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THE STORY OF RINGS OF CONTINUOUS FUNCTIONS IN AHVAZ: FROM $C(X)$ TO $C_c(X)$

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Dedicated to Prof. O. A. S. Karamzadeh

Abstract. The author narrates how the study and research on the rings of continuous real-valued functions on a topological space initiated in Iran. In particular, almost completely all the important work of the authors, in this field, during the last four decades, whose related research are carried out in Ahvaz are referred to and where necessary are commented on, by the author. Also included are some related anecdotes and the contributions of Karamzadeh to the promotion of mathematics in Iran, during the past half-century.

1. Introduction

First of all, I would like to thank the editors of the Journal of the Iranian Mathematical Society, for their right decision to dedicate this volume of the journal to Professor Omid Ali Shahni Karamzadeh (briefly, O.A.S. Karamzadeh) for his effective role for the advancement of mathematics in Iran, during the past half-century. And also for giving me the opportunity of writing this article. First, I should say a few words about my connection to Karamzadeh. While I was an undergraduate student studying engineering at Ahvaz university in 1987 (note, by Ahvaz university, I mean Shahid Chamran University of Ahvaz), I used to attend the two courses in set theory and topology, which were given by him at that time. This prevailed upon me to do my master's degree in mathematics. I did this at Tehran University in 1994, and have written my dissertation under his supervision at Ahvaz university, on a survey article by Matlis, [66], which contained an open problem. Incidentally, later the problem

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was settled by Karamzadeh in two ways, see [48, 51]. Indeed, genealogically I am his grandson, in the sense, that I did my Ph.D. in $C(X)$ under the supervision of Professor Azarpanah (note, I was his first student), who in turn did his Ph.D. in $C(X)$ under the supervision of Karamzadeh (his first student in this field, too). I believe we can give no better expression of our high regard than dedicating this volume to him. However, as for what he has done for the improvement of mathematics in our country, I also believe, only his close friends, some of his students, and some of his colleagues, might be aware of, not necessarily the reader. Therefore, before dealing with the title, I would like to digress for a moment and briefly write a few lines about what he has really done, for the country in this regard. With respect to the role of mathematicians in the advancement of mathematics in their own country, I believe they should not be measured only by the depth and creativity of their contributions to their field of expertise (although on these grounds alone, Karamzadeh is also an internationally well-known figure, see [22]). Because, the excellent work of some of these mathematicians which might even win them Fields Medals are usually used by only few researchers in any country and might have no serious impact on the education and on the advancement of mathematics in their country, in general. In my opinion, some other perhaps more important ways to contribute to the advancement of mathematics in the country lies in what I will describe shortly. However, at the same time, I do not expect the contributions of every mathematician should be evaluated using only the criteria which follows. These criteria perhaps could be applied mostly to those who are still active. We begin with: for what they have been doing for better teaching and the learning methods of mathematics in the country, for giving popular talks and written expositions in mathematics for the teachers and the students alike; for leading and encouraging others and, in particular, the newcomers to the field; and it is also for being a member of different committees concerning mathematics for a long time. Also for visiting public places like schools, mathematics houses (especially in Iran) every now and then to meet with the parents of students and talking to them in order to communicate mathematics to the masses and to school kids and their teachers, see [49] and [44]. Even it is also for being patient, in general, with the mathematical cranks and for being kind to them and also for not ignoring and avoiding them, while they are falsely claiming a proof of some of the well-known unsolved problems or giving an incorrect solution to some well-known facts in mathematics. Indeed, Karamzadeh to me, has always been such a mathematician, who is still fulfilling all these requirements. I have even recently noticed, with my surprise, that he has the phone numbers, of some of the mathematical cranks of our country, saved in his iPhone, see also his comments about the mathematical cranks in [44]. And once or twice, when I was sitting beside him, I had been overhearing his discussions, on the phone, with them. I should also emphasize on his active and effective role as a member of committees for any kind of mathematics competitions in Iran, among either university students or school kids, for a long time in the past. In particular, his excellent and unforgettable role in the earlier activities with regard to the IMO in Iran. In those days, when not many mathematicians in our country were familiar with the literature of IMO, he used to teach as an indispensable member, in the training camps, which used to be held either in Ahvaz (the city of his residency) or in Tehran for weeks, in consecutive years, to prepare the selected students of our national team before the IMO competition

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abroad. Also his conspicuous role as a member of the team leaders of the latter students while abroad for the competition, must be emphasized on. Especially, when Maryam Mirzakhani who won two gold medals successively in Hong Kong and Canada Olympiads, in 1994 and 1995, respectively. And also when in 1998 Iran's team reached the first rank, in the world, at the IMO, held in Taiwan, which was a great achievement for the country and it was also very encouraging and inspirational for the mathematics students of our country at that time, too. All these contributions were not possible, without his kindness, his generosity for sharing his mathematical thoughts and knowledge, and especially for his rare and deep love for mathematics since his childhood. These characteristics enabled him to give his time, originality, and creative insights, freely, wholeheartedly, and without expecting any rewards or receiving any extra fee or salary except his monthly university salary (note, all the rewards given to him are given rather recently). One last word: He used to say, jokingly, if the government, for any reason, decides not to pay his regular monthly salary, he will still be doing the same work in mathematics, because he enjoys mathematics much more than of what he might receive from the government. I do believe he was not joking! Finally, why did I recall all these facts about Karamzadeh, which might look irrelevant to the above story in the title? Because I just want to emphasize that, in my opinion, there might rarely exist a single mathematician, anywhere in the world, who has been involved in similar endeavors to the above activities, for years, and at the same time, being a very good teacher and a researcher, and supervising several Ph.D. students, working separately in two or three different subjects. And especially before accepting any Ph.D. students, feeling obliged to find an appropriate subject, if necessary, apart from his expertise (note, briefly we will see why $C(X)$ became this subject) to communicate the subjects with various students and more colleagues. Moreover, he is also well-known in Ahvaz, our city, for being a very kind social activist, without being a member of any organization or a political party, who used to be invited regularly to some appropriate, and non-political meetings, for the sake of the people, which required much of his time and devotion. Before his retirement and almost during the entire 45 years of his work at our university, any students (not necessarily mathematics students), young colleagues, and some of the workers at the university, with any kind of difficulties would come to his office for help, and he used to receive them happily and was very helpful to most of them. He is well-known among ordinary people here for his latter kind of activities not for his mathematics. So in conclusion, when everything is considered, I would say that, it is a combination of humbleness, confidence, prowess, perseverance, deep love for mathematics, and love for the people and his country, that made the devotion to all of these activities, possible for him. Let us now return to the title and first recall a quote, in what follows briefly, from the late Melvin Henriksen, who died in 2009-2010. Henriksen was best known for his work leading to the creation of a new field of study in mathematics, called "rings of continuous functions" involving the interplay of algebra and topology. Let us also, denote this field of study by just "$C(X)$", the ring of real-valued continuous functions on a completely regular Hausdorff space X (note, X is also called a Tychonoff space). He was perhaps one of the rare precursors in the field who remained very active until his death at the age of 82, and collaborated with the students and scholars alike around the world. Presumably, his last joint article with a student was the one in [36], whose co-author is Amir Nikou, who was then

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an MSc student of Karamzadeh in Ahvaz. Without further ado, let us mention the above promised quote from Henriksen. In [35, p. 553] he wrote "a lot of papers were written on the subject of $C(X)$ in the 1950s, and the state of the art up until 1960 is summarized in the Gillman-Jerison [31] published in 1960. A conference dedicated to the forthcoming 25th anniversary of the publication of this book was held in 1982 in Cincinnati Ohio and its proceedings appeared in 1985 in [6]. He continued by claiming that since that time, the volume of papers on $C(X)$, in the journals devoted mainly to general topology, has diminished in the United States while increasing in some other countries. Because of the scope of his essay he only mentioned three sample papers, one in Iran (Ahvaz) [14], and the other two in Russia and Spain, see [24, 82] respectively, for more details". Although I am still not dealing with the title, but I cannot also help recalling an anecdote which is somehow connected to the title. Reading from the preface and acknowledgments of the above book, the authors admit the book had its origin in a seminar, organized by Henriksen, which was held at Purdue University during 1954-1955, and conducted jointly by him and the authors. The idea of writing such a book, using the seminar notes as the starting point was suggested by Henriksen, who at first collaborated with the authors. However, later he had to withdraw, for a personal reason, from the formal collaboration, but as the authors admit he continued his aid and advice. Nearly half a century later, at the 35th Annual Iranian Mathematics Conference in Ahvaz, while delivering his keynote address, Henriksen confessed the withdrawal from that collaboration was a big mistake in his life. Before returning to the story, I should also bring, a statistical information from the MathSciNet, to the attention of the reader , which is related to our story too, namely, "**More than %50 of all the articles written in the context of $C(X)$, in the world, which have been reviewed by the Mathematical Reviews, during 1995-2023, are written in Ahvaz**". I must admit before writing this story I was not aware of this surprising fact. Now as for our story in the title, I must say that this story, like any other story usually, must have a protagonist. Certainly, the reader after reading the whole story, will easily pinpoint who the protagonist of the story is. However, I would like to anticipate and let the reader know beforehand that Karamzadeh, as the leading character of the story, who initiated the study of $C(X)$ in Iran, too, and who has also been pursuing its main goal, is the protagonist of the story, by definition, metaphorically (note, I should apologize to the reader, for being a spoiler, and also to Karamzadeh, for he won't be happy to be considered as some sort of hero). Although I am not a good storyteller, however, I do my best to convey the details to the reader. To this end, without further ado, let us set the stage and make some preparations even for the non-experts. Rings are always commutative with the identity, unless otherwise stated. A space X is called Tychonoff if, and only if, for each closed set $F \subset X$ and $x \notin F$, there is $f \in C(X)$ such that $f(x) = \{0\}$ and $f(F) = \{1\}$. It is well-known that for every topological space X, there is a Tychonoff space X' which is a continuous image of X such that $C(X)$ and $C(X')$ are isomorphic, see [31, Theorem 3.9]. This manifestly suggests: To study the algebraic structure of $C(X)$ one may assume, without loss of generality, that X is Tychonoff. We may also follow this blanket assumption in this article. For $f \in C(X)$ the zeroset of f is denoted by $Z(f)$, where $Z(f) = \{x \in X : f(x) = 0\}$, and its cozeroset is denoted by $\mathrm{coz}f$, where $cozf = X \setminus Z(f)$. For any zeroset $Z(f)$ we have $Z(f) = \bigcap_{n=1}^{\infty}\{x \in X : |f(x)| < \frac{1}{n}\}$, which shows that every zeroset is

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a G_δ set, i.e., it is a countable intersection of open sets. The set of all zerosets of elements of $C(X)$ in X is denoted by $Z(X)$, which is closed under finite union, and under countable intersection, for if $f_i \in C(X)$ for $i \geq 1$, then $\bigcup_{i=1}^{n} Z(f_i) = Z(f)$, where $f = f_1 f_2 \cdots f_n$, and $\bigcap_{i=1}^{\infty} Z(f_i) = Z(g)$, where $g = \sum_{i=1}^{\infty} \frac{|f_i| \wedge 1}{2^i}$. An element $f \in C(X)$ is unit (i.e., it is invertible) if, and only if $Z(f) = \emptyset$. However, this is not true in $C^*(X)$, the subring of $C(X)$ consisting of bounded functions (note, let $X = \{x_1, x_2, \cdots, x_n, ...\}$ be an infinitely countable discrete space, and $f(x_n) = \frac{1}{n}$, then $Z(f) = \varnothing$), but f is not a unit in $C^*(X)$. An ideal I in $C(X)$ is called a z-ideal if $Z(f) \in Z[I]$ implies that $f \in I$, where $Z[I] = \{Z(g) : g \in I\}$. A topological space X is called a P-space (resp., F-space), if every finitely generated ideal in $C(X)$ is a principal ideal generated by an idempotent (i.e., $C(X)$ is a regular ring) (resp., if every finitely generated ideal in $C(X)$ is principal). These two spaces, which have interesting properties appear in many articles in the context of $C(X)$, see [31]. Moreover, X is a P-space (resp., F-space) if, and only if, for any $f, g \in C(X)$ we have $(f, g) = (f^2 + g^2)$ (resp., $(f, g) = (|f| + |g|)$, or X is a P-space if, and only if, every zeroset $Z(f)$ is open, equivalently if, and only if, every countable intersection of open sets in X is open. Although, more concepts and notations should be introduced, however we prefer to postpone them to subsequent sections, wherever they are needed, see also [31] for undefined ones. Let us now follow $C(X)$ in Ahvaz. I will try briefly to give short synopses of the main parts of some of the papers, not necessarily in chronological order, whose authors are connected, by the Mathematics Genealogy, to Ahvaz, and I will also give some comments, whenever necessary. I leave it to the interested reader to search for the details and for the papers of these kinds, which are still ongoing.

2. What made Karamzadeh attracted towards $C(X)$?

For this part I am just repeating his own words: When I finished my Ph.D. in non-commutative ring theory in June, 1974, at Exeter University (England), under the supervision of Professor David Rees, who was an eminent commutative ring theorist, I came to Chamran University (note, it was then called Jundi-Shapour University) in August, 1974. At that time, all my Iranian colleagues (note, there were some non-Iranian colleagues, too), at the department, except one, who was the dean of the college and had a Ph.D. degree in applied mathematics, were instructors holding master's degrees and were also too busy with their very heavy teaching loads. Pretty soon I found that most of them were not keenly interested in non-commutative ring theory and also I could not make them interested in the subject. Therefore I had to seek out another subject, apart from my own expertise, to communicate with some of them, in order, to attract their attention to the research work too. Especially for those who had already gone abroad before the revolution, to complete their studies, and after the revolution some of them had to leave their studies unfinished and return to the country. One day while I was browsing in the library I came across the book [31]. I started flicking through it and found it interesting to be read carefully. Therefore I borrowed the book to read it through. After a few months, I noticed some similarities of the results in the book, with some of the results in algebraic geometry. For examples, by weak Nullstellensatz, every maximal ideal M in $R = K[x_1, \cdots, x_n]$, where K is an algebraically closed field, is of the form $M = M_x = (x_1 - a_1, \cdots, x_n - a_n)$, for some point $x = (a_1, \cdots, a_n)$ in K^n

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(i.e., M consists of of all the polynomials which are zero at the point x). In particular, the maximal ideals in R are in $1-1$ correspondence with the points of K^n. Similarly, every maximal ideal M in $C(X)$, where X is a compact space, is of the form $M = M_x$, consisting of functions $f \in C(X)$, which are zero at the point $x \in X$. In particular, similar to the above ring R, the maximal ideals of $C(X)$ are also in one-one correspondence with the points in X. I should also bring to the attention of the reader, similarly to $R = K[x_1, \cdots, x_n]$, in which, the ideal $(x_1, x_2, \cdots, x_n)$ or more generally, the ideal $(x_1 - a_1, x_2 - a_2 \cdots, x_n - a_n)$ is maximal, without the assumption that the field K being algebraically closed, the ideal M_x above is also maximal in $C(X)$ without X being a compact space. This immediately shows the Jacobson radical of $C(X)$ is zero, for the intersection of its ideals of the form M_x, where $x \in X$ is zero (note, the latter ideals are called fixed maximal ideals or maximal fixed ideals, maximal ideals which are not fixed are called free). Consequently the intersection of its prime ideals is zero, too (i.e., it is a semiprime ring, i.e., a reduced ring). I noticed more interesting connections of algebraic properties of $C(X)$ with corresponding topological properties of X and vice versa. For example, X is connected if, and only if, $\mathbf{0}, \mathbf{1}$ are the only idempotents in $C(X)$, which in a sense means that, $C(X)$ must also be connected too, i.e., it cannot be written as the direct sum of two nonzero ideals. Also $C(X)$, has a peculiar property in common with different kind of rings, such as the ring of integers, division rings, and local rings, even the non-commutative local rings, in the sense that, in all these rings every element can be written as a finite sum of units. For the latter rings this is obvious, and in the case of $C(X)$, one may write $f = g + h$, where $f \in C(X)$ is arbitrary, $g = f \vee 0 + 1$, and $h = f \wedge 0 - 1$. Let me recall some more facts which were responsible for my interest in $C(X)$. After the appearance of Cantor-Schröder-Bernstein Theorem, in set theory, some interested authors were after similar facts in relation to other structures, such as, groups, rings, modules, topological spaces, etc.. For example, C. Kuratowki in [60], has given two compact spaces X, Y, each homeomorphic to an open subset of the other, but with X not homeomorphic to Y. Therefore by Theorem 4.9, in [31], $C(X)$ is not isomorphic to $C(Y)$. This together with the fact in [31, 1B], gives two rings, each isomorphic to a subring of the other, but yet not being isomorphic to one another. He used to say it was interesting for him that the subject of $C(X)$ has resolved two problems in rings and topological spaces, connected with Cantor-Schröder-Bernstein theorem, simultaneously, see [71, pp. 16-38]. At the same time he is still interested in characterizing rings and topological spaces, similarly to the injective modules, for which the latter theorem holds, see [22, Theorem 3.3A]. One final point in his own words, in this regard: When I was reading through [31], I came across the problems [31, 4B.(1, 2)] and noticed those two problems have standard solutions in ring theory. For example, for 4B.1, one may notice that every maximal ideal in $C(X)$ is idempotent (note, prime ideals, semiprime ideals and z-ideals are also idempotent in $C(X)$). It is also well-known that a finitely generated ideal I in a ring R, which is idempotent, is generated by an idempotent element (note, if $I^2 = I$, then by a proof similar to the proof of Nakayama's Lemma, we infer that there exists $e \in I$ with $(1-e)I = (0)$, i.e., $I = (e)$, and e is clearly idempotent), let alone when M_x is a principal ideal in $C(X)$, in which case, it is evident that M_x is generated by an idempotent, and hence $x \in X$ must be an isolated point. Similarly, for $4B.2$, which is true, more generally, when maximal ideals in $C(X)$ (resp., $C^*(X)$) are

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finitely generated (i.e., each maximal ideal is generated by an idempotent element). To see this, even a non-commutative ring R, is Artin semisimple if, and only if, every maximal right ideal is generated by an idempotent, see [61, Ex. 6], and in case R is a commutative ring, R is a finite direct product of fields. Moreover, if $R = C(X)$, then $C(X)$ is a finite direct product of the field, $\mathbb{R}$, and X must also be finite. These observations together with some other observations in [31], made it easy for me to come to a decision that the subject of $C(X)$ is an appropriate subject for which, I was looking for, and on which, I could share my thoughts, with more colleagues and students with different fields of interest in mathematics. Moreover, pegging a particular topological property of X to an algebraic property of $C(X)$ and vice versa, which is the main objective of $C(X)$, as it is also emphasized, in [31, 1.5.], and for which one needs to present the results, in this context, in the form of "if, and only if" (note, this form for presenting results in mathematics was my favorite from high school days until now), was another good reason for my attraction to the field.

3. Ahvaz, the place where $C(X)$, for the first time, is noticed to be compatible with some non-commutative concepts!

Although $C(X)$ is as commutative as the field $\mathbb{R}$, for the product fg, where $f, g \in C(X)$, is defined by $fg(x) = f(x)g(x)$, with $x \in X$, and $f(x), g(x) \in \mathbb{R}$, however if we assume any usual conditions on $C(X)$, such as being Artinian or being Noetherian, or having Krull dimension, or any other similar properties, which are usually studied in commutative rings, it generally implies that X must be a finite space, which is not a desirable result in the context of $C(X)$. Let us before mentioning the concepts of the socle of a ring and essential ideals in the context of $C(X)$, which are essentially introduced for the first time in this context in [57], make some comments. Surprisingly, there does not exist a single popular textbook, either elementary, or advanced, on commutative rings, let alone on $C(X)$, which deals with these two concepts. However, it is folklore that these two concepts have fundamental roles in the structure theory of non-commutative associative algebra. By the socle of a commutative ring R, denoted by $\mathrm{socle}(R)$, we mean the sum of its minimal ideals which like the non-commutative case is, in fact, the direct sum of these ideals, it also coincides with the intersection of essential ideals of the ring. And by an essential ideal in R, we mean an ideal A which intersects every nonzero ideal of R non-trivially, in the sense that $A \bigcap B \neq (0)$, where B is a nonzero ideal. Let us recall that for any nonzero ideal I in R, $I + \mathrm{Ann}(I)$ is essential in R, where $\mathrm{Ann}(I) = \{a \in R : aI = 0\}$. This immediately implies that, in reduced rings (e.g., $C(X)$) an ideal I is essential if, and only if $\mathrm{Ann}(I) = 0$, for $(I \bigcap \mathrm{Ann}(I))^2 = 0$. Let us digress for a moment and bring to the attention of the reader a point which is also useful in $C(X)$. In [30] the concept of strongly essential submodules is introduced and while obtaining many useful results it is observed that many known essential ideals, in the literature, are in fact strongly essential ideals. It is easy to see an ideal I in a ring R is essential if for a given finite set $S \subseteq R$ containing a nonzero element, there exists an element $r \in R$ with $0 \neq Sr \subseteq I$ and an ideal I, by definition, is strongly essential if for any given subset $T \subseteq R$ containing a nonzero element there exists $r \in R$ with $0 \neq Tr \subset R$. Thus for an ideal I in a ring R, $I + Ann(I)$ is, in fact, strongly essential and consequently in reduced rings (including $C(X)$) essential ideals and strongly essential

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ideals coincide. We should remind the reader that originally essential ideals in the literature were not assumed to be proper ideals, otherwise they may not exist in every ring (note, any finite direct product of fields has no proper essential ideal, see [53, 54]). In [57], the socle of $C(X)$ characterized as the set consisting of those elements of $C(X)$ which are zero everywhere except on a finite number of points in X (note, the socle of $C(X)$ coincides with that of $C^*(X)$). This implies that the socle of $C(X)$ is a z-ideal and it is a free ideal if, and only if, X is a discrete space. Minimal ideals in $C(X)$ and maximal ideals related to the socle, which are called isolated maximal ideals are also studied in [57]. It is observed that these are, in fact, the isolated points in $\mathrm{Max}(C(X))$, the space of maximal ideals of $C(X)$ with the Zariski topology (also, called Stone topology or hull-kernel topology)). These maximal ideals had already been introduced and systematically studied in [52], where they were called nice maximal ideals). By taking an intersection of a finite number of these ideals as a neighborhood of the zero element of $C(X)$ in [57], the In-topology is defined and studied on $C(X)$, which makes $C(X)$ a topological ring (not necessarily Hausdorff). To see when the latter space is Hausdorff we cite the following result from [57].

Theorem 3.1. *The following statements are equivalent.*

(1) $I(X)$*, the set of isolated points of* X*, is dense in* X*.*
(2) The ideal S*, which is the socle of* $C(X)$*, is essential in* $C(X)$*.*
(3) $\mathrm{Ann}(S) = 0$*.*
(4) $C(X)$ *with In-topology is Hausdorff* (*i.e.,* $J_0 = 0$*, where* J_0 *is the intersection of all isolated maximal ideals*)*.*

This theorem immediately implies that if $I(X)$ is dense in X, then the set of zero divisors is an open set in $C(X)$ with In-topology.

I would like to attract the attention of the reader to the equivalency of the above two first parts in the theorem, which emphasizes on the unity of mathematics. It means that, when we say some object is dense in a space X, it is the same thing as saying some object is essential in the ring $C(X)$. We should also notice that, this unity indicates in any structure in mathematics, it is having a non-trivial intersection with each basic member of that structure that matters (note here, open sets and ideals are the basic elements in the above two structures, with each of which, $I(X)$, and the socle of $C(X)$ have nontrivial intersection, respectively).

Another salient feature of [57], is the following fact, which is [57, Lemma 2.6].

Fact. Let $|I(X)| \leq \mathfrak{c}$, where $\mathfrak{c}$ is the cardinality of the continuum. Now if $C(X)$ is isomorphic to $C(Y)$, where Y is a discrete space, then so too is X.

This immediately yields a simple proof of the well-known fact that every discrete space of cardinality less than or equal to the cardinality of the continuum is a realcompact space (note, A space is realcompact if, and only if, it is homeomorphic with a closed subspace of a direct product of real lines), see [31, Corollary 8.18, 8G. 8] for two different proofs of this fact. Incidentally, the simplest possible proof of this fact, among the known proofs, is given recently in [50]. We should remind the reader that

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although it is well-known that a discrete space X is realcompact if, and only if, its cardinal is nonmeasurable, and the cardinality of the continuum is also nonmeasurable , see [31, Theorem 12.2], but the above simple proofs are desirable without referring to the latter complicated result. Since [57], initiated the study of $C(X)$ in Iran, I like to recall more important facts from it. The support of $f \in C(X)$ is defined to be the closure of $X \setminus Z(f)$ and let $C_K(X) = \{f \in C(X) : \text{support f is compact}\}$. The equality of $C_K(X)$ with the intersection of the free maximal ideals (note, an ideal I in $C(X)$ is called free if $\bigcap Z[I] = \varnothing$, otherwise it is called fixed) was first proved for discrete spaces by Kaplansky, who asked if the equality holds in general. Later Kohls generalized Kaplansky's result to P-space and in turn, by applying the concept of the socle, Kohls' result is extended too in [57]. The topological spaces for which the above equality holds for $C_K(X)$, is recently characterized in [81], and the author is an Ex-student of Azarpanah (i.e., one of those authors connected to Ahvaz). In [57] it is also observed that if X is a pseudo-finite space (a space in which every compact subspace is finite), then the socle of C(X) coincides with $C_K(X)$. Similarly to the question of Kaplansky, the equality of $C_K(X)$ with the socle of $C(X)$, in general, is raised as an open question in [57]. This question was later settled by Azarpanah in his Ph.D. thesis "On the essential ideals in $C(X)$ in Ahvaz, see also [8]. Indeed, he proved the following interesting theorem. Let us first recall that $C_\infty(X)$ is the ideal in $C^*(X)$ consisting of functions f which vanish at infinity, in the sense that, $\{x \in X : |f(x)| \geq \frac{1}{n}\}$ is compact for each positive integer n , see [31, 7F.]. Incidentally, in the next theorem the notation $C_F(X)$, for the socle of $C(X)$, is suggested by Azarpanah.

Theorem 3.2. $C_K(X) = C_F(X)$ (resp., $C_\infty(X) = C_F(X)$) if, and only if, X is pseudo discrete, i.e., every compact subspace of X has finite interior (resp., X is pseudo-discrete with only finite number of isolated points).

Before concluding our summary of [57], let us recall that the Stone-Čech compactification of a space X, which is denoted by βX, is perhaps the most important compactification of a space X, and it is an indispensable object in topology, and in particular in $C(X)$. It is well-known that $C^*(X) \cong C(\beta X)$ and βX is homeomorphic with $Max(C(X))$, the space of maximal ideals of $C(X)$ with the Zariski topology under a map that leaves X pointwise fixed, see [31, Chapter 6], for more details. A maximal ideal M of $C(X)$ (resp., $C^*(X)$) is called a real maximal ideal if $\frac{C(X)}{M} \cong \mathbb{R}$ (resp., $\frac{C^*(X)}{M} \cong \mathbb{R}$). Clearly every fixed maximal ideal M_x of $C(X)$, is real maximal. And every maximal ideal in $C^*(X)$ is real, by the above isomorphism with $C(\beta X)$. A non-real maximal ideal is called hyper-real. This is a proper place to remind the reader of an equivalent definition for a realcompact space, namely, if every real maximal ideal in $C(X)$ is fixed, then X is said to be realcompact. And the subspace υX of $Max(C(X))$, consisting of real maximal ideals of $C(X)$ is called the realcompactification of X and $C(X) \cong C(\upsilon X)$, see also [Chapter 8] [31], hence we have $X \subset \upsilon X \subset \beta X$. Now we highlight the next simple proposition, in [57], which is, in fact, purely an algebraic simple proof of a well-known topological fact that a point $x \in X$ is isolated if, and only if, it is isolated in βX (resp., υX), see [31, 3.15. (c)] and [31, P. 90 (d)], for two topological proofs. Before stating the proposition let us record its consequences: $I(X) = I(\beta X) = I(\upsilon X)$, and in view of Theorem 3.1, $I(X)$ is dense in X, if,

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and only if, the socle of $C(\beta X)$ (resp., $C(\upsilon X)$) is essential in $C(\beta X)$ (resp., $C(\upsilon X)$). In this case the In-topology of $C(\beta X)$ (resp., $C(\upsilon X)$) is also Hausdorff. Moreover, in the same way that if the socle of a ring is essential, it is, in fact, the smallest essential ideal in the ring, we infer that when $I(X)$ is dense in X, it is, in fact, the smallest dense subset of all spaces X, βX, and υX, see also [4, Theorem 5.2].

Proposition 3.3. *A maximal ideal M in $C(X)$ is isolated (i.e., it is generated by an idempotent) if, and only if, it is an isolated point in $Max(C(X))$.*

Finally, this section is a proper place to recall the concept of super socle of $C(X)$ which is introduced in [28] and it is a complement to the socle of $C(X)$. By the super socle of $C(X)$, denoted by $SC_F(X)$, we mean the set of functions in $C(X)$ which are zero everywhere except on a countable set (i.e., $f \in SC_F(X)$ if, and only if $X \setminus Z(f)$ is countable). Clearly, $SC_F(X)$ is a z-ideal containing, $C_F(X)$, the socle of $C(X)$. In [28], among other things the one-point Lindelöffication of an uncountable discrete space is algebraically characterized via the super socle as follows.

Theorem 3.4. *The following statements are equivalent for an infinite space X.*
1. X is the one-point Lindelöffication of an uncountable discrete space.
2. $SC_F(X)$ is a regular ideal with $SC_F(X) = O_x$ for some $x \in X$, where for each $x \in X$, O_x consists of functions $f \in C(X)$ such that $Z(f)$ is a neighborhood of x.

In contrast to $C_F(X)$ which is never a prime ideal in $C(X)$, $SC_F(X)$ is prime for some spaces X. The following theorem in [29] characterizes these spaces.

Theorem 3.5. *$SC_F(X)$ is a prime ideal in $C(X)$ if, and only if, either X is the one-point Lindelöffication of an uncountable discrete space or it is a countable set, in which case $SC_F(X) = C(X)$.*

4. Goldie dimension in $C(X)$

The above dimension which is also called uniform dimension is another non-commutative concept which is first introduced and studied by Azarpanah in the context of $C(X)$, in his 1994 Ph.D. thesis, at Ahvaz university. An ideal A, which may not be proper, in a ring R, is said to have finite Goldie dimension if it has no infinite set of independent of ideals within itself. A set of nonzero ideals $\{A_i\}_{i\in I}$ in A is said to be independent if $A_j \bigcap \sum_{j\neq i\in I} A_i = (0)$. And the smallest cardinal number, λ say, which is greater or equal to the cardinality of any independent set of nonzero ideals in A is called the Goldie dimension of A and is denoted by $\text{Gdim}\, A = \lambda$. If a ring R has finite Goldie dimension, then it can be proved that there is an ideal U in R which is of the form $U = U_1 \oplus U_2 \oplus \cdots \oplus U_n$ such that n is the largest possible positive integer, and each U_i is a nonzero ideal. Clearly, U is an essential ideal and each ideal U_i is a uniform ideal (i.e., any two nonzero ideals inside U_i intersect non-trivially, i.e., each nonzero ideal in U_i essential inside U_i). Moreover, it can also be shown that if $U = U_1 \oplus U_2 \oplus \cdots \oplus U_n$ and $V = V_1 \oplus V_2 \oplus \cdots \oplus V_m$ are two essential ideals in R, where U_i, V_j are uniform ideals for all i, j, then $m = n$. These observations imply that when a ring R has finite Goldie dimension, then there is a unique positive integer n and an essential ideal which is the direct

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sum of n uniform ideals. This n is called Goldie dimension of R denoted by $\operatorname{Gdim} R$. Let us also recall the Souslin number, which is also called the cellularity, of the space X. The smallest cardinal number c say, such that every family of pairwise disjoint nonempty open subsets of X has a cardinality less than or equal to c, is called the Souslin number, or cellularity of the space X and is denoted by $S(X)$. If $S(X)$ is countable, then we say that X has the Souslin property or X has the countable chain condition (briefly, *c.c.c.*), for example every separable space has *c.c.c.*. The question of whether the product of any two spaces with the Sousolin properties has the Sousolin property, is independent of ZFC, it is true under Martin's Axiom (briefly, MA) plus the negation of CH, but it fails if CH holds (note, by MA, we simply assume that, no compact Hausdorff space with the *c.c.c.* is the union of $< 2^\omega$ closed nowhere dense subsets. Surprisingly, it is proved if this property is preserved by products of two factors, it is preserved by arbitrary products. Azarpanah in his thesis observed that uniform ideals in $C(X)$ are the minimal ideals and $C(X)$ has finite Goldie dimension if, and only if, X is finite, and he also proved that the Goldie dimension of $C(X)$ coincides with the Suouslin number X, see also [7]. In consequence, if X is dense in Y, then $\operatorname{Gdim} C(X) = \operatorname{Gdim} C(Y)$ (e.g., $\operatorname{Gdim} C(X) = \operatorname{Gdim} C(\upsilon X) = \operatorname{Gdim} C(\beta X)$) Therefore the previous topological question can be translated into an algebraic question, namely: If $\operatorname{Gdim} C(X) = \operatorname{Gdim} C(Y) = \aleph_0$, is $\operatorname{Gdim} C(X \times Y) = \aleph_0$? Goldie dimension and essential ideals which are related concepts (note, $\operatorname{Gdim} R = \operatorname{Gdim} A$ for any essential ideal in a ring R), play a fundamental role in the structure theory of non-commutative rings. These two concepts are topologically characterized in Azarpanah's thesis. One is already shown above and for the essential ideals we have:

Theorem 4.1. *The following statements are equivalent in $C(X)$.*

(1) An ideal A is essential in $C(X)$.
(2) A intersects every nonzero z-ideal non-trivially.
(3) $Z(A) = \bigcap Z[A]$ has empty interior.

We should remind the reader that part (3) of the above theorem is also observed in [P. 14] [23] (note, dense ideals D in [23] (i.e., $\operatorname{Ann}(D) = 0$), coincide with essential ideals in $C(X)$). Immediate consequences of part (3) above are the facts that every free ideal in $C(X)$ is essential and when X has no isolated points every prime ideal is essential. Moreover, in any space X, not necessarily without isolated points, there is an essential ideal which is a z-ideal, hence an intersection of prime ideals, but it is not prime, see [7]. In contrast to the fact that the socle of every ring is the intersection of the essential ideals, this intersection in $C(X)$ for a large class of topological spaces X, including compact spaces, is reduced to the intersection of certain essential ideals, which are in a sense topological objects, namely, the ideals $O_x = \{x \in Z(f) : Z(f) \text{ is a neighborhood of } x\}$, where x runs through the set of nonisolated points of X. Due to the importance of the socle and essential ideals in $C(X)$ these concepts and the quotient ring $\frac{C(X)}{C_F(X)}$, where $C_F(X)$ is the socle of $C(X)$, are fully investigated in [15], see also [21], where the latter quotient ring is first studied. The following results are all appeared in [15]. Let us begin with the simple observation that if the set of isolated points, $I(X)$, of X is finite then $\frac{C(X)}{C_F(X)} \cong C(Y)$, where $Y = X \setminus I(X)$. Perhaps, the starting point towards the main goal of $C(X)$

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is the simple fact that X is connected if, and only if $\mathbf{0}, \mathbf{1}$ are the only idempotents in $C(X)$, which implies that X is connected if, and only if, υX (resp., βX) is connected. The next result generalizes this starting point, from which, for a discrete space X we immediately infer that $\beta X \setminus X$ is connected.

Proposition 4.2. *Let D be the set of isolated points of X. Then $\beta X \setminus D = Y$ is connected if, and only if, for each idempotent $f = f + C_F(X) \in \frac{C^*(X)}{C_F(X)}$, we have either $Y \subseteq Z(f^\beta)$ or $Y \subseteq Z(1 - f^\beta)$, where $f \to f^\beta$, is the isomorphism of $C^*(X)$ onto $C(\beta X)$.*

There are many useful results in [15]. For example every prime ideal in $\frac{C(X)}{C_F(X)}$ is essential (note, this immediately shows that the socle of $\frac{C(X)}{C_F(X)}$ is zero because this ring is reduced), which is also in [21], and for each essential ideal E of $C(X)$, the ideal $\frac{E}{C_F(X)}$ is essential in $\frac{C(X)}{C_F(X)}$ if, and only if, the set of isolated points, $I(X)$, in X is finite. Although in general $\frac{C(X)}{C_F(X)}$ may not be isomorphic to $C(Y)$ for any topological space (Y) , in any case, one encounters a curious similarity between the two rings $C(X)$ and $\frac{C(X)}{C_F(X)}$, and their common properties usually give rise to useful information about X. For example X is a P-space if, and only if $C(X)$ (resp., $\frac{C(X)}{C_F(X)}$) is a regular ring. The inequality $\operatorname{Gdim} C(X) \leq \operatorname{Gdim} \frac{C(X)}{C_F(X)}$ holds and can be strict. It is well-known that unless the Goldie dimension of a ring R is an inaccessible cardinal, $GdimR$ is attained, see [15]. In Examples 8 and 9 of [15], it is also shown that rings with arbitrary Goldie dimensions exist. Incidentally, it is observed that the cardinality of each of the latter rings is the same as its Goldie dimension. In [15, Remark 2.10], by using the sophisticated fact of the infinite additivity of infinite Goldie dimensions, more rings with large cardinality λ are constructed whose Goldie dimension is any infinite cardinal $\mu < \lambda$. In particular, taking advantage of the topological definition of Goldie dimension of $C(X)$, it is shown in [15] that given any inaccessible cardinal number λ, there exists a zero-dimensional compact space X such that the Goldie dimension of $C(X)$ is λ, but unattained. Let us remind the reader that compactness is not an algebraic property, in the sense that $C(X)$ may be isomorphic with $C(Y)$, with Y a compact space but X is not (e.g., for a pseudo-compact space X which is not compact we have $C(X) = C^*(X) \cong C(\beta X)$). However, in view of the following theorem in [15], compact spaces with at most countable number of non-isolated points are algebraic objects.

Theorem 4.3. *A topological space X is a compact space with at most countable (resp., finite) number of non-isolated points if, and only if, $C(X)$ (resp., $\frac{C(X)}{C_F(X)}$) has at most countable (resp., finite) number of essential maximal ideals.*

Let us bring to the attention of the reader that we also have the next interesting fact in [15], too.

Theorem 4.4. *If $S(X) = \aleph_0$, then the following are equivalent.*

(1) $\frac{C(X)}{C_F(X)} \cong C(Y)$, *where Y is a space with $S(Y) = \aleph_0$.*
(2) $\operatorname{Gdim} \frac{C(X)}{C_F(X)} = \operatorname{Gdim} C(X)$.
(3) X has at most finite number of isolated points.

By showing that the Jacobson radical of $\frac{C(X)}{C_F(X)}$ is zero if, and only if every compact subspace of X has at most finite number of isolated points, it is inferred in [15] that the cardinality of a discrete

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space X is nonmeasurable (i.e., X is realcompact) if, and only if, υX is first countable. We conclude this section with a comment on the following proposition in [15].

Proposition 4.5. *Let $I = (f_1, f_2, \cdots, f_n, \cdots)$ be a countably generated ideal in $C(X)$. Then there exists $g \in C(X)$ such that $I \subseteq \bigcap_{n=1}^{\infty}(g^n)$. Moreover, I is essential in (g^n) for all n and $Z(g) = \bigcap_{n=1}^{\infty} Z(f_n)$.*

This proposition has some interesting consequences some of which are pointed out in [15]. We also observe, it improves the well-known fact the set of zerosets in X, i.e., $Z(X)$, is closed under countable intersection, by showing not only is the intersection of any countable zerosets $\{Z(f_i)\}_{i=1}^{\infty}$, a zeroset $Z(g)$, for some $g \in C(X)$, but also one may choose this function g to be such that the ideal generated by these f_i's is essential in (g), too.

5. Intersection of essential ideals in $C(X)$

Clearly $C_K(X) \subseteq C_\infty(X)$ and the former ideal is the intersection of the free ideals in $C(X)$, as well as, the intersection of the free ideals in $C^*(X)$, and the latter ideal is intersection of the free maximal ideals in $C^*(X)$, see [31, 7F]. Hence by Theorem 3.4, they are both the intersection of some essential ideals. The intersection of the essential ideals may be zero in $C(X)$ (e.g., when X has no isolated point). Therefore the essentiality of an intersection of essential ideals, in particular, that of the ideal $C_K(X)$ or $C_\infty(X)$ is a natural question. We have already noticed in Theorem 3.1, the intersection of the essential ideals in $C(X)$ (i.e., its socle) is essential if, and only if, the set of isolated points is dense in X or equivalently the socle of $C(X)$ is essential. The following theorem in [8] answers the previous question.

Theorem 5.1. *The following statements are equivalent.*

(1) X is an almost locally compact space (note, a Hausdorff space X is said to be almost locally compact if every non-empty open set of X contains a non-empty open set with compact closure.)

(2) $C_K(X)$ is an essential ideal in both $C(X)$ and $C^*(X)$.

(3) $C_\infty(X)$ is an essential ideal in $C^*(X)$ (note C_∞ may not be an ideal in $C(X)$, however it is also essential in $C(X)$, in the sense that $C_\infty(X) \bigcap A \neq (0)$, where A is an ideal in $C(X)$).

Theorem 5.2. *Let X be a compact space. Then every countable intersection of essential ideals of $C(X)$ is an essential ideal if, and only if every first category subset of X is nowhere dense in X.*

The ideal $C_\infty(X)$ is fully studied in [3], where, it is observed, inter alia, similarly to $C(X)$, that this ideal has finite Goldie dimension if, and only if, X is finite. Moreover, for any Hausdorff space X there is a locally compact Hausdorff space Y such that $C_\infty(X) \cong C_\infty(Y)$. And it is also observed that, for any two locally compact Hausdorff spaces X, Y, $X \cong Y$ if, and only if $C_\infty(X) \cong C_\infty(Y)$. It is already emphasized that $C_\infty(X)$ may not be an ideal in $C(X)$. Motivated by this, spaces X for which $C_\infty(X)$ is an ideal in $C(X)$, are characterized in [20]. To be precise $C_\infty(X)$ is an ideal in $C(X)$ if, and only if, every open locally compact subset of X is bounded. In particular, if X is a locally compact Hausdorff space, $C_\infty(X)$ is an ideal of $C(X)$ if, and only if, X is a pseudocompact space.

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6. Goldie dimension revisited in the ring of fractions of $C(X)$

Let us first cite the following proposition from [2].

Proposition 6.1. *The following statements are equivalent.*

(1) *X is an F-space.*
(2) *$C(X)$ is locally a domain* (i.e., $C(X)_P$ is a domain for each prime ideal P).
(3) $C(X)_M$ is a domain for every maximal ideal M of $C(X)$.
(4) $C(X)$ is locally uniform (i.e., $\operatorname{Gdim} C(X)_P = 1$ for every prime ideal P).

This immediately implies that X is an F-space if, and only if the primary ideals of $C(X)$ contained in any maximal ideal of $C(X)$, are comparable. It is well-known that if R is a Noetherian ring, which is locally a domain, and without nontrivial idempotents, then R must be a domain, see [39, Theorem 168]. In reference [6] of [2] it is asked if there is a non-Noetherian ring without nontrivial idempotents which is locally a domain, but not a domain. A decade later in reference [23] of [2], a rather complicated example is given. Applying the above simple proposition together with the starting point in the context of $C(X)$ (i.e., X is connected if, and only if, $C(X)$ has no nontrivial idempotents) it is rightly claimed in [2] that for an infinite connected F-space X, $C(X)$ is a natural example for the above required ring (note, for any infinite space X, $C(X)$ is neither a Noetherian ring nor a domain). The rank of a point $x \in \beta X$, denoted by $rk(x)$, is defined to be the number of minimal prime ideals contained in M^x, if the set of all such minimal prime ideals is finite, and $rk(x) = \infty$ otherwise (note, $M^x = \{f \in C(X) : x \in cl_{\beta X} Z(f)\}$ and $\{M^x : x \in \beta X\}$ is the set of all maximal ideals of $C(X)$). The number $rk(x)$ is called the rank of x. The notion of rank of a point is first introduced and studied, by Henriksen and Larson, see reference [9] in [2]. The following result in [2], motivates the authors to define the rank of a point $x \in \beta X$ for infinite cardinals.

Theorem 6.2. *Let the rank of a point $x \in \beta X$ be finite. Then $rk(x) = \operatorname{Gdim} C(X)_{M^x}$.*

This easily paves the way for the authors of [2] to define $rk(x) = \operatorname{Gdim} C(X)_{M^x}$, where $x \in \beta X$ and $rk(x)$ may be any cardinal, finite or infinite.
It is observed that part (4) of Proposition 6.1, i.e., locally uniform, cannot be replaced by locally Goldie finite-dimensional (i.e., $C(X)_P$ has finite Goldie dimension, for every prime ideal P in $C(X)$). It is also observed in [2, Remark 3.6], that in case $rk(x)$ is infinite for some $x \in \beta X$, then $\operatorname{Gdim} C(X)_{M^x}$ may no longer be the cardinality of the set of minimal prime ideals in M^x. The authors in [2] also observe that for any cardinal number λ, there is a space X and a multiplicatively closed set $S \subset C(X)$ with $\operatorname{Gdim} S^{-1}C(X) = \lambda$. One last word in this section, Finite Goldie dimensional rings are not usually dealt with, in popular textbooks on commutative rings, let alone infinite Goldie dimensional rings. The contents of [2] might be of interest to experts in commutative ring theory, and it contains some challenging problems for them, too.

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7. $\aleph_0$-selfinjectivity vs. regularity in $C(X)$

A module A over a ring R is called injective (resp., $\aleph_0$-injective) if any homomorphism $f : I \to A$, where I is an ideal (resp., a countably generated ideal) of R can be extended to R, i.e., there exists $a \in A$ with $f(r) = ra$ for all $r \in I$. If $A = R$ in this definition, then R is called a selfinjective (resp., $\aleph_0$-selfinctive) ring.

Although the selfinjectivity of rings is a common concept between the commutative and the non-commutative rings, however it is more popular and used in the non-commutative setting, see [22]. Let us recall the statement of the open problem in Matlis' paper, which is settled by Karamzadeh, in [48]: Is $\frac{\prod_{i\in I} F_i}{\sum_{i\in I} \oplus F_i}$, where each F_i is a field, a self-injective ring?. Indeed, he proved, with two methods, that this quotient ring is never self-injective. But as he once admitted, since $C(X) \cong \prod_{i\in I} R_i$, where X is a discrete space, and $R_i = \mathbb{R}$ for all $i \in I$, with $|X| = |I|$ and $C_F(X) = \sum_{i\in I} \bigoplus R_i$, he also became interested in considering the selfinjectivity (resp., the $\aleph_0$-selfinjectivity) of the quotient ring $\frac{C(X)}{C_F(X)}$ for a general space X. He suggested the problem to his Ph.D. student, A.A. Estaji, for the topic of his thesis. Consequently, among other interesting results, such as no prime ideal in $\frac{C(X)}{C_F(X)}$ can be finitely generated, and in case X is a P-space this prime ideal cannot even be countably generated, or $C_F(X)$ the socle of $C(X)$ cannot be a prime ideal, etc., the following major theorems concerning the selfinjectivity are also obtained, in [21]. Before stating the theorems, let us recall an anecdote to see how non-commutative ring theory and $C(X)$ interacted in Ahvaz. Once, in a private conversation with Karamzadeh, he told me that Nakayama's Lemma is a very useful result (note, the fact that for a right finitely generated module M over a non-commutative ring R, with $MI = M$, where $I \neq (0)$ is a right ideal inside the Jacobson radical of R, implies that $M = (0)$, is called Nakayama's Lemma), see [22, Lemma 3.35, and the comments following it], to see how highly this lemma is praised, even as important as Zorn's Lemma. However, Karamzadeh claimed this lemma is not stated in its best possible form in the literature. He continued by saying that, this lemma is, in fact, valid for all modules M which have maximal submodules over any ring (to see this, let N be a maximal submodule of $M \neq (0)$, then $\frac{M}{N}.I = (0)$ (note, $\frac{M}{N}$ is a simple module) implies that $MI \subseteq N$, i.e., $MI \neq M$), which includes f.g. modules, and modules with a minimal generating sets (a set $S \subseteq M$ is a minimal generating set for a module M, if $(S - \{x\}) \neq M = (S)$ for any $x \in S$. We should also emphasize that free modules and semisimple modules have minimal generating sets, i.e., the above lemma is also valid for all these modules, not necessarily f.g. modules. A ring R, commutative or not, over which every nonzero module has a maximal submodule is called a Max-ring. Let us also recall that a non-commutative ring R is called a right V-ring if every simple module over R is injective, see [22, 3.19A]. Kaplansky proved that a commutative ring R is a V-ring if, and only if, R is a regular ring, see [22, 3.19B]. A ring R (not necessarily commutative) is said to have Artin-Rees property (briefly, AR-property) if given any two ideals A, B, where A is finitely generated, then there exists a positive integer n such that $A^n \bigcap B \subseteq AB \bigcap BA$ (e.g., Noetherian rings, regular rings (commutative or not), see also [41]). Karamzadeh also reminded me that in his conversation with A.R. Aliabad, while advising their joint Ph.D. student, S. Afrooz, in a tutorial class, they noticed that, in order for

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$C(X)$, to be a V-ring (i.e., for each module $\frac{C(X)}{M}$, where M is a maximal ideal, to be injective over $C(X)$), it suffices to take M to be a fixed maximal ideal, i.e., $M = M_x$ for some $x \in X$. Let us state the following theorem, whose proof can be found in [1]. Since the theorem is a characterization of P-spaces and these spaces are the main objects in this section, I may also digress for a moment and first recall an interesting anecdote concerning the creation of these spaces (note, I have learned this short story like many others from Karamzadeh). L. Gillman, who jointly with M. Henriksen, created these spaces, once said Henriksen and he were roommates at Purdue University. Henriksen came later than Gillman to Purdue and used to try many times to interest Gillman in the subject of his expertise (Note, this reminds me of Karamzadeh, who at his early time in Ahvaz, was trying, without any success, to make his colleagues interested in non-commutative ring theory). Reading a paper by Kaplansky in which Kaplansky showed that in the ring of real functions on a discrete space, every prime ideal is maximal, Henriksen was interested to see whether conversely, this algebraic property for $C(X)$ characterizes X as a discrete space. At first, at Purdue University, Gillman was busy with his own work, however, soon they were working together on the question, that answered it (No) and obtained the correct characterization: all zerosets of continuous functions are open (as well as closed). They called these spaces P-spaces, and the letter P stands for pseudo-discrete or for prime or, as they joked, for "professor", because apparently they both had just been promoted from instructors to assistant professor, then.

Theorem 7.1. *The following statements are equivalent.*

(1) *X is a P-space.*
(2) *$\frac{C(X)}{M_x}$ is injective as a module over $C(X)$.*
(3) *Every ideal of $C(X)_P$, where $C(X)_P$ is the localization of $C(X)$ at a prime ideal P, has a minimal generating set.*
(4) *Every ideal of the ring $\frac{C(X)}{P}$, where P is an prime ideal, has a minimal generating set.*
(5) *For each prime ideal P, $C(X)_P$ has Krull dimension in the sense of Gabriel-Rentschler, see [22, 14.26].*
(6) *Every finitely embedded module* (i.e., having a finitely generated essential socle) over $C(X)$, is injective.
(7) Every Artinian module over $C(X)$ is an injective semisimple module.
(8) $\frac{C(X)}{C_F(X)}$ has AR-property.
(9) For any $f \in C(X)$, the rings $\frac{C(X)}{(f)}$ and $C(X)_f$ are both regular rings, where $C(X)_f = S^{-1}C(X)$ and S is the multiplicative set consisting of the powers of f.

The next corollary is now, in order, and since it is overlooked in [1], its proof is given. Incidentally, the corollary can be considered as a proof of (1) $\Longleftrightarrow$ (4) in the above theorem, too.

Corollary 7.2. *Let I be an ideal containing a prime ideal P in $C(X)$, such that every ideal in $\frac{C(X)}{I}$ has a minimal generating set. Then I is maximal.*

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Proof. Clearly I is contained in a unique maximal ideal, say M, and $\frac{M^2}{I} = \frac{M}{I}$. But $\frac{M}{I}.\frac{M}{I} = \frac{M}{I}$ and the Jacobson radical of the local ring $\frac{C(X)}{I}$ is $\frac{M}{I}$. Now by the Nakayama's Lemma we get $\frac{M}{I} = (0)$ (note, $\frac{M}{I}$ has a maximal submodule over the ring $\frac{C(X)}{I}$, by the above comment of Karamzadeh), i.e., $I = M$ and we are done. □

Next, we give the promised major theorems.

Theorem 7.3. *The following statements are equivalent.*

1. X is a P-space (i.e., $C(X)$ is a regular ring).

2. $C(X)$ is an $\aleph_0$-selfinjective ring.

3. $\frac{C(X)}{C_F(X)}$ is an $\aleph_0$-injective ring.

The next theorem with a different proof is also in [13].

Theorem 7.4. $C(X)$ *is a selfinjective ring if, and only if, the space X is an extremally disconnected* (i.e., every open set in X has an open closure) P-space.

Theorem 7.5. *The following statements are equivalent.*

1. *The ring $\frac{C(X)}{C_F(X)}$ is selfinjective.*

2. *X is an extremally disconnected P-space with only a finite number of isolated points.*

For the set-theoretical proof of part [(1) implies (2)], of the above theorem, Karamzadeh once admitted privately that he and his co-author have benefited much from Barbara Osofsky's set-theoretical proof of her celebrated result in module theory, namely, if every cyclic module over a non-commutative ring R is injective, then R is Artin semisimple. He believes that, in the above part of the proof and in his solution of Matlis' question, Axiom of Choice plays a key role. In his opinion, the fact in Maltis' question, i.e., the non-selfinjectivity of the quotient ring $\frac{\prod_{i\in I} F_i}{\sum_{i\in I}\bigoplus F_i}$, where each F_i is a field, in the question, is equivalent to this axiom, too. This is a good challenge for the interested reader.
It is well-known that each F-space is either extremally disconnected or its Susolin number is uncountable, see [75, 6L, 8]. It is natural to ask for a counterpart property for P-spaces. The next facts in [21], gives this counterpart property.

Proposition 7.6. *If X is a P-space and I an ideal of $C(X)$ such that $\frac{C(X)}{I}$ is not a selfinjective ring, then the Goldie-dimension of $\frac{C(X)}{I}$ is uncountable.*

Corollary 7.7. *If X is a P-space, then either X is an extremally disconnected space with at most a countable number of isolated points or both $C(X)$ and $\frac{C(X)}{I}$ have uncountable Goldie-dimensions.*

Concerning Theorem 7.3, I should remind the reader that not every commutative regular ring is $\aleph_0$-injective, see [33, Ex. 14.7]. It is manifest that if X is an infinite space in Theorem 7.4, then by [31, 12H., 6, 7], its cardinal must be measurable. And if the cardinal of X in Theorem 7.4, is nonmeasurable, then $C(X)$ is selfinjective if, and only if, X is discrete. Motivated by the new property of $\aleph_0$-selfinjectivity for $C(X)$ in Theorem 7.3, which was overlooked in the literature before the appearance of this theorem, we also recall another important property for $C(X)$ when it is a regular

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ring that had also been overlooked, in the context of $C(X)$, in the literature for many years, before its appearance in [13]. First, we recall that in every regular ring R the intersection of any finite number of principal ideals is a principal ideal. But in case $C(X)$ is regular this intersection can be extended to any countable number of principal ideals. To see this, by Proposition 4.5, and [33, Corollary 14.4], $C(X)$ is $\aleph_0$-continuous regular ring, too (note, for the definition of continuous regular rings and $\aleph_0$-continuous regular ring, see the latter reference and [13]. Consequently, by [33, Proposition 14.2], and the comment that follows it, every countable intersection of principal ideals in $C(X)$, in this case, is a principal ideal, see also [13]. This unusual algebraic property for $C(X)$ may be compared with the unusual topological property of the P-space X, namely, every countable intersection of open sets is open. Moreover, in a private conversation with Karamzadeh, I have learned that although the ring of matrices over a commutative continuous regular ring is not usually right or left $\aleph_0$-continuous ring, see [33, Ex. 14.7], but the ring of matrices over $C(X)$, where X is a P-space, is both an $\aleph_0$-selfinjective ring and an $\aleph_0$-continuous ring, for $C(X)$ has the latter two properties too, see the above comments and Theorem 7.3, see also [33, Theorem 14.19], from which we may infer that, in fact, for any finitely generated projective R-module A, $End_R(A)$ is a right $\aleph_0$-selfinjective, regular, right $\aleph_0$-continuous ring, where $R = C(X)$ is a regular ring (note, this is another connection of $C(X)$ with non-commutative rings).

Let us conclude this section with emphasis again on the compatibility of $C(X)$ with the non-commutative rings. It is well-known that for a prime ideal P, in a right self-injective regular ring R, all the ideals of R that contain P are linearly ordered under inclusion, and are all prime ideals in R, see [22, Theorem 4.11]. Similarly for a prime ideal P in $C(X)$, all the prime ideals of $C(X)$ that contain P, are linearly ordered under inclusion, see [31, 14.3.(c)] (note, however the former theorem for a commutative regular ring R becomes obvious, for in that case P is maximal and the theorem is vacuously true).

8. $\mathbf{z^\circ}$-ideals in $\mathbf{C(X)}$

Although z-ideal is a topological object, it also has an algebraic definition, namely, an ideal I in $C(X)$ is called a z-ideal whenever $a \in I$, then the intersection of all maximal ideals containing a, is subset of I, see [31, 4A. 5]. Similarly to this definition one may call an ideal I in a ring R a z°-ideal if it consists of zero divisors and for each $a \in I$, $P_a \subseteq I$, where P_a is the intersection of all the minimal prime ideal of R containing a. Clearly each ideal P_a for any $a \in R$ is a z°-ideal, which is called basic z°-ideal. z°- ideals play an important role in the studying the ideal theory of rings which are not domains, see [16], for some basic properties of these ideals. As for $C(X)$, it turns out, an ideal I is a z°-ideal if $intZ(f) = intZ(g)$ with $f \in I$ it implies $g \in I$. In particular, these ideals are also applied to give algebraic characterizations of some important disconnected topological spaces including almost P-spaces (note, X is called almost P-space if for each $f \in C(X)$, $intZ(f) \neq \emptyset$, or equivalently if every element of $C(X)$ is either a unit or a zero divisor, see [9] for some useful properties of these spaces). We should bring to the attention of the reader although each z°-ideal in $C(X)$ is clearly a z-ideal but the converse may not be true, for the ideal $\{f \in C(X) : [0,1] \bigcup \{2\} \subset Z(f)\}$, where X is

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the real line, and maximal ideals containing it are all z-ideals which are not z°-ideals. Interestingly, by transfinite induction in [16], it is shown that every ideal consisting of zero divisors is contained in z°-ideal and consequently every maximal ideal consisting of zero divisors is a z°-ideal (note, it is interesting to know that no maximal ideal in the ring $R[x_1, x_2, \cdots, x_n]$ can consist entirely of zero divisors, see [39, Theorem 150]) . Clearly every finitely generated z-ideal in $C(X)$ is z°-ideal, for it is generated by an idempotent, by the comments concerning [31, 4B.2], at the end of Section 2. Every countably generated fixed maximal ideal in $C(X)$ is a z°-ideal, by Proposition 4.5, and the previous comments. More generally for a compact space X, any countably generated z-ideal I in $C(X)$ is a z° ideal, and if X is not compact, the ideal $C_K(X)$, the functions with compact supports, is always z°-ideal, see [31]. The following algebraic characterizations of some disconnected spaces, which are towards the main goal of the subject, are now in order. The next theorem is in [14].

Theorem 8.1. *The following statements are equivalent.*

1 *X is a P-space.*
2 *Every ideal in $C(X)$ consisting of zero divisors is a z°-ideal.*
3 *Every non-unit element of $C(X)$ is a zero divisor and P_f is a principal ideal in $C(X)$ for all $f \in C(X)$.*

The next theorem except its last statement is in [13] and the equivalency of the last part with the first part of the theorem is in [14]

Theorem 8.2. *The following statements are equivalent.*

(1) *The space X is basically disconnected* (*i.e., every cozero-set has an open closure*).
(2) *Every nonzero countably generated ideal is essential in a principal ideal generated by an idempotent.*
(3) *If $S \subset C(X)$ is a countable set then $\mathrm{Ann}(S) = (e)$, where $e = e^2$.*
(4) *If $g \in C(X)$, then $\mathrm{Ann}(g) = (f)$, where $f = f^2$* (i.e., C(X) is a p.p. ring (i.e.,every principal ideal is projective)).
(5) Every z°-ideal is generated by a set of idempotents.
(6) If $A \subseteq \beta X$, then $O^A = \{f \in C(X) : A \subseteq int_{\beta X} cl_{\beta X} Z(f)\}$ is generated by idempotents.
(7) $A \subseteq X$, then $O_A = \{f \in C(X) : A \subseteq intZ(f)\}$.
(8) Every basic z°-ideal in $C(X)$ is principal.

The next result, which is an algebraic characterization of almost P-spaces, immediately shows that the sum of z°-ideals in $C(X)$, where X is an almost P-space, is either a z°-ideal or the whole of $C(X)$.

Theorem 8.3. *The following statements are equivalent.*

(1) *X is an almost P-space.*
(2) *Every z-ideal in $C(X)$ is a z°-ideal.*
(3) *Every maximal ideal (resp., prime z-ideal) in $C(X)$ is a z°-ideal.*
(4) *Every maximal ideal in $C(X)$ consists entirely of zero divisors.*

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(5) *The sum of any two ideals consisting of zero divisors is either* $C(X)$ *or consists of zero divisors.*
(6) *For each non-unit element* $f \in C(X)$*, there exists a nonzero element* $g \in C(X)$ *with* $P_f \subseteq \mathrm{Ann}(g)$.

Let us recall a space X is zero-dimensional if it has a base consisting of clopen sets. The next theorem is in [13].

Theorem 8.4. *X is a zero-dimensional space (resp., strongly zero-dimensional (i.e., βX is zero-dimensional)) if, and only if O_x, where $x \in X$ (resp., O^x, where $x \in \beta X$) is generated by a set of idempotents.*

The above theorem yields the following interesting fact.

Theorem 8.5. *X is a strongly zero-dimensional F-space if, and only if, every minimal prime ideal is generated by a set of idempotents.*

The first four statements of the following theorem is in [13] and the last one is in [14].

Theorem 8.6. *The following statements are equivalent.*

(1) *X is an extremally disconnected space.*
(2) *$C(X)$ is a Baer ring (i.e., the annihilator of every set is generated by an idempotent).*
(3) *Every nonzero ideal of $C(X)$ is essential in a principal ideal generated by an idempotent. $C(X)$ is a CS-ring (i.e., each closed ideal in $C(X)$ is generated by an idempotent, and an ideal is said to be closed if it is not essential in a larger ideal).*
(4) *$C(X)$ is a p.p. ring which is complete (i.e., the Boolean algebra $B(R)$ of its idempotents is complete).*
(5) *$C(X)$ is a p.p. ring in which every orthogonal set of idempotents has a supremum.*
(6) *Every intersection of basic z°-ideals is principal.*

We should remind the reader that the equivalencies of (2) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) hold in any ring R. The previous theorem immediately yields the next purely topological result.

Theorem 8.7. *The following statements are equivalent.*

(1) *X is an extremally disconnected space.*
(2) *X is basically disconnected space and the Boolean algebra of clopen sets is complete.*
(3) *X is basically disconnected and the family of disjoint clopen sets has a supremum.*

The next obvious lemma, Theorem 8.2 and Theorem 8.6, immediately yield the next corollary which in turn gives a unified proof of the two well-known results that follow it.

Lemma 8.8. *Let $S \subseteq C^*(X)$. Then $\mathrm{Ann}_{C(X)}S = (e)$ if, and only if, $\mathrm{Ann}_{C^*(X)}S = eC^*(X)$, where $e = e^2$.*

Corollary 8.9. *$C(X)$ is a Baer* (resp., p.p.) ring if, and only if, $C^*(X)$ is a Baer (resp., p.p.) ring.

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Corollary 8.10. *A space X is extremally* (resp., basically) disconnected if, and only if, so too is βX.

Let us conclude our algebraic characterizations of some disconnected spaces with a few more results of these kinds.

Theorem 8.11. *X is an extremally disconnected P-space with a dense set of isolated points if, and only if, $C(X)$ is isomorphic to a direct product of fields.*

The proof of the above theorem in [13] shows that whenever $C(X)$ is isomorphic to a direct product of fields, it is in fact, isomorphic to the direct products of the field of real numbers.
Without referring to the nonmeasurability of any cardinal numbers, by invoking the above theorem we have the next fact.

Proposition 8.12. *Let X be an extremally disconnected P-space with a dense set of isolated points $I(X)$, where $|I(X)| \leq \aleph_1$. Then X is discrete.*

Finally in this section, we admit that the conspicuous salient feature of [14] is the construction of two peculiar almost P-spaces, X_1 and X_2 say, which are not P-spaces, and for each of which there is a 'rare species' of a prime z°-ideal in their corresponding rings of continuous functions (note, the authors in [14] call these species, rare animals). To be precise, in one of them, say $C(X_1)$, they show that every prime z°-ideal is either a minimal prime ideal or a maximal ideal. And in $C(X_2)$ it is shown that there is a prime z°-ideal which is neither a minimal prime nor a maximal ideal. It seems it was the capture of these rare species in [14], which might caught Henriksen's eye, and this was perhaps a good reason for him to include this reference as one of those three, apparently worthwhile sample papers, among the papers in the context of $C(X)$, at that time. I should remind the reader of his quote that I have already recalled at the beginning of telling this story.

9. Universalization of some of the concepts towards the unity in mathematics

Finiteness of a set means its cardinality is less than $\aleph_0$. But we know that for any set there is a cardinal number greater than the cardinality of that set. Therefore the finiteness condition should not be a barrier for moving forward in mathematics. Here I should recall an anecdote concerning some comments of Karamzadeh and three problems from his set theory course, in 1989, that prevailed upon me to choose pure mathematics for my further study. It is also related to the set theoretic subject of this section. He used to say during that course, if anyone has clear understanding of countability, cardinals, and ordinals, then he or she would clearly understands any part of mathematics, that is intended to be studied. Towards the end of the course, he proved that a set X is countable if, and only if, there exists a bijective map $f : X \to X$ such that $f(A) \neq A$, for any proper nonempty subset A of X. And he claimed, in particular, this shows that $\mathbb{Z}$ is a better prototype than $\mathbb{N}$ for a countable set X to be compared with, see [43, pp. 34-38]. He also emphasized the previous fact may be used as a definition for a countable set X, which involves only the set X itself. He asked if we can similarly characterize any infinite set X with a given cardinality. He continued by saying if any students either settle this question or solve the next two problems then these students will never have any difficulty

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using Zorn's Lemma appropriately, in the future. And as an encouragement he emphasized that the successful students needed no sitting for the final exam and will give them excellent marks.

(1) For every non-singleton set X there is a bijective map $f : X \to X$ with no fixed points.
(2) Let $f : X \to Y$ and $g : Y \to X$ be any two maps. Show that X and Y can each be written as disjoint unions, $X = A \bigcup B$, $Y = C \bigcup D$ such that $f(A) = C$ and $g(D) = B$.
Motivated by his comments, I spent a week or so working on the questions with a full success on the latter two and no success, at all, with the former. I believe the former question which was raised by Karamzadeh himself, for the first time in that class, is still open.

Let us now start with general rings before dealing with $C(X)$. By Hilbert Basis Theorem, a ring R is Noetherian if, and only if, $R[x]$ (resp., $R[x_1, x_2, \cdots, x_n]$), is Noetherian. If R is a ring, then an ideal I of R is $\mathfrak{a}$-generated, where $\mathfrak{a}$ is a cardinal number, if it admits a generating set S with $|S| \leq \mathfrak{a}$. The least element in the set of cardinal numbers of all generating sets of I is denoted by $g(I)$. If $g(I) < \mathfrak{b}$ for each ideal I of R, where $\mathfrak{b}$ is the least regular cardinal with this property, then R is called $\mathfrak{b}$ -Noetherian. Thus a ring R is Notherian if, and only if, it is $\aleph_0$-Noetherian, where $\aleph_0$ is the first infinite cardinal number. It is manifest that every ring R is $\mathfrak{a}$-Noetherian for some regular cardinal number $\mathfrak{a}$. Now in [42], it is proved that for every ring R, R and $R[x]$ are $\mathfrak{a}$-Noetherian for the same regular cardinal number $\mathfrak{a}$. This means that Hilbert Basis Theorem is true for any ring, not necessarily for the Noetherian ones. This is what we mean by universalization of a result. If we put $\mathfrak{a} = \aleph_0$ we get Hilbert Basis Theorem as a special case. As for $C(X)$, in [42] it is proved that X is an infinite space if, and only if, $C(X)$ is $\mathfrak{a}$-Noetherian, where $\mathfrak{a} > \aleph_1$ and $\aleph_1$ is the least uncountable cardinal number. This means that if X is infinite, then there exists at least an ideal in $C(X)$ which cannot be generated by a countable number of functions in $C(X)$. Motivated by this universalization of Hilbert Basis Theorem, the authors in [55] are to universalize some topological results in the context of $C(X)$. Le us first recall that A topological space X (not necessarily Hausdorff) is said to be λ-compact, whenever each open cover of X can be reduced to an open cover of X whose cardinality is less than λ, where λ is the least infinite cardinal number with this property. We note that compact spaces (resp., Lindelöf noncompact spaces) are $\aleph_0$-compact (resp., $\aleph_1$-compact spaces) and in general every topological space X is λ-compact, for some infinite cardinal number λ. It is also observed, in [55] that given any infinite cardinal number λ there exists a space Y which is λ-compact and if $\lambda \geq \aleph_1$ is a regular cardinal, then Y is a P-space too (note, there are no infinite compact P-spaces). Now in order to universalize the well-known fact "X is compact if, and only if, each ideal in $C(X)$" is fixed, the authors notice that every ideal I with $g(I) < \aleph_0$ is always fixed in any space X. Hence one may recast the above well-known result by asserting that X compact if, and only if every ideal I of $C(X)$ with $g(I) \geq \aleph_0$ is fixed, whenever every subideal A of I with $g(A) < \aleph_0$ is fixed (note, by the above comment the fact that A is fixed, is always valid in $C(X)$). Motivated by this restatement of the above classical result, we define an ideal I of $C(X)$ with $g(I) \geq \lambda$ to be λ-fixed, where λ is an infinite cardinal number, whenever each subideal A of I with $g(A) < \lambda$ is fixed. Consequently, X is compact if, and only if every $\aleph_0$-fixed ideal is fixed. We are now ready to universalize the above classical result as follows.

Theorem 9.1. *X is a λ-compact space if, and only if every λ-fixed ideal in $C(X)$ is fixed and λ is the least infinite cardinal number with this property.*

The above theorem like Hilbert Basis Theorem shows that the above classical result is, in fact, valid in any space X, not necessarily the compact ones. Now let there exist a ring isomorphism between two rings R, S. This isomorphism takes maximal ideals to maximal ideals and in particular takes any real maximal ideal in R to a real maximal ideal in S (note, M is a real maximal in R, if $\frac{R}{M} \cong \mathbb{R}$). Consequently, $Max(R)$ and $Max(S)$ with their Zariski topologies are homeomorphic spaces and in particular their subspaces consisting of real maximal spaces are homeomorphic too. Now if we replace R and S by $C(X)$ and $C(Y)$, respectively, then we infer that an isomorphism between $C(X)$ and $C(Y)$ implies that βX (resp., υX) is homeomorphic with βY (resp., υY) and, in particular, υX and υY are λ-compact for the same cardinal λ and $\beta(\upsilon X) = \beta X$ (note, $C(X) \cong C(\upsilon X)$). This immediately gives a unified proof of the well-known facts that if X, Y are two compact (resp., realcompact) spaces, then they are homeomorphic if, and only if $C(X) \cong C(Y)$ (note, the previous comments are due to Karamzadeh). This means for the latter spaces, $C(X)$ determines X. Considering the fact that $C(X) \cong C(\upsilon X)$, we may raise a natural question. Can $C(X)$ determine X without X being realcompact? In [37] it is shown that the answer is sometimes "yes", by introducing a class of locally compact spaces which properly contains locally compact realcompact spaces and give a positive answer to the question. In what follows we aim to universalize this fact too. Let us first define an isomorphism $\phi : C(X) \to C(Y)$ to be λ-fixed isomorphism, where λ is a cardinal number, if whenever M is a fixed maximal ideal in $C(X)$ (resp., $C(Y)$), then for any subideal $I \subseteq M$ of M (note, ideals here are always proper) with $g(I) < \lambda$, $\phi(I)$ (resp., $\phi^{-1}(I)$) is fixed. If there is such an isomorphism between $C(X)$ and $C(Y)$, we say that X and Y are λ-fixedly isomorphic). Clearly, every isomorphism $\phi : C(X) \to C(Y)$ is $\aleph_0$-fixed isomorphism. It is also interesting to note that if X, Y are compact (resp., realcompact) spaces, then any isomorphism between $C(X)$ and $C(Y)$ is λ-fixed isomorphism for every cardinal number λ (i.e., if X, Y are both compact or realcompact spaces, then they are λ-fixedly isomorphic for any λ). We should also remind the reader that if $\theta : X \to Y$ is a homeomorphism between two spaces X and Y , then it is manifest that the natural isomorphism $\phi : C(X) \to C(Y)$, where $\phi(f) = f\theta^{-1}$ for all $f \in C(X)$, is λ-fixed isomorphism for any cardinal number λ. In the following theorem we universalize the previous fact which is true for compact and realcompact spaces, to all spaces.

Theorem 9.2. *Two spaces X and Y are homeomorphic if, and only if they are λ-fixedly isomorphic, where X, Y are λ-compact spaces.*

It is well-known that compact Hausdorff spaces are maximal spaces with respect to the compactness. In the following theorem we universalize this fact, too. First we need a definition. For an infinite cardinal number λ, regular or not, a space X is called P_λ-space, if given any family F of open sets in X with $|F| < \lambda$, then $\bigcap F$ is open in X. Clearly any P-space is $P_{\aleph_1}$-space and every compact

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space (i.e., $\aleph_0$-compact) like any other space is $P_{\aleph_0}$-space. Manifestly, any P_λ-space is P_μ-space for any infinite cardinal $\mu < \lambda$, but the converse may not be true, for no infinite compact space, which is $P_{\aleph_0}$-space, can be a $P_{\aleph_1}$-space.

Theorem 9.3. *Let X be a λ-compact Hausdorff space, where λ is a regular cardinal. Then the following statements are equivalent.*

(1) *X is a P_λ-space.*
(2) *X is a maximal λ-compact space.*
(3) *Every γ-compact subspace of X, with $\gamma \leq \lambda$, is closed.*

10. R-sequences in $C(X)$

Let M be a module over a ring R. The sequence $a_1, a_2, \cdots, a_n$ of elements in R is said to be an R-sequence on M if
1. a_1 is not a zero-divisor on M (i.e., $a_1 m \neq 0$ for any nonzero element $m \in M$), a_2 is not a zerodivisor on $\frac{M}{a_1 M}$, ..., a_n is not a zerodivisor on $\frac{M}{(a_1,\cdots,a_{n-1})M}$.
2. $(a_1, \cdots, a_n)M \neq M$.

In the case that $M = R$, this sequence is called an R-sequence, or simply a regular sequence in R. If $M \neq 0$, the supremum of integers n (if it exists) such that there is an R-sequence of length n on M is called the depth of M, denoted by depth(M). If $M = (0)$ we put *depth*$(M) = \infty$. If R is Noetherian then *depth*(R) does exist. The interested reader is referred to [39, Chapter 3]. Apparently, Wiegand at the 4th Seminar on Algebra and its Application, August 2016, Ardabil-Iran, by knowing a fact for $C(X)$, where $X = (0, 1)$ or $X = [0, 1]$, namely, the depth of $C(X)$ is less than 2, raised the question of whether $C(X)$, for some X, contains a regular sequence of length 2. The authors in [12, Corollary 2.7] show that for any topological space X, not necessarily the above intervals, $C(X)$ cannot contain a regular sequence of length greater than or equal to 2 (note, $C(X)$ is highly non-Noetherian for an infinite space X). They also present simple proofs for the case of those intervals. Motivated by this, they study the regular sequences in the factor rings of $C(X)$ and the depth of ideals in $C(X)$. In particular, the authors show that the depth of any maximal ideal in $C(X)$ is less than 2 and admit that they cannot prove this fact for an arbitrary ideal of $C(X)$. However, they conjecture that it must be true for all ideals in $C(X)$. Incidentally, they later show that it is indeed true, see [47]. Let us cite what follows from [46]. It is worth recalling Kaplansky's opinion about the concept of the "depth" of a ring, in his book [39]. It seems he was somewhat pessimistic about the possible role of this concept in the context of non-Noetherian rings. However, let us compare the following two results.
(a). It is well known in the context of rings that a Noetherian local ring R has depth zero (i.e., every non-unit is a zero-divisor) if, and only if each proper finitely generated ideal has a nonzero annihilator, or equivalently if, and only if its maximal ideal has nonzero annihilator.
(b). It is also well known in the context of rings of continuous functions that $C(X)$ has depth zero if, and only if X is an almost P-space (i.e., each non-empty zero set in X has non-empty interior). Indeed,

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it is clear that a ring R is classical (i.e., every non-unit is a zero-divisor) if, and only if $depth(R) = 0$. It is also manifest that $C(X)$ is classical if, and only if X is an almost P-space. As we mentioned earlier, $C(X)$ is not Noetherian for any infinite space X. However, similarly to the Noetherian local rings, it has depth zero if, and only if every proper finitely generated ideal of $C(X)$ has a nonzero annihilator; see the property A in [16] . Moreover, as it is observed in [12], $C(X)$ has depth zero if, and only if every maximal (resp., essential) ideal has depth zero. Since by definition X is an almost P-space if, and only if $depth(C(X)) = 0$, one immediately infers that $depth(C(X)) = 1$ if, and only if, X is a non-almost P-space (note, $depth(C(X)) \leq 1$).

11. The m-topology in Ahvaz and keeping up with the Joneses

Hewitt, defined the m-topology on $C(X)$, denoted by $C_m(X)$. He showed that certain classes of topological spaces X can be characterized in terms of topological properties of $C_m(X)$. In particular, he proved that X is pseudocompact if, and only if, $C_m(X)$ is first countable (note, since $C_m(X)$ is a Hausdorff topological ring, it is a metrizable space in this case, by Birkhoff-Kakutani theorem), see [32], [38], for a short history of recent activities of few authors on $C_m(X)$, especially after a rather long lapse of times of inactivities on this topology, since its introduction in 1948. It seems to me, these inactivities of the authors were due to the drawback of this topology when X is not pseudocompact. Because in the latter case the m-topology is not preserved under the isomorphism from $C^*(X)$ to $C(\beta X)$. This is why the authors in [38] claim that their new topology on $C(X)$, called r-topology, denoted by $C_r(X)$, which is also finer than m-topology, behaves better than m-topology when applied to $C^*(X)$ (note, it is also shown that the r-topology and the m-topology coincide if, and only if, X is almost P-space). Recently, some of our colleagues at Ahvaz university have also returned to the m-topology, see [18, 19, 63–65, 70, 78, 83] and in the same vein introduced and studied some variants of it on $C(X)$. However, I must admit all these new topologies similarly to the m-topology itself, still have the same drawbacks. Of course at the same time, I have to also admit that most of the authors in their recent articles have succeeded to connect some appropriate useful topological properties of X to that of $C(X)$ with these new topologies, and vice versa. However, unfortunately, in the process of making these transitions no serious algebraic properties of $C(X)$ are used. Let us recall that a space X is called a weak P-space if every countable subset is closed (e.g., P-spaces). In [32], it is observed that $C_m(X)$ is never a P-space and in [32, Theorem 3.14], interestingly, a characterization of X is given such that $C_m(X)$ becomes a weak P-space, and vice versa (note, being a weak P-space seems to be close to having an algebraic property). However, it seems among all the natural topologies defined already on a function space such as $C(X)$, the space $C_P(X)$ which is $C(X)$ with the following family of sets as a base for its open sets, is becoming more popular, "Let $x_1, x_2, \cdots, x_n$ be any finite set in X, $f \in C(X)$, $r > 0$, and define $W(f, x_1, x_2, \cdots, x_n, r) = \{g \in C(X) : |g(x_i) - f(x_i)| < r, i = 1, 2 \cdots, n\}$" to be these sets. Then $C_P(X)$ is a topological algebra and it is naturally a topological subspace of $\mathbb{R}^X$, which is called the topology of pointwise convergence on $C(X)$. It is interesting to note that $C_P(X)$ is homeomorphic with $\mathbb{R}^X$ if, and only if, X is a discrete space, see [5]. Since $C_P(X)$ is dense

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in $\mathbb{R}^X$, we infer that its Souslin number is countable, by the introduction of our Section 4, i.e., by Azarpanah's result, we infer that Gdim $C(C_P(X)) \leq \aleph_0$. $C_P(X)$ has some interesting advantages over similar topologies defined already on $C(X)$, see [5]. A conspicuous advantage which certainly will be approved, by any open-minded mathematician, is the theorem of Nagata, which says: Two spaces X, Y are homeomorphic if, and only if, $C_P(X)$ and $C_P(Y)$ are topologically isomorphic, see [5, Theorem 0.6.1], i.e., $C_P(X)$ determines X. This is a remarkable theorem because there are no any constraints in the statement of the theorem on X or Y, whatsoever, except the blanket assumption of complete regularity of these spaces. This reminds me of our Theorem 9.2, in which, the powerful tools of C_P-theory in Nagata's Theorem is traded off with only the constraint of λ- fixedly isomorphic concept. Since we will briefly notice that the compact scattered spaces are important objects in the next section, we like to emphasize that X is a compact scattered space if, and only if, $C_P(X)$ is a k-space (note, a space X is called a k-space, if every set for which, its intersection with an arbitrary compact set in X is closed in that compact set, it is also closed in X, see [5]). Let us conclude this section in the hope that our story in the next decade will be complemented by our young colleagues and the students with their work in C_P-theory. And hoping that the title of this section shall be replaced then, by "$C_P(X)$ topology in Ahvaz and keeping up with the Jonseses (or rather metaphorically, keeping up with Arkhangel'skiĭ and his school)"

12. $C_c(X)$, a genuinely proper subring of $C(X)$ (metaphorically, a chip off the old block) A happy ending

Since this section is the happy ending part of our story, the reader deserves to be well informed of the true history and the incidents behind the definition of $C_c(X)$. Therefore, I anticipate and let the reader know beforehand that we are to have naturally a rather lengthy introduction for this section. Moreover, we may record, whenever possible, the memories of our living mathematicians of what they have experienced in the past. In particular, we should record the important incidents in their lives which are related to mathematics, for the sake of the oral history of mathematics in our country. Like the other sections of this article, I owe, in particular, the anecdotes of this section to Karamzadeh, too. Let us first start with a definition of scattered spaces. X is called scattered (or, dispersed) if for any nonempty subspace Y of X, $I(Y) \neq \varnothing$ as a subspace of X, where $I(Y)$, denotes the set of isolated points of Y. We emphasize that Y may taken to be just a closed subspace. And a subset Z of X is called perfect if, and only if it equals to Z', the set of its accumulation points. Hence X is scattered if it has no nonempty perfect subset. Let us also recall another equivalent definition for a scattered space X, which is also related to its derived dimension, denoted by $d(X)$. The latter dimension is also called the Cantor-Bendixon dimension or rank, in the literature (note, as we promised earlier in the introduction of the article, we are setting the stage for the non-experts in $C(X)$, too). For an ordinal α, the α-derivative of a space X is defined by transfinite induction: $X_0 = X$, $X_{\alpha+1} = X'_\alpha$, and $X_\alpha = \bigcap_{\beta<\alpha} X_\beta$, if α is a limit ordinal. Clearly each X_α is a closed subspace of X and if for an ordinal α, $X_\alpha = \varnothing$, then X is called scattered, see [41, Lemma 3]. If X is scattered and α is the least ordinal such that $X_\alpha = \varnothing$, then α is called the derived dimension of X and as mentioned above, denoted by

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$d(X) = \alpha$. It is manifest that if X is compact, then $d(X)$ is a non-limit ordinal. Now as the title suggests, this section is to be the last part of this story. And since I have also had a role in this part, see [26], I like to reveal the true story behind the introduction of $C_c(X)$, the subring of $C(X)$ consisting of functions with countable images. Karamzadeh who first defined this concept for a topological space X admits that he was motivated by [76, main theorem $(0) \Leftrightarrow (1)$)] (resp., [77, Theorems 1, 2]), which asserts that a compact Hausdorff space X is scattered if, and only if, $f(X)$ is scattered for every function $f \in C(X)$ (resp., if there exists a continuous mapping f of X, a Hausdorff compact scattered space, onto a compact Hausdorff space Y, then Y is scattered. In particular, if $f \in C(X)$, then $f(X)$ is countable). Indeed, the former theorem (especially, the part $(0) \Leftrightarrow (1)$) is the converse of the latter for the compact spaces. Hence, if we combine these theorems and call it RPS-Theorem, we infer that a compact space X is scattered if, and only if, $C(X) = C_c(X)$. Ghadermazi who was then a Ph.D. student of Karamzadeh and I was his co-advisor, was advised by Karamzadeh that he could algebraically characterize the scattered spaces, not necessarily the compact ones, for his thesis (note, still this is an open problem). And also advised him to read [77] and [76]. In the meanwhile he defined $\mathfrak{a}$-scattered spaces, where $\mathfrak{a}$ is a regular cardinal and suggested to us, similarly to Hilbert bases Theorem, see Section 9, that we may extend Rudin's Theorems 1, 2, in [77]. Pretty soon with his advice and help we succeeded to obtain our main theorem, in this regard, see [27, Theorem 3.8], which says every continuous function takes any compact $\mathfrak{a}$-scattered space onto a $\mathfrak{b}$-scattered space, where $\mathfrak{b} \leq \mathfrak{a}$. This immediately yields the corresponding Rudin's theorems in [77], by putting $\mathfrak{a} = \aleph_0$. I cannot help recalling the next incident concerning our main theorem in [27]. One day, we (i.e., Ghadermazi and I) were in Karamzadeh's room and were talking about our recent article in [27]. While I was complaining, unintentionally, about our decision concerning the home of our previous article, I claimed that we could publish it in a better journal (note, the journal of [27] for some purpose, was suggested by Karamzadeh). Then, a young colleague of ours, who was also present in the room, turned to me and sarcastically said: "if you believe that your recent result is more general than Rudin's, how come your article is published in a much less prestigious journal?" Karamzadeh promptly turned to him and replied: "in fact, this is a good example that manifestly justifies an old idiom which says: never judge a book by its cover". Let us go back a little further to see how the main source of scattered spaces and consequently the definition of $C_c(X)$ in Ahvaz, also goes back to non-commutative rings. While studying the classical Krull dimension of non-commutative rings in [41], Karamzadeh for a not necessarily commutative ring R and an ideal I of R defines $V(I) = \{P : I \subseteq P\}$, where $P \in Spec(R)$ and $Spec(R)$ is the set of all prime ideals of R. He then takes the collection of $\{V(I)\}$ where I runs through the set of all ideals of R as an open base for a topology on $X = Spec(R)$ and calls it the V-topology on X (note, in fact, he just interchanged the open and closed bases in the Zariski topology, with each other). This topology is not necessarily Hausdorff with $I(X) = Max(R)$, where $I(X)$ is the set of isolated points. He proves in [41], the interesting fact that the derived dimension of X exists (i.e., X is scattered) if, and only if, the set of prime ideals of R satisfies the ascending chain condition (i.e., the classical Krull dimension of R, denoted by $Cl.K.\dim(R)$, exists, see [22, Theorem 14.25], with $d(X) = Cl.K.\dim(R)$, where $d(X)$ is a limit ordinal, and $d(X) = Cl.K.\dim(R) + 1$,

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where $d(X)$ is not a limit ordinal, see [41], for details. Incidentally, Karamzadeh admits that when he came across [84, 13E, P. 90], he was motivated by [84, Definition 4.9] to call the above dimension, the derived dimension of X, however apparently this dimension had already a more popular name in the literature, namely, "Cantor-Bendixon dimension or rank". Karamzadeh later also admitted that the title "Krull dimension vs. Cantor-Bendixon dimension" was perhaps a more appropriate title for his article [41]. Because, if one searches in the literature for the Cantor-Bendixon related articles, his article does not appear in Google Search results. Incidentally, we should remind the reader that, in his article it is shown that the two important dimensions in mathematics, one in commutative rings (i.e., classical Krull dimension) and the other one in set theory and topology (i.e., Cantor-Bendixon rank) are, in fact, the same concept. He continued by saying that, these dimensions in mathematics should be called "Cantor-Bendixon-Krull dimension", for the sake of unity in mathematics. Indeed, this was the title of his talk in 1984, at the Dept. of mathematics of Exeter University, while on a sabbatical leave there. By the way, this was his only sabbatical. After the appearance of Karamzadeh's main result in [41], in which, he used derived dimension instead of Cantor-Bendixon rank, some authors later obtained similar results, in the same vein, about the lattices and frames using the phrase ''Cantor-Bendixon rank'' instead, see for example, [80, the comments following the main theorem]. The above result, of which, Karamzadeh should be proud (note, Karamzadeh is usually a humble person and too shy to toot his own horn, but in this case he has all the rights to be proud), not only for the importance of the result, per se , but also because of the amazing story behind this result. Especially, its story behind the exchange of the letters with the editors of Fundamenta Mathematicae, concerning the related article, which was written during the Iraq-Iran war. And also its story of reading the final acceptance letter, while inside the building of the main post office in Ahvaz, together with Professor M. Motamedi. In this regard, whenever he tries to recall the story, he admits the euphoria of reading that acceptance letter didn't last much, then. Because when he turned to Motamedi, who was standing behind him, to let him know of the good news in the letter, an air bombardment of the bridge, in the vicinity of that building, carried out by the Iraqi planes, shattered all the windows of the building and some of the flying shrapnel got into the building too. Both of them were thrown forcibly to the ground floor of the building and were unconscious for a few minutes. Incidentally, this incident needs to be narrated and recorded separately. Fortunately, they both had a close call (note, during the war when the university was closed some of the colleagues used to go to the post office to collect their possible urgent letters, personally). Some part of this story is already revealed in his talk, which was organized by the women's committee of the Iranian mathematical society, during the 52nd AIMC, under the title "Why do I like some of my results which were written during the war? Since the idea of suggesting the algebraic characterization of scattered spaces to Ghadermazi and consequently the definition of $C_c(X)$ stemmed from the above result of Karamzadeh, the phrase " A happy ending'' seems to be so appropriate to be added in the title of this section. It was only after that narrow escape of the protagonist of our story in that air strike, we have been able to have access to $C_c(X)$, which was defined by him much later. As an interplay between algebra and topology, which is the main goal of $C(X)$, the above result of Karamzadeh is an interesting prototype, in this respect. To see this, we

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may ask, given any non-limit ordinal $\alpha = \beta + 1$, is there a compact space X with $d(X) = \alpha$? (note, in [84, 13E(3), P. 90], it is asked for showing that for any positive integer n, there exists a set $A \subsetneq \mathbb{R}$ such that $d(A) = n$). Let R be a Noetherian ring with $Cl.K.\dim(R) = \beta$ (note, there is such a ring, see [41, Note added in the proof]) and now consider the compact scattered space $X = Spec(R)$ with the V-topology of Karamzadeh on X, then by Karamzadeh's result we have $d(X) = \alpha$, see [41]. We are now ready to deal with $C_c(X)$. The reason we call it a genuinely proper subring of $C(X)$ is the fact it may not be isomorphic to any $C(Y)$, in contrast to $C^*(X)$, which is always isomorphic to $C(\beta X)$, i.e., $C^*(X)$ and $C(X)$ are the rings of continuous functions on some topological spaces, i.e., they are of the same type, although we may have $C^*(X) \subsetneq C(X)$. This is why we used the phrase"The description of a proper subring of $C(X)$ fits $C_c(X)$, to a tee" in the introduction of [26]. Before, publishing our article [26], Karamzadeh used to say to Ghadermazi and me, that you may rewrite the book [31], by just replacing $C(X)$ with $C_c(X)$, and in most cases the proofs of corresponding results can be carried over mutatis mutandis to the new setting. Before giving more details about $C_c(X)$ I have copy-pasted below a relevant comment of an unknown referee of [26]: "For a topological space X, $C(X)$ denotes the ring of all continuous real-valued functions defined on X. A major theme in the study of rings of continuous functions is to investigate how the topological properties of X relate to the algebraic properties of $C(X)$. This is an interesting paper that explores the subring $C_c(X)$ of $C(X)$ consisting of those continuous functions whose image is countable, as well as a few other subrings of $C(X)$. The authors show that the subring $C_c(X)$ shares many properties with $C(X)$, relate topological properties of X to algebraic properties of $C_c(X)$, and show $C_c(X)$ is not necessarily itself a $C(X)$. In particular, the authors show the analog to z-ideals in $C_c(X)$ play the same role as z-ideals do in a $C(X)$. Also, every $C_c(X)$ is isomorphic to a $C_c(Y)$ for a zero-dimensional space Y which allows one, without loss of generality, to assume the space is zero-dimensional when studying $C_c(X)$. This result parallels the well-known result that for every space X, the ring $C(X)$ is isomorphic to a $C(Y)$, where Y is a completely regular space. Some properties of $C(X)$ that are not shared by $C^*(X)$, the subring of all bounded continuous functions, are also shown to hold in $C_c(X)$. The paper is readable and the results are correct. In the future other authors will be interested in taking up further investigation of the properties of the subring $C_c(X)$. I recommend publication of the article after some revision".

I have inserted the above comment to save us from repeating it, and it is exactly what Karamzadeh predicted about the future of $C_c(X)$. Anyway, although $C_c(X)$ is not algebraically defined, however similarly to $C^*(X)$ it is preserved under isomorphism, i.e., $C(X) \cong C(Y)$ implies that $C_c(X) \cong C_c(Y)$ and also $C^F(X) \cong C^F(Y)$, where $C^F(X)$ is the subring of $C_c(X)$, a fortiori, a subring of $C(X)$ consisting of functions in $C(X)$ with finite image. The reason for the above preservation stems from the interesting fact that for any homomorphism $\phi : C(X) \to C(Y)$, $Im(\phi(f)) \subseteq Im(f)$, see the comment following [25, Corollary 3.5]. Also, we should remind the reader that both $C_c(X)$ and $C^F(X)$ are algebraically closed in $C(X)$, see [25, Proposition 3.1]. Moreover, $C_c(X)$ is also an algebraic subring of $C(X)$, in the sense that it contains all constant functions and $f^2 \in C_c(X)$ implies $f \in C_c(X)$ for any function $f \in C(X)$. We should also emphasize that on the well-known fact that $C^F(X)$ is always a regular ring, which is also the smallest algebraic subring of $C(X)$, see [31, 16.29] and [17, Proposition

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2.1]. In what follows, I like to convince the reader that $C_c(X)$ is, indeed, more than just a perfect replica of its origin, $C(X)$.

By reading [26], [25], [34], [50], [56], [69] and [17], the interested reader can learn the basic and important results about $C_c(X)$, in order to get the hang of working with it. One can also easily notice that $C_c(X)$ and some of its locally related objects, namely, $L_c(X)$ (resp., $L_{cc}(X)$) see [56] (resp., [69]) for definitions of these objects, are as good as $C(X)$ to most related intentions and purposes. For example, all these rings are regular, if, and only if, they are $\aleph_0$-selfinjective, see Theorem 7.3, and [56], [69]. However, although $C^F(X)$ is always regular it cannot be $\aleph_0$-selfinjective, see the introduction of Section 7, in [26]. As we emphasized earlier compactness is not an algebraic property in general, however the one-point compactification of a discrete space is also an algebraic property. For X is such a space, if, and only if, $C_F(X)$ is a unique proper essential ideal in $C^F(X)$, see [25, Theorem 6.7], see also [54, Proposition 3.1] and [16, Proposition 5.4]. In particular, in some cases $C_c(X)$ has some advantages over $C(X)$. To see this, let us recall that a ring R is called clean if every element in R can be written as a sum of a unit element and an idempotent. It is well-known that X is strongly zero-dimensional (i.e., βX is zero-dimensional) if, and only if, $C(X)$ (resp., $C^*(X)$) is clean. In contrast to the latter result, $C_c(X)$ (resp., $C_c^*(X)$) is always clean, without any constraint on X whatsoever, see [17, Corollary 2.8], and its preceding comments, see also [45, Reviewer's Comments]. We should also remind the reader that $C(X)$ can never be Artin semisimple (i.e., being finite direct product of some fields) without X being a finite set, whereas both $C_c(X)$ and its local version, $L_c(X)$, can become Artin semisimple without X being finite, see [25], [56], respectively. Let us also recall that βX is known to most people working in $C(X)$ as, $\mathrm{Max}(C(X))$, the set of maximal ideals of $C(X)$, with Zariski topology. It is interesting to emphasize that $\beta_0 X$, the Banaschewski Compactification see [75, Sec. 4.7, Corollary (f)], is also the set, $\mathrm{Max}(C_c(X))$, of maximal ideals of $C_c(X)$ with Zariski topology, see [17, Remarks 3.6, 3.7], where it is observed that $C_c(X)$, $C^F(X)$, $C_c(\beta_0 X)$ and $C(\beta_0 X)$ all have the same structure spaces homeomorphic with $\beta_0 X$, see also [74]. This useful and important property of $C_c(X)$ and its latter related rings is comparable with the similar property of $C(X)$ with regards to its structure space which is homeomorphic with βX. Moreover, the fact that $C_c(X)$ is a natural example of a clean ring without any constraint, whatsoever, on X, may be contrasted with the cleanness property of $C(X)$ which depends on the zero-dimensionality of βX. These observations alone, should suffice to convince the most skeptical, for raison d′être of $C_c(X)$ in mathematics.

Concluding Remarks: I believe the work of my colleagues on $C(X)$, in Ahvaz, should be collected in the form of a book, which would be a very useful complement to [31].

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This story can be considered as a consequence of my conversations with Karamzadeh, almost every day, at our department, during the last three decades. I have learned many good stories, not only in mathematics, but also in real life, from him. Words cannot express my gratitude, I simply say thank you, Omid. The writing of the story wouldn't have been also possible, without my collaboration with some of the colleagues. I would also like to thank them all, and in particular, express my sincere

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