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COMMUTATORS AND HYPONORMAL OPERATORS ON A HILBERT SPACE

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Abstract. Let $\mathcal{H}$ be an infinite-dimensional Hilbert space over the field $\mathbb{C}$, $\mathcal{B}(\mathcal{H})$ be the $*$-algebra of all linear bounded operators on $\mathcal{H}$, let $|X| = \sqrt{X^*X}$ for $X \in \mathcal{B}(\mathcal{H})$. An operator $A \in \mathcal{B}(\mathcal{H})$ is a commutator, if $A = [S, T] = ST - TS$ for some $S, T \in \mathcal{B}(\mathcal{H})$. Let $X, Y \in \mathcal{B}(\mathcal{H})$ and $X \geq 0$. If the operator XY is a non-commutator, then X^pYX^{1-p} is a non-commutator for every $0 < p < 1$. Let $A \in \mathcal{B}(\mathcal{H})$ be p-hyponormal for some $0 < p \leq 1$. If $|A^*|^r$ is a non-commutator for some $r > 0$, then $|A|^q$ is a non-commutator for every $q > 0$. Let $\mathcal{H}$ be separable and $A \in \mathcal{B}(\mathcal{H})$ be a non-commutator. If A is hyponormal (or cohyponormal), then A is normal. We also present results in the case of a finite-dimensional Hilbert space.

1. Introduction

Let $\mathcal{H}$ be a Hilbert space over the field $\mathbb{C}$, $\mathcal{B}(\mathcal{H})$ be the $*$-algebra of all linear bounded operators on $\mathcal{H}$. For a C^*-subalgebra $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ put

$$\mathcal{A}_0 = \{X \in \mathcal{A} : \ X = \sum_{n \geq 1} [X_n, X_n^*] \ \text{ for } \ (X_n)_{n \geq 1} \subset \mathcal{A}, \ \text{ the series } \ \| \cdot \| - \text{converges}\}.$$

It is proved in [26, Theorem 2.6] that $\mathcal{A}_0$ coincides with the zero-space of all finite traces on $\mathcal{A}^{\mathrm{sa}}$. For a wide class of C^*-algebras that contains all von Neumann algebras, we can consider only finite sums of the indicated form, see [28]. Elements of unital C^*-algebras without tracial states, can be represented as finite sums of commutators. Moreover, the number of commutators involved in these

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sums is bounded and depends only on the given C^*-algebra [35]. The characterization of traces on C^*-algebras is an urgent problem and attracts the attention of a large group of researchers. Commutation relations allowed us to obtain characterizations of the traces in a broad class of weights on von Neumann algebras and C^*-algebras [8, 10, 11]. An interesting problem is the representation of elements of C^*-algebras via commutators of special form [3–5], [17–20, 24, 33, 34].

Our paper continues article [21], that possesses the following results: Let $\mathcal{H}$ be a Hilbert space, $\dim \mathcal{H} = +\infty$.

(1) Let a Hermitian operator $X \in \mathcal{B}(\mathcal{H})$ be a non-commutator and $\sigma(X)$ be the spectrum of X. Then $f(X)$ is a non-commutator for every continuous function $f : \sigma(X) \to \mathbb{R}$ with $f(x) \neq 0$.
(2) Let $X = U|X|$ be the polar decomposition of an operator $X \in \mathcal{B}(\mathcal{H})$. Then the following conditions are equivalent:
   (a) X is a non-commutator;
   (b) U and $|X|$ are non-commutators.
(3) For a Hermitian operator $X \in \mathcal{B}(\mathcal{H})$, the following conditions are equivalent:
   (a) X is a commutator;
   (b) the Cayley transform $\mathcal{K}(X)$ is a commutator.
(4) Let $\mathcal{H}$ be a Hilbert space, and $\dim \mathcal{H} \leq +\infty$, $A, B \in \mathcal{B}(\mathcal{H})$ and $P \in \mathcal{B}(\mathcal{H})$, $P = P^2$. If $AB = \lambda BA$ for some $\lambda \in \mathbb{C} \setminus \{1\}$, then the operator AB is a commutator. The operator AP is a commutator, if and only if PA is a commutator.

Our results here concern the facts stated above. Let $\dim \mathcal{H} = +\infty$. The algebra $\mathcal{B}(\mathcal{H})$ is known to possess a proper uniformly closed ideal $\mathcal{J}$, that contains all other proper uniformly closed ideals of $\mathcal{B}(\mathcal{H})$.

Let $X, Y \in \mathcal{B}(\mathcal{H})$ and $X \geq 0$. If the operator XY is a non-commutator, then $A = X^p Y X^{1-p}$ is a non-commutator for every $0 < p < 1$ (Theorem 3.1).

Differences of idempotents in C^*-algebras are naturally related to the quantum Hall effect [1, 2, 13, 14, 29]. Let $P, Q \in \mathcal{B}(\mathcal{H})$ be idempotents, $P^\perp = I - P$. Then $P - Q$ is a non-commutator, if and only if exactly one of the following conditions holds: (i) $Q, P^\perp \in \mathcal{J}$; (ii) $P, Q^\perp \in \mathcal{J}$ (Theorem 3.6). Let $A = A_+ - A_-$ be the Jordan decomposition of a Hermitian operator $A \in \mathcal{B}(\mathcal{H})$. Then A is a non-commutator, if and only if exactly one of A_+ or A_- is a non-commutator (Theorem 3.9). Let $A \in \mathcal{B}(\mathcal{H})$ be p-hyponormal for some $0 < p \leq 1$. If $|A^*|^r$ is a non-commutator for some $r > 0$, then $|A|^q$ is a non-commutator for every $q > 0$ (Theorem 3.10). Let $\mathcal{H}$ be separable and $A \in \mathcal{B}(\mathcal{H})$ be a non-commutator. If A is hyponormal (or cohyponormal), then A is normal (Theorem 3.11).

We also present results in the setting of $\dim \mathcal{H} < +\infty$. For instance, for any unitary matrix $U \in \mathbb{M}_n(\mathbb{C})$ there exists $\varphi \in [-\pi, \pi]$, such that the inverse Cayley transform of $e^{i\varphi}U$ possesses zero trace (Proposition 3.15).

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2. Preliminaries

Let $\mathcal{A}$ be an algebra, and let $\mathcal{A}^{\mathrm{id}} = \{A \in \mathcal{A} : A^2 = A\}$ be the set of all idempotents in $\mathcal{A}$. An element $X \in \mathcal{A}$ is a *commutator*, if $X = [A, B] = AB - BA$ for some $A, B \in \mathcal{A}$. A C^*-algebra is a complex Banach $*$-algebra $\mathcal{A}$, such that $\|A^*A\| = \|A\|^2$ for all $A \in \mathcal{A}$. For a C^*-algebra $\mathcal{A}$ by $\mathcal{A}^{\mathrm{pr}}$, $\mathcal{A}^{\mathrm{sa}}$, and $\mathcal{A}^+$ we denote its projections ($A = A^* = A^2$), Hermitian elements, and positive elements, respectively. If $A \in \mathcal{A}$, then $|A| = \sqrt{A^*A} \in \mathcal{A}^+$. As is well known, in a unital C^*-algebra $\mathcal{A}$ the Cayley transform

$$\mathcal{K}(X) = \frac{X + \mathrm{i}I}{X - \mathrm{i}I} = (X - \mathrm{i}I)^{-1}(X + \mathrm{i}I) = (X + \mathrm{i}I)(X - \mathrm{i}I)^{-1}$$

of an element $X \in \mathcal{A}^{\mathrm{sa}}$ is a unitary element of $\mathcal{A}$. The inverse Cayley transform of a unitary element U of $\mathcal{A}$ is $\mathcal{K}^{-1}(U) = 2\mathrm{i}(I - U)^{-1} - \mathrm{i}I$, if $(I - U)^{-1} \in \mathcal{A}$. If $P \in \mathcal{A}^{\mathrm{id}}$, then $P^{\perp} := I - P \in \mathcal{A}^{\mathrm{id}}$.

Let $\mathcal{H}$ be a Hilbert space over the field $\mathbb{C}$, $\mathcal{B}(\mathcal{H})$ be the $*$-algebra of all linear bounded operators on $\mathcal{H}$. An operator $X \in \mathcal{B}(\mathcal{H})$, is called *$p$-hyponormal* for some $0 < p \leq 1$, if $(A^*A)^p \geq (AA^*)^p$; *p-cohyponormal*, if A^* is p-hyponormal. By Gelfand–Naimark Theorem every C^*-algebra is isometrically isomorphic to a concrete C^*-algebra of operators on a Hilbert space $\mathcal{H}$ [22, II.6.4.10]. For $\dim \mathcal{H} = n < \infty$, the algebra $\mathcal{B}(\mathcal{H})$ can be identified with the full matrix algebra $\mathbb{M}_n(\mathbb{C})$.

Lemma 2.1. *For $X \in \mathcal{B}(\mathcal{H})$, $\dim \mathcal{H} = n < \infty$, the following conditions are equivalent:*

(i) *X is a commutator;*

(ii) $\mathrm{tr}(X) = 0$*;*

(iii) *X is unitarily equivalent to a matrix with zero diagonal;*

(iv) $\mathrm{tr}(|I + zX|) \geq n$ *for all $z \in \mathbb{C}$.*

Proof. For (i)⇔(ii) see [32, Ch. 24, Problem 230]; for (ii)⇔(iii) see [30, Chap. II, Problem 209]; for (ii)⇔(iv) see [12, Theorem 4.8]. □

Let $\mathcal{H}$ be an infinite-dimensional Hilbert space. The algebra $\mathcal{B}(\mathcal{H})$ is known to contain a proper uniformly closed ideal $\mathcal{J}$ that carries all other proper uniformly closed ideals of $\mathcal{B}(\mathcal{H})$, see [23, Section 6]. In case $\mathcal{H}$ is separable, $\mathcal{J}$ is the ideal of compact operators. Combining Theorems 3 and 4 in [23] we get the following assertion (see also [21, Theorem 2.2]).

Theorem 2.2 (Brown–Pearcy Theorem)**.** *An operator $X \in \mathcal{B}(\mathcal{H})$, $\dim \mathcal{H} = +\infty$, is a non-commutator, if and only if $X = \lambda I + J$ for some $\lambda \in \mathbb{C} \setminus \{0\}$ and $J \in \mathcal{J}$.*

3. Idempotents and commutators in $\mathcal{B}(\mathcal{H})$

If $\dim \mathcal{H} < +\infty$, $X, Y \in \mathcal{B}(\mathcal{H})$ and $X \geq 0$, then the operator XY is a commutator, if and only if X^pYX^{1-p} is a commutator for some (hence, for all) $0 < p < 1$, see equivalence (i)⇔(ii) of Lemma 2.1.

Theorem 3.1. *Let $\dim \mathcal{H} = +\infty$, $X, Y \in \mathcal{B}(\mathcal{H})$ and $X \geq 0$. If the operator XY is a non-commutator, then $A = X^pYX^{1-p}$ is a non-commutator for every $0 < p < 1$.*

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Proof. By Theorem 2.2 we have $XY = \lambda I + J$, for some $\lambda \in \mathbb{C} \setminus \{0\}$, and $J \in \mathcal{J}$. We show that $A = \lambda I + J_0$ for some operator $J_0 \in \mathcal{J}$ (then A is a non-commutator by Theorem 2.2). Obviously,

$$\Big(X + \frac{1}{n}I\Big)Y = \lambda I + \frac{1}{n}Y + J, \ \ n \in \mathbb{N}.$$

Multiply these equalities by the operator $\Big(X + \frac{1}{n}I\Big)^{p-1}$ from the left, and by the operator $\Big(X + \frac{1}{n}I\Big)^{1-p}$ from the right, and obtain

$$\Big(X + \frac{1}{n}I\Big)^{p}Y\Big(X + \frac{1}{n}I\Big)^{1-p} = \lambda I + \frac{1}{n}\Big(X + \frac{1}{n}I\Big)^{p-1}Y\Big(X + \frac{1}{n}I\Big)^{1-p} + J_n, \tag{1}$$

where $J_n = \Big(X + \frac{1}{n}I\Big)^{p-1}J\Big(X + \frac{1}{n}I\Big)^{1-p} \in \mathcal{J}$, $n \in \mathbb{N}$. Since

$$X + \frac{1}{n}I \to X \ \text{ as } \ n \to \infty$$

in the operator norm, we have $(X + \frac{1}{n}I)^q \to X^q$ as $n \to \infty$ by the $\|\cdot\|$-continuity of the functional calculus. Therefore,

$$\Big(X + \frac{1}{n}I\Big)^{p}Y\Big(X + \frac{1}{n}I\Big)^{1-p} \to A \ \text{ as } \ n \to \infty$$

in the operator norm by joint $\|\cdot\|$-continuity of the product operation in $\mathcal{B}(\mathcal{H})$. Let us show that

$$\frac{1}{n}\Big(X + \frac{1}{n}I\Big)^{p-1}Y\Big(X + \frac{1}{n}I\Big)^{1-p} \to 0 \ \text{ as } \ n \to \infty$$

in the operator norm. Consider an Abelian unital C^*-subalgebra $\mathcal{A}$ in $\mathcal{B}(\mathcal{H})$, generated by the operators X and I. Then $\mathcal{A} \simeq C(\Omega)$ for some compact topological space Ω (Gelfand representation) and

$$\frac{1}{(X + \frac{1}{n}I)^{1-p}} = \frac{n^{1-p}}{(nX + I)^{1-p}} \le n^{1-p}I \ \text{ for all } \ n \in \mathbb{N},$$

hence $\|(X + \frac{1}{n}I)^{p-1}\| \le n^{1-p}$, $n \in \mathbb{N}$. Therefore,

$$\begin{aligned}\Big\|\frac{1}{n}\Big(X + \frac{1}{n}I\Big)^{p}Y\Big(X + \frac{1}{n}I\Big)^{1-p}\Big\| &\le \frac{1}{n}\Big\|\Big(X + \frac{1}{n}I\Big)^{p}\Big\| \, \|Y\| \, \Big\|\Big(X + \frac{1}{n}I\Big)^{1-p}\Big\| \\ &\le \frac{n^{1-p}}{n}\,\|Y\|\,\Big\|\Big(X + \frac{1}{n}I\Big)^{1-p}\Big\| \to 0 \ \text{ as } \ n \to \infty.\end{aligned}$$

Now it follows from (1) that the sequence $(J_n)_{n=1}^{\infty} \subset \mathcal{J}$ is $\|\cdot\|$-convergent as $n \to \infty$ to some operator $J_0 \in \mathcal{B}(\mathcal{H})$. Since the ideal $\mathcal{J}$ is $\|\cdot\|$-closed, we get $J_0 \in \mathcal{J}$. Then by Theorem 2.2 the operator A is a non-commutator. □

Since the exponential function $\lambda \in \mathbb{C} \mapsto \exp \lambda \in \mathbb{C}$ is an entire function, we can define $\exp X$ for all elements X of a unital Banach algebra $\mathcal{A}$, see [36, Chap. I, Proposition 2.7]. It also admits the representation as the absolutely convergent power series

$$\exp X = \sum_{n=0}^{\infty} \frac{1}{n!}X^n,$$

where $X^0 = I$ as usual.

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Theorem 3.2. *Let $A, B \in \mathcal{B}(\mathcal{H})$ and $X = AB + (I - B)A$.*

(i) *If $\dim \mathcal{H} < +\infty$ then A is a commutator, if and only if X is a commutator;*

(ii) *If $\dim \mathcal{H} = +\infty$ and A is a non-commutator, then X and e^A are non-commutators;*

(iii) *If $\dim \mathcal{H} < +\infty$ and $A = A^*$, then e^A is a non-commutator. There exists a normal non-commutator T such that e^T is a commutator.*

Proof. The statement is obtained (i). We have $\mathrm{tr}(X) = \mathrm{tr}(AB - BA) + \mathrm{tr}(A)$ and apply equivalence (i)⇔(ii) of Lemma 2.1.

(ii). By Theorem 2.2, we have $A = \lambda I + J$ for some $\lambda \in \mathbb{C} \setminus \{0\}$ and $J \in \mathcal{J}$. Then

$$X = (\lambda I + J)B + (I - B)(\lambda I + J) = \lambda I + J_1$$

for $J_1 = JB - BJ + J \in \mathcal{J}$ and we apply Theorem 2.2.

We also have $e^A = e^{\lambda} I + J_2$, for some $J_2 \in \mathcal{J}$. Indeed, consider the partial sums

$$I + A + A^2 + \frac{A^2}{2!} + \cdots + \frac{A^n}{n!} = I + \lambda I + \frac{\lambda^2}{2!} I + \cdots + \frac{\lambda^n}{n!} I + \\ + J + \lambda J + \frac{\lambda^2}{2!} J^2 + \cdots + \frac{\lambda^n}{n!} J^n, \quad n \in \mathbb{N}.$$

Since $I + \lambda I + \frac{\lambda^2}{2!} I + \cdots + \frac{\lambda^n}{n!} I \to e^{\lambda} I$ as $n \to \infty$ in the operator norm, the sequence $J + \lambda J + \frac{\lambda^2}{2!} J^2 + \cdots + \frac{\lambda^n}{n!} J^n$ is also $\|\cdot\|$-convergent to some operator $J_2 \in \mathcal{B}(\mathcal{H})$ as $n \to \infty$. Since the ideal $\mathcal{J}$ is $\|\cdot\|$-closed, we get $J_2 \in \mathcal{J}$. Then by Theorem 2.2 the operator e^A is a non-commutator.

(iii). If $\dim \mathcal{H} = n < +\infty$ and $A = A^*$ then without loss of generality put

$$A = \mathrm{diag}(a_1, \ldots, a_n)$$

with $a_1, \ldots, a_n \in \mathbb{R}$. Then $e^A = \mathrm{diag}(e^{a_1}, \ldots, e^{a_n})$, $\mathrm{tr}(e^A) = e^{a_1} + \cdots + e^{a_n} > 0$ and we apply equivalence (i)⇔(ii) of Lemma 2.1.

Finally, put $T = \mathrm{diag}(0, \mathrm{i}\pi)$ in $\mathbb{M}_2(\mathbb{C})$ and apply Lemma 2.1. □

Let $X \in \mathcal{B}(\mathcal{H})$, $P \in \mathcal{B}(\mathcal{H})^{\mathrm{id}}$ and $S = 2P - I$. Consider the following conditions:

(A) X is a non-commutator;

(B) $PX + XP^{\perp}$ is a non-commutator;

(C) $X + SXS$ is a non-commutator.

Theorem 3.3. *Let operators $X, P, S \in \mathcal{B}(\mathcal{H})$ be as above.*

(i) *If $\dim \mathcal{H} < +\infty$, then* (A) ⇔ (B) ⇔ (C).

(ii) *If $\dim \mathcal{H} = +\infty$, then* (A) ⇒ (B) ⇒ (C) *and in the general case the implications* (C) ⇒ (B), (C) ⇒ (A) *and* (B) ⇒ (A) *are false.*

Proof. (i). Follows from equivalence (i)⇔(ii) of Lemma 2.1.

(ii), (A) ⇒ (B). Consider a non-commutator $X = \lambda I + J$, for some $\lambda \in \mathbb{C} \setminus \{0\}$ and $J \in \mathcal{J}$, see Theorem 2.2. Then

$$PX = \lambda P + PJ, \quad XP^{\perp} = \lambda P^{\perp} + JP^{\perp}.$$

We sum these equalities term-by-term, conclude that $PX+XP^{\perp}=\lambda I+J_1$, where $J_1=PJ+P^{\perp}J\in\mathcal{J}$, and apply Theorem 2.2.

(ii), (B) $\Rightarrow$ (C). Consider a non-commutator $PX+XP^{\perp}=\lambda I+J$ with some $\lambda\in\mathbb{C}\setminus\{0\}$ and $J\in\mathcal{J}$, see Theorem 2.2. Then

$$PXP=(PX+XP^{\perp})P=\lambda P+JP,\ \ P^{\perp}XP^{\perp}=P^{\perp}(PX+XP^{\perp})=\lambda P^{\perp}+JP^{\perp}.$$

By summing these equalities term-by-term we get

$$PXP+P^{\perp}XP^{\perp}=\frac{1}{2}(X+SXS)=\lambda I+J_2,$$

where $J_2=JP+P^{\perp}J\in\mathcal{J}$, and apply Theorem 2.2.

Now we show that for an infinite dimensional separable Hilbert space $\mathcal{H}$ implications (C) $\Rightarrow$ (B), (C) $\Rightarrow$ (A) and (B) $\Rightarrow$ (A) are false. Fix some $X\in\mathcal{B}(\mathcal{H})^{\mathrm{pr}}$ with $\dim X\mathcal{H}=\dim X^{\perp}\mathcal{H}=+\infty$. Then in the direct sum $\mathcal{H}=X\mathcal{H}\oplus X^{\perp}\mathcal{H}$ we have $X=\mathrm{diag}(1,0)$ and for

$$P=\frac{1}{2}\begin{pmatrix}1&1\\1&1\end{pmatrix}$$

with $S=2P-I$ we obtain $SXS=\mathrm{diag}(0,1)=X^{\perp}$. Hence $X+SXS=I$ and condition (C) holds by Theorem 2.2. It is clear that condition (A) does not hold by Theorem 2.2. Since

$$PX+XP^{\perp}=\begin{pmatrix}1&-1/2\\1/2&0\end{pmatrix},$$

condition (B) is false by Theorem 2.2.

Consider now $P\in\mathcal{B}(\mathcal{H})^{\mathrm{pr}}$ with $\dim P\mathcal{H}=\dim P^{\perp}\mathcal{H}=+\infty$. Then in the direct sum $\mathcal{H}=P\mathcal{H}\oplus P^{\perp}\mathcal{H}$ we have $P=\mathrm{diag}(1,0)$, $P^{\perp}=\mathrm{diag}(0,1)$. For

$$X=\begin{pmatrix}\lambda&b\\c&\lambda\end{pmatrix}$$

with $\lambda\in\mathbb{C}\setminus\{0\}$, $b\in\mathcal{J}$ and $c\notin\mathcal{J}$, the operator

$$PX+XP^{\perp}=\begin{pmatrix}\lambda&2b\\0&\lambda\end{pmatrix}$$

is a non-commutator by Theorem 2.2, i.e., condition (B) holds. Since X is a commutator by Theorem 2.2, condition (A) does not hold. For $S=2P-I$ we obtain $X+SXS=2\lambda I$ and condition (C) holds by Theorem 2.2. □

Let $\mathcal{A}$ be an algebra, let $A,B\in\mathcal{A}$ be such that $AB=-BA$, i.e., A and B *anticommute*. Then AB and BA are commutators: $AB=[\frac{A}{2},B]$, $BA=[B,\frac{A}{2}]$.

Example 3.4. *Let $\mathcal{A}$ be a unital algebra. Then*

(i) *if $P,Q\in\mathcal{A}^{\mathrm{id}}$ then $A=P-Q$ and $B=I-P-Q$ anticommute;*

(ii) *if $P\in\mathcal{A}^{\mathrm{id}}$, $X\in\mathcal{A}$ then $A=2P-I$ and $B=[X,P]$ anticommute;*

(iii) *if $X,Y,T\in\mathcal{A}$ and T is left invertible then $T[X,Y]T_l^{-1}=[TXT_l^{-1},TYT_l^{-1}]$.*

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Note that even matrices with zero trace may not only anticommute but enjoy more peculiar properties, cf. item (i) of [21, Theorem 3.19].

Example 3.5. *For* $A = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & \lambda & 0 & \dots & 0 \\ 0 & 0 & 0 & \lambda^2 & \dots & 0 \\ & & \dots & & & \\ 0 & 0 & 0 & 0 & \dots & \lambda^{n-2} \\ 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & \mu & 0 & \dots & 0 \\ 0 & 0 & 0 & \mu^2 & \dots & 0 \\ & & \dots & & & \\ 0 & 0 & 0 & 0 & \dots & \mu^{n-2} \\ 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}$, $\lambda, \mu \in \mathbb{C}$, *we have* $\lambda AB = \mu BA$.

Theorem 3.6. *Let* $\dim \mathcal{H} = +\infty$ *and* $P, Q \in \mathcal{B}(\mathcal{H})^{id}$. *Then* $A = P - Q$ *is a non-commutator if and only if exactly one of the following conditions holds:*

(i) $Q, P^{\perp} \in \mathcal{J}$;

(ii) $P, Q^{\perp} \in \mathcal{J}$.

Proof. Assume that $A = P - Q$ is a non-commutator. Then by Theorem 2.2 we have

$$P - Q = \lambda I + J \tag{2}$$

for some $\lambda \in \mathbb{C} \setminus \{0\}$ and $J \in \mathcal{J}$. Hence

$$P = Q + \lambda I + J = (Q + \lambda I + J)^2$$

and

$$\lambda I + J = \lambda^2 I + 2\lambda Q + J_1, \tag{3}$$

where $J_1 = J^2 + 2\lambda J + QJ + JQ \in \mathcal{J}$. Consider two cases.

a) If $Q \in \mathcal{J}$ then $\lambda = \lambda^2$. Since $\lambda \neq 0$, we have $\lambda = 1$ and by (2) infer that $P^{\perp} = -J - Q \in \mathcal{J}$. Thus condition (i) holds.

b) If $Q \notin \mathcal{J}$ then by (2) we know that $Q^{\perp} = -J - Q \in \mathcal{J}$ and $\lambda^2 + \lambda = 0$. Since $\lambda \neq 0$, we have $\lambda = -1$ and by (2) achieve the equality $P = -Q^{\perp} + J \in \mathcal{J}$. Thus condition (ii) holds.

Let us show the reverse implication of Theorem. If condition (i) holds then $-I + P - Q := J \in \mathcal{J}$ and $P - Q = I + J$. If condition (ii) holds then $P - Q + I := J \in \mathcal{J}$ and $P - Q = -I + J$. In both of these cases $A = P - Q$ is a non-commutator by Theorem 2.2. □

Let $\mathcal{A}$ be an algebra and $A = A^3 \in \mathcal{A}$. Then $A = P - Q$ for some $P, Q \in \mathcal{A}^{\mathrm{id}}$ with $PQ = QP = 0$, see [7, Proposition 1].

Corollary 3.7. *Let* $\dim \mathcal{H} = +\infty$ *and* $A = A^3 \in \mathcal{B}(\mathcal{H})$, *let* $A = P - Q$ *be a representation as above. Then* A *is a non-commutator if and only if exactly one of conditions (i) or (ii) of Theorem 3.6 holds.*

Corollary 3.8. *Let* $\dim \mathcal{H} = +\infty$ *and* $P, Q \in \mathcal{B}(\mathcal{H})^{id}$. *Then* $A = P + Q$ *is a non-commutator, if and only if exactly one of the following conditions holds:*

(i) $P - Q^{\perp} \in \mathcal{J}$;

(ii) $P^{\perp}, Q^{\perp} \in \mathcal{J}$.

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Proof. Assume that $A = P + Q$ is a non-commutator. Then by Theorem 2.2 we have

$$P + Q = \lambda I + J,$$

for some $\lambda \in \mathbb{C} \setminus \{0\}$ and $J \in \mathcal{J}$. If $\lambda = 1$ then $P - Q^{\perp} := J \in \mathcal{J}$ and condition (i) holds. If $\lambda \notin \{0, 1\}$ then the equality

$$P - Q^{\perp} = (\lambda - 1)I + J$$

allows us to apply Theorem 3.6 to the idempotent pair $\{P, Q^{\perp}\}$. Therefore, $P^{\perp}, Q^{\perp} \in \mathcal{J}$ by item (i) of Theorem 3.6 and condition (ii) of Corollary 3.7 holds. Moreover, condition (ii) of Theorem 3.6 leads us to $P, Q \in \mathcal{J}$ and corresponds to the prohibited value $\lambda = 0$.

Let us show the reverse implication of Corollary 3.8. If condition (i) holds then $P - Q^{\perp} := J \in \mathcal{J}$ and $P + Q = I + J$. If condition (ii) holds then $P^{\perp} + Q^{\perp} := J \in \mathcal{J}$ and $P + Q = 2I - J$. In both of these cases, $A = P + Q$ is a non-commutator by Theorem 2.2. □

Theorem 3.9. *Let* $\dim \mathcal{H} = +\infty$ *and* $A = A_+ - A_-$ *be the Jordan decomposition of an operator* $A \in \mathcal{B}(\mathcal{H})^{sa}$. *Then* A *is a non-commutator if and only if exactly one of* A_+ *or* A_- *is a non-commutator.*

Proof. "⇒". Assume that $A \in \mathcal{B}(\mathcal{H})^{\mathrm{sa}}$ and $A = A_+ - A_-$ with $A_+, A_- \in \mathcal{B}(\mathcal{H})^+$, $A_+A_- = 0$. Let P_+ and P_- be the support projections (=carriers) of A_+ and A_-, respectively; put $S := P_+ - P_-$. Then the polar decomposition of A is $A = S|A|$ with $|A| = A_+ + A_-$. By [21, Theorem 3.15] the operators S and $|A|$ are non-commutators. Thus by Theorem 3.6 we have one of the following conditions: either (i) $P_-, P_+^{\perp} \in \mathcal{J}$, or (ii) $P_+, P_-^{\perp} \in \mathcal{J}$. In case (i), the projection P_+ is a non-commutator by Theorem 2.2 and $A_+ = P_+A$ is a non-commutator by [21, Lemma 3.5]. In case (ii), the projection P_- is a non-commutator by Theorem 2.2 and $A_- = -P_-A$ is a non-commutator by [21, Lemma 3.5]. □

Theorem 3.10. *Let* $\dim \mathcal{H} = +\infty$ *and* $A \in \mathcal{B}(\mathcal{H})$ *be p-hyponormal for some* $0 < p \leq 1$. *If* $|A^*|^r$ *is a non-commutator for some* $r > 0$ *then* $|A|^q$ *is a non-commutator for every* $q > 0$.

Proof. By [21, Remark 3.14] the operator $|A^*|^{2p} = (|A^*|^r)^{\frac{2p}{r}}$ is also a non-commutator. By Theorem 2.2 we have

$$|A|^{2p} \geq |A^*|^{2p} = \lambda I + J \tag{4}$$

for some $\lambda > 0$ and $J \in \mathcal{J}^{\mathrm{sa}}$, $J \geq -\lambda I$.

If $A = U|A|$ is the polar decomposition of A then $A^* = U^*|A^*|$ is the polar decomposition of A^* and $A = (A^*)^* = |A^*|U$, hence $|A| = U^*A = U^*|A^*|U$. Therefore, $|A|^n = U^*|A^*|^n U$ for all $n \in \mathbb{N}$. By the Weierstrass Theorem there exists a sequence $\{f_n\}_{n=1}^{\infty}$ of polynomials, which converges uniformly on the interval $[0; 2\|A\|]$ to the function $f(t) = t^q$ as $n \to \infty$. Hence $|A|^q = U^*|A^*|^q U$ for all $q > 0$. Therefore, by (4) we have

$$|A|^{2p} = \lambda U^*U + U^*JU = \lambda P + J_1$$

with the projection $P = U^*U$ and $J_1 = U^*JU \in \mathcal{J}^{\mathrm{sa}}$, $J_1 \geq -\lambda P$. Thus,

$$\lambda P + J_1 \geq \lambda I + J = \lambda P + \lambda P^{\perp} + J$$

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and $0 \le \lambda P^{\perp} \le J_1 - J_2 \in \mathcal{J}^+$. If $X, Y \in \mathcal{B}(\mathcal{H})^+$ and $X \le Y$ then $X = VYV^*$ for some $V \in \mathcal{B}(\mathcal{H})$ with $\|V\| \le 1$, see [27, Chap. 1, Sect. 1, Lemma 2]. So, $\lambda P^{\perp} \in \mathcal{J}^+$ and $|A|^{2p} = \lambda P + J_1 = \lambda I + J_2$ with $J_2 = J_1 - \lambda P^{\perp} \in \mathcal{J}$. Hence $|A|^{2p}$ is a non-commutator by Theorem 2.2 and $|A|^q = (|A|^{2p})^{\frac{q}{2p}}$ is a non-commutator by [21, Remark 3.14]. □

If $\mathcal{H}$ is separableand $\dim \mathcal{H} = +\infty$, then there exists a hyponormal operator $A \in \mathcal{B}(\mathcal{H})$ such that A^*A is a non-commutator, but AA^* is a commutator (hint: consider an isometry $A \in \mathcal{B}(\mathcal{H})$ with $\dim(\mathrm{Ker}(AA^*)) = +\infty$).

Theorem 3.11. *Let $\mathcal{H}$ be separable, $\dim \mathcal{H} = +\infty$ and $A \in \mathcal{B}(\mathcal{H})$ be a non-commutator. If A is hyponormal (or cohyponormal) then A is normal.*

Proof. By Theorem 2.2 we have

$$A = \lambda I + J$$

for some $\lambda \in \mathbb{C} \setminus \{0\}$ and $J \in \mathcal{J}$. Since $A^*A \ge AA^*$, we obtain $J^*J \ge JJ^*$. Since $\mathcal{J}$ is the set of compact operators (when $\mathcal{H}$ is separable), by Ando–Berberian–Stampfli Theorem (see [32, Chap. 21, Problem 206]) we obtain $J^*J = JJ^*$. Therefore, $A^*A = AA^*$, i.e., A is normal.

If A is cohyponormal ($A^*A \le AA^*$) then A^* is hyponormal. If $\mathcal{A}$ is a $*$-algebra, then $X \in \mathcal{A}$ is a commutator, if and only if X^* is a commutator (hint: if $X = [Y, Z]$ then $X^* = [Z^*, Y^*]$). □

Theorem 3.12. *Let $P_1, \ldots, P_n \in \mathcal{B}(\mathcal{H})^{id}$ and $P_1 + \cdots + P_n = I$. Put $\mathcal{P}(A) = \sum_{k=1}^n P_k A P_k$ for $A \in \mathcal{B}(\mathcal{H})$.*

(i) *If $\dim \mathcal{H} < +\infty$ then A is a commutator if and only if $\mathcal{P}(A)$ is a commutator;*

(ii) *If $\dim \mathcal{H} = +\infty$ and A is a non-commutator then $\mathcal{P}(A)$ is a non-commutator.*

Proof. (i). If $\dim \mathcal{H} < +\infty$ then $\mathrm{tr}(A) = \mathrm{tr}(\mathcal{P}(A))$ for all $A \in \mathcal{B}(\mathcal{H})$ by [15, Lemma 1] and the assertion follows by equivalence (i)⇔(ii) of Lemma 2.1.

(ii). By Theorem 2.2 we have $A = \lambda I + J$ for some $\lambda \in \mathbb{C} \setminus \{0\}$ and $J \in \mathcal{J}$. Then

$$\mathcal{P}(A) = \lambda I + \sum_{k=1}^{n} P_k J P_k = \lambda I + J_1$$

with $J_1 = \sum_{k=1}^n P_k J P_k \in \mathcal{J}$ and $\mathcal{P}(A)$ is a non-commutator by Theorem 2.2. □

Recall that for $P_1, \ldots, P_n \in \mathcal{B}(\mathcal{H})^{\mathrm{pr}}$, the mapping $\mathcal{P}$ coincides with the block projection operator, which was investigated in [9, 25, 31] and [16].

Example 3.13. *For an infinite dimensional separable Hilbert space $\mathcal{H}$, consider $P_1 \in \mathcal{B}(\mathcal{H})^{pr}$ with $\dim P_1\mathcal{H} = \dim P_1^{\perp}\mathcal{H} = +\infty$, and put $P_2 = P_1^{\perp}$, $\mathcal{P}(X) = P_1 X P_1 + P_2 X P_2$ for all $X \in \mathcal{B}(\mathcal{H})$. Let $A \in \mathcal{B}(\mathcal{H})^+$ admit the operator matrix $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ in the direct sum $\mathcal{H} = P_1\mathcal{H} \oplus P_2\mathcal{H}$. Then $P_1 = \mathrm{diag}(1, 0)$, $P_2 = \mathrm{diag}(0, 1)$ and A is a commutator, but $\mathcal{P}(A) = I$ is a non-commutator by Theorem 2.2.*

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Note that the Cayley transform of a commutator in the finite dimensional case, is not necessarily a matrix with zero trace, cf. the infinite dimensional case of [21].

Example 3.14. (i) *Scalar multiples of the Pauli matrices are the unitary matrices with zero trace whose inverse Cayley transform also possesses zero trace.*

$$\mathrm{i}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \mathrm{i}\begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix},\ \mathrm{i}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(ii) *A set of unitary matrices with zero trace whose inverse Cayley transform also possesses zero trace: for $a, b \in \mathbb{R}$ with $a^2 + b^2 \leq 1$, $c := \sqrt{1 - a^2 - b^2}$, we have*

$$\begin{pmatrix} \mathrm{i}a & b - \mathrm{i}c \\ -b - \mathrm{i}c & -\mathrm{i}a \end{pmatrix} \mapsto \begin{pmatrix} -a & c + \mathrm{i}b \\ c - \mathrm{i}b & a \end{pmatrix}.$$

(iii) *Unitary matrices with zero trace whose inverse Cayley transform possesses nonzero trace:*

$$\begin{pmatrix} \mathrm{e}^{\mathrm{i}\varphi} & 0 \\ 0 & -\mathrm{e}^{\mathrm{i}\varphi} \end{pmatrix} \mapsto \begin{pmatrix} \frac{\sin(\varphi)}{\cos(\varphi)-1} & 0 \\ 0 & \frac{\sin(\varphi)}{\cos(\varphi)+1} \end{pmatrix},\ \varphi \neq \pi k/2,\ k \in \mathbb{Z}.$$

$$\begin{pmatrix} 0 & \mathrm{e}^{\mathrm{i}\alpha} & 0 \\ 0 & 0 & \mathrm{e}^{\mathrm{i}\beta} \\ \mathrm{e}^{\mathrm{i}\gamma} & 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} \mathrm{i}\frac{\mathrm{e}^{\mathrm{i}(\alpha+\beta+\gamma)}+1}{1-\mathrm{e}^{\mathrm{i}(\alpha+\beta+\gamma)}} & \frac{2\mathrm{i}\mathrm{e}^{\mathrm{i}\alpha}}{1-\mathrm{e}^{\mathrm{i}(\alpha+\beta+\gamma)}} & \frac{2\mathrm{i}\mathrm{e}^{\mathrm{i}(\alpha+\beta)}}{1-\mathrm{e}^{\mathrm{i}(\alpha+\beta+\gamma)}} \\ \frac{2\mathrm{i}\mathrm{e}^{\mathrm{i}(\gamma+\beta)}}{1-\mathrm{e}^{\mathrm{i}(\alpha+\beta+\gamma)}} & \mathrm{i}\frac{\mathrm{e}^{\mathrm{i}(\alpha+\beta+\gamma)}+1}{1-\mathrm{e}^{\mathrm{i}(\alpha+\beta+\gamma)}} & \frac{2\mathrm{i}\mathrm{e}^{\mathrm{i}\beta}}{1-\mathrm{e}^{\mathrm{i}(\alpha+\beta+\gamma)}} \\ \frac{2\mathrm{i}\mathrm{e}^{\mathrm{i}\gamma}}{1-\mathrm{e}^{\mathrm{i}(\alpha+\beta+\gamma)}} & \frac{2\mathrm{i}\mathrm{e}^{\mathrm{i}(\alpha+\gamma)}}{1-\mathrm{e}^{\mathrm{i}(\alpha+\beta+\gamma)}} & \mathrm{i}\frac{\mathrm{e}^{\mathrm{i}(\alpha+\beta+\gamma)}+1}{1-\mathrm{e}^{\mathrm{i}(\alpha+\beta+\gamma)}} \end{pmatrix},$$

$\alpha + \beta + \gamma \neq \pi k,\ k \in \mathbb{Z}.$

Despite last of these examples we have a

Proposition 3.15. *For any unitary matrix $U \in \mathbb{M}_n(\mathbb{C})$, there exists $\varphi \in [-\pi, \pi]$ such that the inverse Cayley transform of $\mathrm{e}^{\mathrm{i}\varphi}U$ possesses zero trace.*

Proof. Indeed, if we diagonalize U so that $U = \mathrm{diag}\{z_1, \ldots, z_n\}$, $|z_k| = 1$ the inverse Cayley transform of U is also a diagonal real matrix $\mathcal{K}^{-1}(U) = \mathrm{diag}\{\mathrm{i}\frac{1+z_1}{1-z_1}, \ldots, \mathrm{i}\frac{1+z_n}{1-z_n}\}$. Consider the adjacent numbers z_k, z_{k+1} of the unit circle $\mathbb{S}^1$. Now for $z_k \to 1$ from below the number $\mathrm{i}\frac{1+z_k}{1-z_k} \to -\infty$ and for $z_{k+1} \to 1$ from above the number $\mathrm{i}\frac{1+z_{k+1}}{1-z_{k+1}} \to +\infty$. The function $\mathrm{tr}(\mathcal{K}^{-1}(U))$ is continuous. Hence there exists $\varphi \in [-\pi, \pi]$ so that the trace of $\mathcal{K}^{-1}(\mathrm{e}^{\mathrm{i}\varphi}U)$ equals zero, thus $\mathcal{K}^{-1}(\mathrm{e}^{\mathrm{i}\varphi}U)$ is a commutator by equivalence (i)⇔(ii) of Lemma 2.1. □

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