has Noetherian dimension $\leq \alpha$. Furthermore, the concepts of qn-submodules (a proper submodule N of M is called a qn-submodule if $\frac{M}{N}$ has Noetherian dimension) and qn-modules are introduced. It is proved that if M is quotient finite-dimensional and each of its submodules has at least a qn-submodule, then M has Noetherian dimension.

1. Introduction

Motivated by the concept of the Krull dimension of modules, Lemmonier in [12], introduced the concept of the deviation and it's dual, the codeviation for any arbitrary poset $(E, \leq)$. The codeviation of E which is also called the Noetherian dimension (the dual Krull dimension in other texts), is just the deviation of E^0, the opposite poset of E. Note that the Noetherian dimension of an R-module M, which is denoted by n-dim M, measures the deviation of M from being Noetherian. It is convenient, when we are dealing with the above dimensions, to begin our list of ordinals with -1. For more details and some basic facts about the deviation and the codeviation refer to [2, 7, 9–13].

Communicated by Alireza Abdollahi

MSC(2020): Primary: 16P20; Secondary: 16P40, 16P60.

Keywords: Noetherian dimension; small submodules; α-SN-modules; qn-modules.

Received: 27 May 2023, Accepted: 14 June 2023.

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DOI: https://dx.doi.org/10.30504/JIMS.2023.399248.1118

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An R-module M is called a SN-module, if it has a small submodule S such that $\frac{M}{S}$ is Noetherian, see [4]. More generally, for any ordinal α, M is called an α-SN-module , if it has a small quotient module with Noetherian dimension at most α and α is the least ordinal with this property. Using this concept, we extend some of the basic results of SN-modules to α-SN-modules. We should remind the reader that by a quotient finite-dimensional (briefly, qfd) module, we mean an R-module M such that $\frac{M}{N}$ has finite Goldie dimension, for every submodule N of M. We shall see that if M is qfd, and all of its submodules are α-SN, for some α, then the Noetherian-dimension of M exists and $n\text{-dim}\, M \leq \alpha$. Further, if α is countable, then every submodule of M is countably generated. An R-module M with only finitely many non-zero small submodules is said to be an fs-module. A ring R is called an fs-ring, if it is an fs-module as an R-module, see [6]. We show that an fs-module M is α-SN, if and only if $\frac{M}{Rad(M)}$ is β-SN, for some $\beta \leq \alpha$. Furthermore, we prove that if an fs-module M is α-SN, then $n\text{-dim}\, M \leq \alpha$. A proper submodule N of M, is called a qn-submodule (α-qn-submdule) and denoted by $N \overset{qn}{\subsetneq} M$ ($N \overset{\alpha-qn}{\subsetneq} M$) if $\frac{M}{N}$ has Noetherian dimension ($\leq \alpha$). Moreover, M is called a qn-module (α-qn-module), if any of its submodules is contained in a qn-submodule (an α-qn-submodule). It is easy to see that M is an α-qn-module, if and only if for each proper submodule N of M, there exists a proper submodule K of M, such that $N \subseteq K$ and $\frac{M}{K}$ has Noetherian dimension $\leq \alpha$. In other words, every nonzero quotient module of M has a proper α-qn-submodule.

Throughout this paper, rings are associative with $1 \neq 0$, and modules are unital right modules. $N \subseteq_e M$ denotes that N is an essential submodule of M, that is $N \cap A \neq 0$, for all non-zero submodules A of M. M has finite Goldie (resp., hollow) dimension, if for any ascending (resp., descending) chain $N_1 \subseteq N_2 \subseteq \cdots$ (resp., $N_1 \supseteq N_2 \supseteq \cdots$) of submodules of M, there exists an integer $n \geq 1$, such that $N_n \subseteq_e N_k$ (resp., $\frac{N_n}{N_k} \ll \frac{M}{N_k}$) for all $k \geq n$. The radical of M is denoted by $Rad(M)$ and is equal to the intersection of all of the maximal submodules of M. Equivalently, $Rad(M)$ is the sum of all the small submodules of M. If M fails to have maximal submodules, we set $Rad(M) = M$ and in case, M fails to have a proper nonzero small submodule, then $Rad(M) = 0$.

2. Preliminaries

We begin with the following definition.

Definition 2.1. *Let M be an R-module. A proper submodule S of M is small in M, if $S + N \neq M$ for every proper submodule N of M. We indicate that S is a small submodule of M, by notation $S \ll M$.*

In the following, we recall some basic properties of small submodules, see [15].

Lemma 2.2. *Let M be a module with submodules $K \subseteq N \subseteq M$, and let $P \subseteq M$. Then*

(1) *$N \ll M$, if and only if $K \ll M$ and $\frac{N}{K} \ll \frac{M}{K}$.*
(2) *$P + K \ll M$, if and only if $P \ll M$ and $K \ll M$.*
(3) *If $K \ll N$, then $K \ll M$.*
(4) *$Rad(M)$ is the sum of all the small submodules of M.*

DOI: https://dx.doi.org/10.30504/JIMS.2023.399248.1118

We cite the following facts from [6].

Definition 2.3. *Let R be a ring. An R-module M with only finitely many non-zero small submodules is said to be an fs-module. In particular, R is called an fs-ring, if as an R-module, it is an fs-module.*

We note that if M is an fs-module, then $S \ll M$ if and only if $S \subseteq Rad(M)$.

Proposition 2.4. *Let R be a ring and M be an R-module. Then M is an fs-module, if and only if $Rad(M)$ has only finitely many submodules.*

Corollary 2.5. *If M is an fs-module, then*

(1) *$Rad(M)$ has finite length, so it is both Artinian and Noetherian.*
(2) *M is Noetherian (Artinian) if and only if $\frac{M}{Rad(M)}$ is Noetherian (Artinian).*

Theorem 2.6. *Let M be an fs-module with finite hollow dimension over a ring R. The following holds.*

(1) *M is an Artinian module.*
(2) *M is a Noetherian module.*
(3) *$Rad(M)$ is finitely-generated.*
(4) *M has a finite-composition series.*
(5) *M has finite-length.*
(6) *M has finite-Goldie-dimension.*

Definition 2.7. *Let M be an R-module, and let $S \ll M$, then $\frac{M}{S}$ is called a small quotient module of M.*

The following result shows that the class of all fs-modules are closed under sumodules with non-zero small submodule and small quotient modules.

Proposition 2.8. *The following are equivalent for any R-module M.*

(1) *M is an fs-module.*
(2) *Every submodule of M, with non-zero small submodule, is an fs-module.*
(3) *Every small quotient module of M is an fs-modules.*

3. α-SN-modules and their properties

According to [4], an R-module M is called a SN-module if $\frac{M}{S}$ is Noetherian, for some small submodule S of M. In this section, we generalize this concept and introduce the concept of α-SN-modules. We extend the basic results of SN-modules to α-SN-modules.

Next, we give our definition of α-SN-modules.

Definition 3.1. *A non-zero R-module M is called an α-SN-module, if it has a small quotient module with Noetherian-dimension at most α and α is the least ordinal with this property.*

DOI: https://dx.doi.org/10.30504/JIMS.2023.399248.1118

Clearly 0-SN-modules are just SN-modules.

Remark 3.2. (1) *If M is an R-module with $n\text{-dim}\, M = \alpha$, then M is β-SN for some $\beta \leq \alpha$. But the converse is not true in general. For example, every local ring (i.e., a ring with a unique one sided maximal ideal) R is a SN-ring, since $\frac{R}{Rad(R)}$ is Noetherian, but R need not to have the Noetherian-dimension.*

(2) *An essential extension of any α-SN-module is not necessarily a β-SN-module for some ordinal β. For example, as $\mathbb{Z}$-module, we have $\mathbb{Z} \subseteq_e \mathbb{Q}$ and $\mathbb{Z}$ is a SN-module, but $\mathbb{Q}$ is not an α-SN-module, for some α.*

Definition 3.3. *Let L and N be submodules of M. Then N is called a supplement of L in M, if N is minimal with respect to $N + L = M$. A submodule N of M is called supplement submodule, if it is supplement of some submodule L of M.*

Note that, N is a supplement of L in M, if and only if $N + L = M$ and $N \cap L \ll N$. Also, if $K \subseteq N$ and $K \ll M$, then $K \ll N$, see [15, ch. 41].

Remark 3.4. *Let M be an α-SN- module. There exists a small submodule S such that $\frac{M}{S}$ has Noetherian-dimension at most α. Now, let N be a supplement in M containing S. Then $S \ll N$ and $\frac{N}{S}$ has Noetherian-dimension. Hence N is a β-SN-module for some ordinal $\beta \leq \alpha$.*

Proposition 3.5. *Let M be an α-SN-module. Then every factor module of M is β-SN, for some $\beta \leq \alpha$.*

Proof. Let N be a proper submodule of M. Since M is α-SN, it has a small submodule S such that $n\text{-dim}\, \frac{M}{S} \leq \alpha$. Clearly $\frac{S+N}{N} \ll \frac{M}{N}$ and $\frac{M/N}{(N+S)/N} \simeq \frac{M}{N+S} \simeq \frac{M/S}{(N+S)/S}$. Thus $n\text{-dim}\, \frac{M/N}{(N+S)/N} = n\text{-dim}\, \frac{M}{N+S} = n\text{-dim}\, \frac{M/S}{(N+S)/S} \leq n\text{-dim}\, \frac{M}{S} \leq \alpha$. □

Theorem 3.6. *The following statements are equivalent for every R-module M and every ordinal α.*

(1) *M is an α-SN-module.*
(2) *$\frac{M}{S}$ is an α-SN-module, for every small quotient $\frac{M}{S}$.*
(3) *$\frac{M}{S}$ is an α-SN-module, for some small quotient $\frac{M}{S}$.*

Proof. (1) ⇒ (2) Suppose that M is an α-SN-module and $S \ll M$. Thus $\frac{M}{S}$ is β-SN, for some $\beta \leq \alpha$, by the previous proposition. That is, there exists a small submodule $\frac{N}{S}$ of $\frac{M}{S}$ such that $n\text{-dim}\, \frac{M}{N} = n\text{-dim}\, \frac{M/S}{N/S} \leq \beta$. But Lemma 2.2(1) implies that $N \ll M$ and by definition $\alpha \leq \beta$ and so $\alpha = \beta$.
(2) ⇒ (3) It is clear.
(3) ⇒ (1) By assumption, there exists a small submodule $\frac{N}{S}$ of $\frac{M}{S}$ such that $n\text{-dim}\, \frac{M/S}{N/S} \leq \alpha$. Since $S \ll M$, $\frac{N}{S} \ll \frac{M}{S}$, thus $N \ll M$, by Lemma 2.2(1), and we have $n\text{-dim}\, \frac{M}{N} \leq \alpha$. It follows that M is β-SN for some $\beta \leq \alpha$. On the other hand $\alpha \leq \beta$, by Proposition 3.5. This shows that $\alpha = \beta$. □

Corollary 3.7. *Let M be an R-module and $Rad(M) \ll M$. Then M is an α-SN if and only if $\frac{M}{Rad(M)}$ is α-SN.*

DOI: https://dx.doi.org/10.30504/JIMS.2023.399248.1118

In view of Proposision 3.5 and Theorem 3.6, we conclude the following result.

Corollary 3.8. *Let $0 \to N \to M \to K \to 0$ be a short exact sequence of R-modules where $N \ll M$. Then*

1. *If M is α-SN, then K is β-SN, for some $\beta \leq \alpha$.*
2. *If K is α-SN, then M is α-SN.*

Corollary 3.9. *Let M be an fs-module. Then M is α-SN if and only if $\frac{M}{Rad(M)}$ is β-SN, for some $\beta \leq \alpha$.*

The proof of the following two results is straightforward.

Lemma 3.10. *Let M be an R-module and $Rad(M) = 0$. Then M is an α-SN-module, if and only if $n\text{-}\dim M \leq \alpha$.*

Lemma 3.11. *Let M be an R-module and $Rad(M) \ll M$. Then $\frac{M}{Rad(M)}$ is an α-SN -moddule, if and only if $n\text{-}\dim \frac{M}{Rad(M)} \leq \alpha$.*

The following fact is also presented in [12, Theorem 2.4] and [1, Proposition 2.2].

Proposition 3.12. *The following statements are equivalent for any R-module M and any ordinal $\alpha \geq 0$.*

1. *M is qfd and for any submodules $N \subset P \subseteq M$, there exists a submodule X with $N \subseteq X \subset P$ such that $n\text{-}\dim \frac{P}{X} \leq \alpha$.*
2. *$n\text{-}\dim M \leq \alpha$.*

Theorem 3.13. *Let M be a qfd-module such that any of its submodules is γ-SN, for some ordinal γ. Then M has Noetherian-dimension and $n\text{-}\dim M \leq \alpha$, where $\alpha = sup\{\gamma : M \text{ has a } \gamma\text{-}SN \text{ submodule}\}$.*

Proof. It suffices to prove that M satisfies the part (1) of Proposition 3.12. For each $N \subset P \subseteq M$, $B = \frac{P}{N}$ has small quotient module, say $\frac{B}{C}$, such that $n\text{-}\dim \frac{B}{C} \leq \alpha$, by Lemma 3.5. □

In [11], it is shown that every submodule of a module with countable Noetherian-dimension is countably generated. Thus, the following result is immediate.

Corollary 3.14. *Let M be a qfd-module such that all of its submodules are α-SN, where α is a countable ordinal. Then, every submodule of M is countably generated.*

Definition 3.15. *An R-module M is said to have property $AB5^*$ (or is said to be an $AB5^*$-module) if, for every submodule N and inverse system $\{M_i\}_{i\in I}$ of submodules of M, $N + \bigcap_{i\in I} M_i = \bigcap_{i\in I}(N + M_i)$.*

Artinian modules and more generally, linearly compact modules are $AB5^*$, see [15, 29.8]. The next fact is in [5, Lemma 6] and [3].

Lemma 3.16. *Let M be an $AB5^*$-module . Then M is qfd, if and only if every submodule of M has finite-hollow-dimension.*

DOI: https://dx.doi.org/10.30504/JIMS.2023.399248.1118

The following result is now immediate.

Corollary 3.17. *Let M be an $AB5^*$-module with finite-hollow-dimension. If any submodule of M is γ-SN, then n-$\dim M \leq \alpha$, where $\alpha = sup\{\gamma : M$ has a γ-SN submodule$\}$.*

Proposition 3.18. *Let M be an fs-module which is α-SN. Then n-$\dim M \leq \alpha$.*

Proof. If M is an α-SN-module, then there exists a small submodule S such that n-$\dim \frac{M}{S} \leq \alpha$. By Proposition 2.4, S is Noetherian. We infer that, n-$\dim(M) = sup\{n$-$\dim \frac{M}{S}, n$-$\dim S\} = n$-$\dim \frac{M}{S} \leq \alpha$. □

4. Quotient Noetherian modules and their properties

It is well-known that an R-module M has Noetherian-dimension, if and only if $\frac{M}{N}$ does too, for every non-zero submodule N of M. Thus, if M failes to have Noetherian-dimension, it has a non-zeo submoule N, such that $\frac{M}{N}$ has not Noetherian-dimension. In this section we focus on submodules N such that $\frac{M}{N}$ has Noetherian-dimension. We are going to see that how far is for M from having Noetherian-dimension. We begin this section by introducing the concept of qn-submodules and qn-modules.

Definition 4.1. *A proper submodule N of M is a quotient Noetherian (briefly, qn-submodule) if $\frac{M}{N}$ has Noetherian-dimension. More generally, N is a α-qn-submodule if $\frac{M}{N}$ has Noetherian-dimension $\leq \alpha$.*

In order to show that N is a qn-submodule (α-qn-submodule), we use the notation $N \overset{qn}{\subsetneq} M$ ($N \overset{\alpha-qn}{\subsetneq} M$).

Definition 4.2. *An R-module M is called a qn-module (α-qn-module), if any of its proper submodules is contained in a qn-submodule (α-qn-submodule) of M.*

Note that, M is a qn-module if and only if every nonzero quotient module of M has a proper qn-submodule.

Remark 4.3. (1) *If $N \overset{qn}{\subsetneq} M$ and $N \subseteq K \subsetneq M$, then $K \overset{qn}{\subsetneq} M$. Because $\frac{M}{K} \simeq \frac{M/N}{K/N}$ and $\frac{M}{N}$ has Noetherian-dimension. That is, every extension of a qn-submodule is qn-submodule.*

(2) *If $N \overset{qn}{\subsetneq} M$ and $K \overset{qn}{\subsetneq} M$, then $N \cap K \overset{qn}{\subsetneq} M$, for $\frac{M}{N \cap K}$ is isomorphism to an submodule of $\frac{M}{N} + \frac{M}{K}$ which it has Noetherian-dimension.*

(3) *If M has Noetherian-dimension and n-$\dim M = \alpha$, then M is a β-qn-module for some $\beta \leq \alpha$. But the converse is not true in general, for example, the $\mathbb{Z}$-module $M = \mathbb{Z}_2 \oplus \mathbb{Q}$ has an qn-submodule $\mathbb{Q}$ but M does not have Noetherian-dimension.*

(4) *Every α-SN has at least an α-qn-submdule, and it is clear that its converse is not true in general.*

DOI: https://dx.doi.org/10.30504/JIMS.2023.399248.1118

(5) *A ring R is a Max-ring if and only if, every nonzero R-module M has a maximal submodule. Hence every module over a Max-ring R, has a qn-submodule. In particular, every ring R with identity as an R-module has a qn-submodule.*

Lemma 4.4. *Let M be a qn-module and N a proper submodule of M. Then so is $\frac{M}{N}$.*

Proof. Let $\frac{K}{N}$ be a proper submodule of $\frac{M}{N}$. Then K is a proper submodule of M and so there exists a qn-submodule D of A such that $K \subseteq D$ and $\frac{M}{D} \cong \frac{M/N}{D/N}$ has Noetherian-dimension. Hence $\frac{D}{N}$ is a qn-submodule of $\frac{M}{N}$ and $\frac{K}{N} \subseteq \frac{D}{N}$. Therefore $\frac{M}{N}$ is a qn-module. □

Lemma 4.5. *Let N be a proper submodule of M. If N and $\frac{M}{N}$ are qn-modules, then so is M.*

Proof. Let K be a proper submodule of M. If $K+N=M$, then $K\cap N \subsetneq N$. So there exists a qn-submodule D of N such that $K\cap N \subseteq D \overset{qu}{\subsetneq} N$. But $\frac{M}{K+D} = \frac{K+N}{K+D} \cong \frac{N}{D}$, so $K+D \overset{qn}{\subsetneq} M$ and $K \subseteq K+D$. Now, if $N+K \subsetneq M$, $\frac{K+N}{N}$ may be zero. Then $\frac{K+N}{N} \subsetneq \frac{M}{N}$, so there exists a qn-submodule $\frac{D}{N}$ of $\frac{M}{N}$ such that $\frac{K+N}{N} \subseteq \frac{D}{N} \overset{qn}{\subsetneq} \frac{M}{N}$. So $K \subseteq K+N \subseteq D \overset{qn}{\subsetneq} M$. Thus M is a qn-module. □

In view of Lemmas 4.4 and 4.5, we conclude the following.

Corollary 4.6. *Let $0 \to K \to M \to N \to 0$ be an exact sequence of R-modules. Then*

(1) *If M is a qn-module, then N is a qn-module.*
(2) *If K and N are qn-modules, then M is a qn-module.*

Corollary 4.7. *Let $M = \sum_{i=1}^{n} \oplus M_i$. If each M_i is a qn-module, then so is M.*

The next important result is now immediate.

Theorem 4.8. *Let M be a qfd-module, such that any of its submodules has at least a γ-qn-submodule. Then M has Noetherian-dimension and n-dim $M \leq \alpha$, where $\alpha = sup\{\gamma : M$ has a γ-qn submodule$\}$. Further, if α is countable, then every submodule of M is countabely generated.*

Proof. It suffices to prove that M satisfies part (1) of Proposition 3.12. For every $N \subset P \subseteq M$, $B = \frac{P}{N}$ has quotient module, say $\frac{B}{C}$, such that n-dim $\frac{B}{C} \leq \alpha$, by Lemma 3.5 and the comment after of Theorem 3.13, every submodule of M is countably generated. □

A module M is called finitely-embedded (briefly, $f.e.$), if $Soc(M)$ is finitely generated and essential in M. It is easy to see that every module has an $f.e.$-factor-module, see [11, Comments, preceding Lemma 1.1]. It is also well-known that over a commutative ring R, every finitely embedded module is Artinian, if and only if R is a locally Noetherian ring (i.e., R_M is Noetherian for every maximal ideal M of R), see [14, Theorem 2]. Moreover, if R is a Noetherian duo ring, then an R-module M is $f.e.$, if and only if it is Artinian, see [8, Theorem 2.4]. In view by this comment and Proposition 3.12, we have the following results.

Theorem 4.9. *Let M be a qfd-module over a locally Noetherian (or, a Noetherian right duo) ring R. Then M has Noetherian-dimension.*

Proof. It suffices to prove that M satisfies the part (1) of Proposition 3.12. For each $N \subset P \subseteq M$, $B = \frac{P}{N}$ has a nonzero quotient module- which is finitely embedded, say $\frac{B}{C}$, by the previous comment. Thus, $\frac{B}{C}$ is an Artinian module and so has Noetherian-dimension and we are done □

Theorem 4.10. *Let N be a small submodule of M. Then $\frac{M}{N}$ is a qn-module, if and only if A is so.*

Proof. We suppose that $\frac{M}{N}$ is a qn-module. Let K be a proper submodule of M. Since N is a small submodule of M, we infer that $K+N$ is a proper submodule of M. Hence, there exists a qn-submodule $\frac{X}{N}$ of $\frac{M}{N}$ such that $\frac{K+N}{N} \subseteq \frac{X}{N}$. Thus X is a qn-submodule of M such that $K \subseteq X$. This shows that M is qn-module. Conversely, let M be a qn-module. By Lemma 4.4, we infer that $\frac{M}{N}$ is a qn-module. □

Acknowledgments

The authors want to thank the referee for careful reading, and useful comments which improved this paper. The research of the second named author was partially supported by a grant from Shahid Chamran University.

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DOI: https://dx.doi.org/10.30504/JIMS.2023.399248.1118

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DOI: https://dx.doi.org/10.30504/JIMS.2023.399248.1118