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REAL POWERS AND LOGARITHMS OF MATRICES

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Abstract. We define the logarithm function and the power function on the algebra of matrices. Additionally, we study further properties of the logarithm and power functions. Utilizing the sign function, we propose a new approach to representing the power function. Furthermore, we compute the power function for various types of matrices, including Hermitian, orthogonal, and symmetric matrices.

1. Introduction

Let $\mathcal{M}_n$ denote the C^*-algebra of all n-square matrices. Two matrices A and B are *similar*, if there exists an invertible matrix T such that $A = T^{-1}BT$. We say that A is *diagonalizable*, if there exist $\lambda_1, \ldots, \lambda_n$ such that $A = T^{-1}\text{diag}(\lambda_1, \ldots, \lambda_n)T$, and A is called *unitarily diagonalizable*, if there is a unitary matrix U such that $A = U^*\text{diag}(\lambda_1, \ldots, \lambda_n)U$. Any solution of the matrix equation $e^X = A$, where e^X denotes the exponential of the matrix X, is called *the logarithm of* A. We say that a matrix A is a *real matrix*, if its elements consist entirely of real numbers. In general, a nonsingular real matrix may have an infinite number of real and complex logarithms. We denote by $\log A$ the principal logarithm of A. From the other side, the matrix C is a *square root* of A, if $A = C^2$. We say that A is a *root-approximable*, if there exists a sequence $\{C_k\}$ such that $C_k \longrightarrow I$ and $C_k^{2^k} = A$, for each $k = 0, 1, 2, \ldots$; see, for example, [2, 6]. Matrix functions are studied in [1, 4, 5, 7]. In this paper, the matrix functions $f(A) = A^\alpha$ (α is a real number) and $f(A) = \log A$ for specific matrices are studied. The logarithms of orthogonal, Hermitian, and in particular real symmetric matrices are

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studied in [8,9].

Fora square matrix A, we define

$$\log(I+A) = A - \frac{A^2}{2} + \frac{A^3}{3} - \frac{A^4}{4} + \cdots, \qquad \rho(A) < 1.$$

Here, ρ denotes the spectral radius, and the condition $\rho(A) < 1$ ensures the convergence of the matrix series. If A is a nilpotent matrix of order k, then $A^k = 0$, and we can write

$$\log(I+A) = A - \frac{A^2}{2} + \frac{A^3}{3} - \frac{A^4}{4} + \cdots + (-1)^k \frac{A^{k-1}}{k-1}.$$

Let Q be an n-square matrix. We use the following product :

$$Q\mathrm{diag}(I, \ldots, I) = \mathrm{diag}(Q, \ldots, Q)$$

We say that an n-square complex matrix A is *involutory*, if $A^2 = I$. We have the following proposition.

Proposition 1.1. [10, Theorem 5.1] *Let A be an n-square complex matrix. Then A is involutory, if and only if A is similar to a diagonal matrix of the form*

$$\mathrm{diag}(1, \ldots, 1, -1, \ldots, -1).$$

We need the following facts for our purposes.

Proposition 1.2. [10, Theorem 6.11] *Let A and B be real square matrices of the same size. If P is a complex invertible matrix such that $P^{-1}AP = B$, then there exists a real invertible matrix Q such that $Q^{-1}AQ = B$.*

Proposition 1.3. [10, Theorem 6.13] *Let A and B be real square matrices of the same size. If $A = UBU^*$ for some unitary matrix U, then there exists a real orthogonal matrix Q such that $A = QBQ^T$.*

Proposition 1.4. [10, Theorem 6.4] *Every real orthogonal matrix is real orthogonally similar to a direct sum of real orthogonal matrices of order at most 2.*

Now, we recall the definition of Jordan matrices and real Jordan matrices.

Definition 1.5. A *Jordan block* is an $r \times r$ matrix, $J_r(\lambda)$, of the form

$$J_r(\lambda) = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 1 \\ 0 & 0 & 0 & \cdots & \lambda \end{pmatrix}, \tag{1.1}$$

where $\lambda \in \mathbb{C}$ with $J_1(\lambda) = (\lambda)$. A real Jordan block is either

(1) a Jordan block as above with $\lambda \in \mathbb{R}$,

or

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(2) a real $2r \times 2r$ matrix, $J_{2r}(\lambda, \mu)$, of the form

(1.2)
$$J_{2r}(\lambda,\mu) = \begin{pmatrix} L(\lambda,\mu) & I & 0 & \cdots & 0 \\ 0 & L(\lambda,\mu) & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & I \\ 0 & 0 & 0 & \cdots & L(\lambda,\mu) \end{pmatrix},$$

where $L(\lambda, \mu)$ is a 2×2 matrix of the form

$$L(\lambda,\mu) = \begin{pmatrix} \lambda & -\mu \\ \mu & \lambda \end{pmatrix},$$

with $\lambda, \mu \in \mathbb{R}$ and I is the 2×2 identity matrix. Note that $J_2(\lambda, \mu) = L(\lambda, \mu)$.

A Jordan matrix J is an $n \times n$ block diagonal matrix of the form

(1.3)
$$J = \begin{pmatrix} J_{r_1}(\lambda_1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & J_{r_m}(\lambda_m) \end{pmatrix},$$

where each $J_{r_k}(\lambda_k)$ is a Jordan block associated with some $\lambda_k \in \mathbb{C}$ and $r_1 + \cdots + r_m = n$. A real Jordan matrix J is an $n \times n$ block diagonal matrix of the form

(1.4)
$$J = \begin{pmatrix} J_{s_1}(\alpha_1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & J_{s_m}(\alpha_m) \end{pmatrix},$$

where each $J_{s_k}(\alpha_k)$ is a real Jordan block either associated with some $\alpha_k = \lambda_k \in \mathbb{R}$ as in (1.1), or associated with some $\alpha_k = (\lambda_k, \mu_k) \in \mathbb{R}^2$, with $\mu_k \neq 0$, as in (1.2), in this case $s_k = 2r_k$.

It is a standard result that any matrix $A \in \mathbb{C}^{n\times n}$ can be expressed in the Jordan canonical form

$$T^{-1}AT = J = \mathrm{diag}(J_{r_1}(\lambda_1), \ldots, J_{r_m}(\lambda_m)),$$

where T is nonsingular and $r_1 + r_2 + \cdots + r_m = n$. The Jordan matrix J is unique up to the ordering of the blocks $J_k = J_k(\lambda_k)$, but the transforming matrix T is not unique. Denote by $\lambda_1, \ldots, \lambda_m$,the eigenvalues of A. Let n_i be the order of the largest Jordan block in which λ_i appears, which is called *the index of* λ_i.

We need the following terminology.

Definition 1.6 (Matrix function via Jordan canonical form)**.** Let f be defined on the spectrum of $A \in \mathbb{C}^{n\times n}$ and let A have the Jordan canonical form. Then

$$f(A) = Tf(J)T^{-1} = T\mathrm{diag}(f(J_{r_1}(\lambda_1)), \ldots, f(J_{r_m}(\lambda_m)))T^{-1}.$$

We use some results in [7] such as the existence of real logarithm as follows.

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Theorem 1.7 (Existence of real logarithm)**.** [7, Theorem 1.23] *The nonsingular matrix $A \in \mathbb{R}^{n\times n}$ has a real logarithm if and only if A has an even number of Jordan blocks of each size for every negative eigenvalue.*

Theorem 1.8 (Principal logarithm)**.** [7, Theorem 1.31] *Let $A \in \mathbb{C}^{n\times n}$ have no eigenvalues on $\mathbb{R}^-$. There is a unique logarithm X of A, all of whose eigenvalues lie in the strip $\{z : -\pi < \mathrm{Im}(z) < \pi\}$. We refer to X as the principal logarithm of A and write $X = \log(A)$. If A is real, then its principal logarithm is real.*

For $A \in \mathbb{C}^{n\times n}$ with no eigenvalues on $\mathbb{R}^-$ and $\alpha \in [-1, 1]$, it follows from [7, Theorem 11.2] that $\log(A^\alpha) = \alpha \log(A)$. In particular, $\log(A^{-1}) = -\log(A)$ and $\log(A^{1/2}) = \frac{1}{2}\log(A)$. Let $B, C \in \mathbb{C}^{n\times n}$ have no eigenvalues on $\mathbb{R}^-$, and let $BC = CB$. It is shown in [7, Theorem 11.3] that if every eigenvalue λ_j of B and the corresponding eigenvalue μ_j of C satisfy $|\arg\lambda_j + \arg\mu_j| < \pi$, then $\log(BC) = \log(B) + \log(C)$.

For $A \in \mathbb{C}^{n\times n}$ and $A = ZJZ^{-1}$, let $J = \mathrm{diag}(J_1, J_2)$, where $J_1 \in \mathbb{C}^{p\times p}$, the eigenvalues of J_1 lie in the open left half-plane, and $J_2 \in \mathbb{C}^{q\times q}$ lie in the open right half-plane. We define $\mathrm{sign}(A) = Z\mathrm{diag}(-I_p, I_q)Z^{-1}$. Now we have the following theorem.

Theorem 1.9. [7, Theorem 5.1] *Let $A \in \mathbb{C}^{n\times n}$ have no pure imaginary eigenvalues, and let $S = \mathrm{sign}(A)$. Then S is involutory and $SA = AS$. Moreover, if A is real, then S is real.*

2. Main results

Let A be a nonsingular real $n \times n$ matrix. We define

$$A^\beta := \exp(\beta \log A)\,, \qquad \beta \in [-1, 1]. \tag{2.1}$$

In the next theorem, we show that A^β is a real matrix.

Theorem 2.1. *Let A be a nonsingular real $n \times n$ matrix, and let $J_{r_1}(\alpha_1), \dots, J_{r_m}(\alpha_m)$ be the list of its Jordan blocks. For every real eigenvalue $\alpha_i < 0$, suppose the number of Jordan blocks identical to $J_{r_i}(\alpha_i)$ is even. Then A^β is a real matrix for all $\beta \in [-1, 1]$.*

Proof. By Proposition 1.2, there is a real invertible matrix Q such that $A = Q^{-1}J'Q$, where

$$J' = \mathrm{diag}(J_{l_1}(\lambda_1), \dots, J_{l_k}(\lambda_k)) \oplus \mathrm{diag}(J_{2p_1}(\mu_1, \zeta_1), \dots, J_{2p_t}(\mu_t, \zeta_t)),$$

in which $\{\lambda_1, \dots, \lambda_k, (\mu_1, \zeta_1), \dots, (\mu_t, \zeta_t)\}$ are eigenvalues of A, $\lambda_i > 0, 1 \le i \le k$, and $(\mu_j, \zeta_j), 1 \le j \le t$, are complex numbers. Let

$$J_{l_i}(\lambda_i) = \lambda_i I_{l_i}(I_{l_i} + N_i), \quad 1 \le i \le k,$$

where $N_i \in \mathbb{R}^{l_i\times l_i}$ is nilpotent. Let $Y_i = S_i + M_i$ where $S_i = \mathrm{diag}(\log(\lambda_i), \dots, \log(\lambda_i))$, and $M_i = \log(I_{l_i} + N_i)$. Then, by [3, Theorem 3.4], Y_i is a logarithm of $J_{l_i}(\lambda_i)$ for $i = 1, 2, \dots, k$. The other real Jordan blocks of J' are of the form $J_{2p_j}(\mu_j, \zeta_j)$ with $\mu_j, \zeta_j \in \mathbb{R}$, $1 \le j \le t$. In this case, if we define $S_j = \mathrm{diag}(S(\rho_j, \theta_j), \dots, S(\rho_j, \theta_j))$, then by [3, Theorem 3.4], $Y_j = S_j + M_j$ is a logarithm of

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$J_{2p_j}(\mu_j,\zeta_j)$ for $j=1,2,\ldots,t$, where M_j is a logarithm of $I_{2p_j}+D_j^{-1}H_j$, $J_{2p_j}(\mu_j,\zeta_j)=D_j+H_j$, $D_j=\mathrm{diag}(L(\mu_j,\zeta_j),\ldots,L(\mu_j,\zeta_j))$ and H_j is a real nilpotent matrix. Therefore, $X=Q^{-1}YQ$ is a real logarithm of A, where $Y=\mathrm{diag}(Y_1,Y_2,\cdots Y_m)$. It follows from [7, Theorem 11.2] that

$$\begin{aligned} A^\beta &= \exp(\beta\log A)=\exp(\beta Q^{-1}\log(J')Q)\\ &= Q^{-1}\exp(\beta\mathrm{diag}(Y_1,\ldots,Y_m))Q\\ &= Q^{-1}\mathrm{diag}(e^{\beta Y_1},\ldots,e^{\beta Y_m})Q\,. \end{aligned} \tag{2.2}$$

Since $S_iM_i=M_iS_i$, we have $e^{\beta Y_i}=e^{\beta S_i}e^{\beta M_i}=\lambda_i^\beta\exp(\beta M_i), i=1,\ldots,k$. Let $\lambda_k+i\mu_k=\rho_k e^{i\theta}$, with $\rho_k>0$ and $\theta_k\in[-\pi,\pi)$ for $j=1,2,\ldots,t$,

$$\begin{aligned} e^{\beta Y_j} &= e^{\beta S_j}e^{\beta M_j}\\ &= \rho_j^\beta\begin{pmatrix}\cos\beta\theta_j & -\sin\beta\theta_j\\ \sin\beta\theta_j & \cos\beta\theta_j\end{pmatrix}\mathrm{diag}(I_2,\ldots,I_2)\exp(\beta M_j). \end{aligned}$$

Hence, $e^{\beta Y_j}$ is a real matrix for each $j=1,2,\ldots,t$. Since Q is a real invertible matrix, (2.2) yields that A^β is real. □

Remark 2.2. Consider $A\in\mathbb{C}^{n\times n}$ with all eigenvalues in $T_m=\{re^{i\theta}; r>0, \frac{-\pi}{m}<\theta<\frac{\pi}{m}\}$, where m is a positive integer. We extend (2.1) for any $\beta\in[-m,m]$, that is,

$$A^\beta=\exp(\beta\log A)\,.$$

Indeed, by Theorem 1.8, [7, Theorem 11.3], and induction, we have $\log A^m=m\log A$, since $A^\beta=(A^m)^{\beta/m}$. From Theorem 2.1, we have $\log A^\beta=\frac{\beta}{m}\log A^m=\frac{\beta}{m}(m\log A)=\beta\log A$ and so $A^\beta=\exp(\beta\log A)$.

Let A be positive definite. Then all eigenvalues of A are positive, so all eigenvalues are in T_m for each $m\in\mathbb{N}$. Hence we can extend (2.1) as follows:

$$A^\alpha=\exp(\alpha\log A)\,,\quad \alpha\in\mathbb{R}.$$

Theorem 2.3. *Let $A\in\mathbb{C}^{n\times n}$ have no pure imaginary eigenvalues, and let $S=\mathrm{sign}(A)$. Then all eigenvalues of SA lie in the open right half-plane.*

Proof. Let $A=ZJZ^{-1}$, $J=\mathrm{diag}(J_1,J_2)$, where $J_1\in\mathbb{C}^{p\times p}$, $J_2\in\mathbb{C}^{q\times q}$, the eigenvalues of J_1 and J_2 lie in the open left half-plane and open right half-plane, respectively. Then $S=\mathrm{sign}(A)=Z\mathrm{diag}(-I_p,I_q)Z^{-1}$. Therefore, Theorem 1.9 implies that $SA=AS=Z\mathrm{diag}(-J_1,J_2)Z^{-1}$. Thus all eigenvalues of SA lie in the open right half-plane. □

Theorem 2.4. *Suppose that S is an involutory matrix, and let $\det(S)>0$. Then, for any $\alpha\in\mathbb{R}$, $S^\alpha:=\exp(\alpha\log S)$ is well-defined.*

Proof. By Proposition 1.1, there is an invertible matrix Z such that $S=Z\mathrm{diag}(-I_p,I_q)Z^{-1}$, where p is an even number. If $\alpha\in\mathbb{R}$, then the equation $\alpha=2[\frac{\alpha+1}{2}]+\beta$

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has a solution β in $[-1,1]$. Therefore $\cos\alpha\pi = \cos\beta\pi$ and $\sin\alpha\pi = \sin\beta\pi$, which gives $\exp(\alpha\pi E_2) = \exp(\beta\pi E_2)$, where $E_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. So

$$\begin{aligned}\exp(\alpha \log S) &= Z(\exp(\alpha\pi E_2)\mathrm{diag}(I_2,\dots,I_2) \oplus I_q)Z^{-1} \\ &= Z(\exp(\beta\pi E_2)\mathrm{diag}(I_2,\dots,I_2) \oplus I_q)Z^{-1} = \exp(\beta \log S) = S^\beta.\end{aligned}$$

Hence $S^\alpha = \exp(\alpha \log S) = S^\beta$ and so, by (2.1), S^α is well-defined. □

It is clearly seen that $\log S = Z[\pi E_2 \mathrm{diag}(I_2,\dots,I_2) \oplus O_q]Z^{-1}$, so we conclude the following result.

Corollary 2.5. *An involutory matrix has an invertible logarithm if and only if it is equal to $-I_p$ where p is an even number.*

Now, by Remark 2.2, Theorem 2.3, and Theorem 2.4, we extend the definition of power function in (2.1) to $[-2,2]$.

Definition 2.6. Let $A \in \mathbb{C}^{n\times n}$ be such that $\det(S) > 0$ where $S = \mathrm{sign}(A)$. Then we define $A^{[\alpha]} = S^\alpha (SA)^\alpha$ for $-2 \le \alpha \le 2$. Also, we set $\mathrm{Log}(A) = \log(SA) + \log S$.

If the eigenvalues of matrix A are real nonzero and the number of negative eigenvalues is even, then the above definition can be generalized to any $\alpha \in \mathbb{R}$. If all the eigenvalues of A lie in the open right half-plane, then $S = I$. So it is clearly seen that $A^{[\alpha]}$ and $\mathrm{Log}A$ are extensions of the definition of A^α and $\log A$, respectively.

Theorem 2.7. *Let A be a nonsingular real $n \times n$ matrix such that A has no eigenvalue on the pure imaginary axis, and let $\det(S) > 0$, where $S = \mathrm{sign}(A)$. Then $\mathrm{Log}A$, and for any $\beta \in [-2,2]$, $A^{[\beta]}$ are a real matrix.*

Proof. Since all eigenvalues of SA lie in the open right half-plane, it follows from Theorem 2.1 that SA is a real logarithm. In addition, Remark 2.2 and Theorem 2.1 entail that $(SA)^\beta$ is real for all $\beta \in [-2,2]$. Thus $\mathrm{Log}A$ and $A^{[\beta]}$ are real. □

Let $A \in \mathbb{R}^{n\times n}$ be a diagonalizable matrixand, such that A has no pure imaginary eigenvalues. If $\det A > 0$, then SA is diagonalizable too, where $S = \mathrm{sign}(A)$. We can write $SA = Q^{-1}DQ$, where

$$D = \mathrm{diag}(\lambda_1,\dots,\lambda_k, L(\mu_1,\zeta_1),\dots,L(\mu_t,\zeta_t)),$$

with $\lambda_i > 0$ and (μ_j,ζ_j) are in open right half-plane, for any $1 \le i \le k$ and $1 \le j \le t$. Suppose $\alpha \in [-2,2]$. Then

$$D^\alpha = \mathrm{diag}\left(\lambda_1^\alpha,\dots,\lambda_k^\alpha, \rho_1^\alpha \begin{pmatrix} \cos\alpha\theta_1 & -\sin\alpha\theta_1 \\ \sin\alpha\theta_1 & \cos\alpha\theta_1 \end{pmatrix},\dots,\rho_t^\alpha \begin{pmatrix} \cos\alpha\theta_t & -\sin\alpha\theta_t \\ \sin\alpha\theta_t & \cos\alpha\theta_t \end{pmatrix}\right),$$

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where $\mu_j + i\zeta_j = \rho_j e^{i\theta_j}$, $\rho_j > 0$ and $-\pi < \theta_j \leq \pi$. We have $(SA)^\alpha = Q^{-1}D^\alpha Q$. Therefore, $A^{[\alpha]} = S^\alpha (SA)^\alpha$ is real. Moreover,

$$\begin{aligned}\mathrm{Log}A &= \log(SA) + \log S\\ &= Q^{-1}\mathrm{diag}(\log\lambda_1, \dots, \log\lambda_k, S(\rho_1,\theta_1), \dots, S(\rho_t,\theta_t))Q + \log S.\end{aligned}$$

Hence the following corollary is proved.

Corollary 2.8. *Let $A \in \mathbb{R}^{n\times n}$ be a diagonalizable matrix such that A has no pure imaginary eigenvalues, and let $\det A > 0$. Then $\mathrm{Log}A$ and $A^{[\alpha]}$ are real matrices for every $\alpha \in [-2,2]$.*

Applying Theorems 1.7 and 2.7, the following corollary can be proved.

Corollary 2.9. *Let A be a nonsingular real $n \times n$ matrix such that A has no eigenvalue on the pure imaginary axis. Let $M_1, \dots, M_k$ be real square matrices, and let A be similar to $\mathrm{diag}(M_1, M_1, \dots, M_k, M_k)$. Then $\log A$, A^β for $\beta \in [-1,1]$, and also $A^{[\beta]}$ for $\beta \in [-2,2]$ are real matrices.*

Corollary 2.10. *Let A be a nonsingular real $n \times n$ matrix such that A has no eigenvalue on the pure imaginary axis. Let $M_1, \dots, M_k$ be real square matrices, and let A be similar to*

$$\begin{pmatrix} O & I_1 \\ M_1 & O \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} O & I_k \\ M_k & O \end{pmatrix},$$

where $I_1, \dots, I_k$ are identity matrices with the same size of $M_1, \dots, M_k$, respectively. Then $\log A$, A^β for $\beta \in [-2,2]$, and also $A^{[\beta]}$ for $\beta \in [-4,4]$ are real.

Proof. The proof is easily obtained from

$$\begin{pmatrix} O & I_i \\ M_i & O \end{pmatrix}^2 = \begin{pmatrix} M_i & O \\ O & M_i \end{pmatrix},$$

and Corollary 2.9. □

Theorem 2.11. *Suppose that A is a Hermitian matrix of order n with $\det(A) > 0$. Then for every $\alpha \in \mathbb{R}$, $\mathrm{Log}A$ and $A^{[\alpha]}$ are well-defined.*

Proof. Since A is a Hermitian, it is unitarily diagonalizable, that is, there exists a unitary matrix U such that

$$U^*AU = \mathrm{diag}(\lambda_1, \dots, \lambda_p, \mu_1, \dots, \mu_q),$$

where $\lambda_i < 0$ and $\mu_j > 0$. In this case, we have $A = U(J_1 \oplus J_2)U^*$, where $J_1 = \mathrm{diag}(\lambda_1, \dots, \lambda_p)$ and $J_2 = \mathrm{diag}(\mu_1, \dots, \mu_q)$. If $S = U(-I_p \oplus I_q)U^*$, then S is an involutory. It follows from Definition 2.6 that

$$\begin{aligned}\log(SA) &= \log[U^*\mathrm{diag}(-J_1, J_2)U]\\ &= U^*[\mathrm{diag}(\log(-\lambda_1), \dots, \log(-\lambda_p)) \oplus \mathrm{diag}(\log\mu_1, \dots, \log\mu_q)]U\end{aligned}$$

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and $\log S = U^*[\pi E_2 \mathrm{diag}(I_2, \ldots, I_2) \oplus O_q]U$. Therefore

$$\begin{aligned}\mathrm{Log} A &= \log(SA) + \log S \\ &= U^*[(\mathrm{diag}(\log(-\lambda_1), \ldots, \log(-\lambda_p)) + \pi E_2 \mathrm{diag}(I_2, \ldots, I_2)) \\ &\quad \oplus \mathrm{diag}(\log \mu_1, \ldots, \log \mu_q)]U.\end{aligned}$$

Note that by Remark 2.2 and Theorem 2.4, S^α and $(SA)^\alpha$ are well-defined. For any $\alpha \in \mathbb{R}$, we have

$$\begin{aligned}S^\alpha (SA)^\alpha &= U[\exp(\alpha\pi E_2)\mathrm{diag}(I_2, \ldots, I_2) \oplus I_q][(-J_1) \oplus J_2]^\alpha U^* \\ &= U[\exp(\alpha\pi E_2)\mathrm{diag}(I_2, \ldots, I_2) \oplus I_q][(-J_1)^\alpha \oplus (J_2)^\alpha]U^* \\ &= U[\exp(\alpha\pi E_2)\mathrm{diag}(I_2, \ldots, I_2)\mathrm{diag}((-\lambda_1)^\alpha, \ldots, (-\lambda_p)^\alpha) \\ &\quad \oplus \mathrm{diag}(\mu_1^\alpha, \ldots, \mu_q^\alpha)]U^*\end{aligned}$$

Hence, $A^{[\alpha]} = S^\alpha (SA)^\alpha$ is well-defined. □

According to Proposition 1.4, for any real orthogonal matrix A, there exist $\alpha_1, \ldots, \alpha_s$ such that A is similar to the matrix

$$C = I_r \oplus -I_l \oplus \begin{pmatrix} \cos\alpha_1 & \sin\alpha_1 \\ -\sin\alpha_1 & \cos\alpha_1 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} \cos\alpha_s & \sin\alpha_s \\ -\sin\alpha_s & \cos\alpha_s \end{pmatrix}, \tag{2.3}$$

where r and s are nonnegative integer numbers. We have the following theorem for real orthogonal matrices.

Theorem 2.12. *Let A be a real orthogonal matrix and l is even, where l is introduced in (2.3). Then A^β is orthogonal for all $\beta \in [-1, 1]$.*

Proof. By the assumption, there is a real orthogonal matrix P such that

$$C = P^T AP = I_r \oplus -I_l \oplus \begin{pmatrix} \cos\alpha_1 & \sin\alpha_1 \\ -\sin\alpha_1 & \cos\alpha_1 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} \cos\alpha_s & \sin\alpha_s \\ -\sin\alpha_s & \cos\alpha_s \end{pmatrix}.$$

Then $\log C = O_r \oplus \pi E_2 \mathrm{diag}(I_2, \ldots, I_2) \oplus E_2 \mathrm{diag}(\alpha_1 I_2, \ldots, \alpha_s I_2)$, which gives

$$\log A = P(O_r \oplus \pi E_2 \mathrm{diag}(I_2, \ldots, I_2) \oplus E_2 \mathrm{diag}(\alpha_1 I_2, \ldots, \alpha_s I_2)P^T.$$

Since, for $\beta \in [-1, 1]$,

$$\begin{aligned}C^\beta = \exp(\beta \log C) = I_r \oplus &\begin{pmatrix} \cos\beta\pi & \sin\beta\pi \\ -\sin\beta\pi & \cos\beta\pi \end{pmatrix} \mathrm{diag}(I_2, \ldots, I_2) \\ \oplus &\begin{pmatrix} \cos\beta\alpha_1 & \sin\beta\alpha_1 \\ -\sin\beta\alpha_1 & \cos\beta\alpha_1 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} \cos\beta\alpha_s & \sin\beta\alpha_s \\ -\sin\beta\alpha_s & \cos\beta\alpha_s \end{pmatrix},\end{aligned}$$

we have $A^\beta = (PCP^T)^\beta = PC^\beta P^T$, where A^β is real orthogonal. □

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If $S = \operatorname{sign}(C)$ and $\det(S) > 0$, then

$$SC = I_{r+l} \oplus \begin{pmatrix} \cos\gamma_1 & \sin\gamma_1 \\ -\sin\gamma_1 & \cos\gamma_1 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} \cos\gamma_s & \sin\gamma_s \\ -\sin\gamma_s & \cos\gamma_s \end{pmatrix},$$

where

$$\begin{cases} \gamma_i = \alpha_i + \pi & \text{if } -\pi < \alpha_i < -\frac{\pi}{2}, \\ \gamma_i = \alpha_i & \text{if } -\frac{\pi}{2} < \alpha_i < \frac{\pi}{2}, \\ \gamma_i = \alpha_i - \pi & \text{if } \frac{\pi}{2} < \alpha_i \leq \pi, \end{cases}$$

$i = 1, 2, \ldots, s$, which gives the following result.

Corollary 2.13. *Let $-2 \leq \beta \leq 2$. Let A be similar to the matrix C in (2.3). If the number of eigenvalues of C on the left half-plane is even, then $A^{[\beta]} = P(S^\beta (SC)^\beta)P^T$.*

Remark 2.14. In Theorem 2.12, if we take $\beta = \frac{1}{2^k}$ and

$$D_k = I_r \oplus \begin{pmatrix} \cos\frac{\pi}{2^k} & \sin\frac{\pi}{2^k} \\ -\sin\frac{\pi}{2^k} & \cos\frac{\pi}{2^k} \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} \cos\frac{\pi}{2^k} & \sin\frac{\pi}{2^k} \\ -\sin\frac{\pi}{2^k} & \cos\frac{\pi}{2^k} \end{pmatrix}$$
$$\oplus \begin{pmatrix} \cos\frac{\alpha_1}{2^k} & \sin\frac{\alpha_1}{2^k} \\ -\sin\frac{\alpha_1}{2^k} & \cos\frac{\alpha_1}{2^k} \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} \cos\frac{\alpha_s}{2^k} & \sin\frac{\alpha_s}{2^k} \\ -\sin\frac{\alpha_s}{2^k} & \cos\frac{\alpha_s}{2^k} \end{pmatrix},$$

then $D_k \longrightarrow I$ as $k \to \infty$ and $(PD_kP^T)^{2^k} = A$, that is, A is root-approximable.

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