Journal of the Iranian Mathematical Society
ISSN (on-line): 2717-1612
J. Iran. Math. Soc. 4 (2023), no. 2, 79-94

ON SOME GROUPS WITHOUT PERFECT FACTORS

L. A. KURDACHENKO, P. LONGOBARDI AND M. MAJ*

Dedicated to Prof. D. J. S. Robinson

Abstract. If G is a group and H, K are normal subgroups of G, $H \leq K$, then K/H is said to be a G-perfect factor if $[K/H, G] = K/H$. If G is a nilpotent group, then every non-trivial factor of G is not G-perfect. Conversely, if G is finite and all non-trivial factors of G are not G-perfect, then G is nilpotent. We study (infinite) groups with no non-trivial G-perfect factors. We prove that if either G is a locally generalized radical group with finite section rank, or G has a normal nilpotent subgroup A such that G/A is a locally finite group with Chernikov Sylow p-subgroups for every prime p, and G has no non-trivial G-perfect factors, then for every prime p there exists a positive integer s_p, such that $\zeta_{s_p}(G)$, the s_p-term of the upper central series of G, contains the Sylow p-subgroups of G, and $G/Tor(G)$ is nilpotent. In particular, G is hypercentral and the hypercentral length of G is at most $\omega + k$, for some positive integer k.

1. Introduction

If G is a group and H, K are normal subgroups of G, $H \leq K$, then the group K/H is called **a factor** of G. The factor K/H is called **perfect** (more precisely, **G-perfect**), if $[K/H, G] = K/H$. Otherwise, we will say that the factor is not perfect. Thus H/K not perfect means that $[K/H, G] \neq K/H$. The factor K/H is called **central** (more precisely, **G-central**), if $[K, G] \leq H$.

Communicated by Alireza Abdollahi

MSC(2020): Primary: 47B20; Secondary: 47B47.

Keywords: G-perfect factors; nilpotent groups; hypercentral groups; upper central series.

Received: 8 June 2023, Accepted: 10 June 2023.

∗Corresponding author

DOI: https://dx.doi.org/10.30504/JIMS.2023.400933.1123

.

If G is a nilpotent group, then every non-trivial factor of G is not perfect. Moreover, it is easy to prove that if G is a finite group, such that all of whose non-trivial factors are not perfect, then G is nilpotent. Therefore the following natural question arises:

What can be said concerning the structure of a group G whose non-trivial factors are not G-perfect?

And, in particular,

When is a group G whose non trivial factors are not G-perfect nilpotent?

If G is a group whose non-trivial factors are not G-perfect, then obviously every quotient of G has this property. Moreover, every chief factor of G is central; recall that groups G with each chief factor central are called **$\bar{Z}$-groups** (see [9]). The class of all $\bar{Z}$-groups is very wide and has been studied by Yu.I. Merzljakov, P. Hall and M.I. Kargapolov (see [9]). Notice that there exist $\bar{Z}-$groups (and even hypercentral groups)that have perfect factors. For example, let D be a divisible abelian 2-group. Then D has an automorphism ν, such that $\nu(d) = d^{-1}$ for each element $d \in D$. Consider the semidirect product $G = D \rtimes \langle b \rangle$, such that $d^b = \nu(d) = d^{-1}$ for every element $d \in D$. Let $a \in D$, since D is divisible, there exists an element $d \in D$, such that $d^2 = a$. We have $[b, d] = b^{-1}d^{-1}bd = d^2 = a$. It follows that $[b, D] = D$ and therefore $[G, D] = D$. The group G is not nilpotent, however the series $\langle 1 \rangle \leq \Omega_1(D) \leq \cdots \leq \Omega_n(D) \leq \Omega_{n+1}(D) \leq \cdots \leq D \leq G$ is central, hence G is an hypercentral abelian-by-finite group.

Notice that it is not true that a group whose non-trivial factors are not G-perfect is nilpotent. In fact such groups can be very far from being nilpotent. Indeed, H. Heineken and A. Mohamed constructed in the paper [4] a p-group H, where p is a prime, satisfying the following conditions: H contains a normal elementary abelian p-subgroup A such that H/A is a Prüfer p-group and every proper subgroup S of H is subnormal in H and nilpotent, $SA \neq H$, and $Z(H) = \langle 1 \rangle$. It is possible to prove that every non-trivial factor of H is not H-perfect. Another example is the group constructed by R.I. Grigorchuk in the paper [3]. This is an infinite finitely generated p-group, whose non-trivial normal subgroups have finite index. It is easy to see that every non-trivial factor of this group is not G-perfect.

Other examples of groups with no non-trivial G-perfect factors are the following groups, which are close to be nilpotent. A periodic group G is said to be **Sylow-nilpotent** if G is locally nilpotent and every Sylow p-subgroup of G is nilpotent, for every prime p. It is not hard to see that every Sylow-nilpotent group has no non-trivial G-perfect factors.

Before to state our results, we recall some definitions. A group G is **generalized radical**, if G has an ascending series, whose factors are either locally nilpotent or locally finite. Notice that every generalized radical group has an ascending series of normal, indeed characteristic, subgroups with locally nilpotent or locally finite factors. Let p be a prime. We say that a group G has **finite section p-rank** ($sr_p(G) = r$), if every elementary abelian p-section of G is finite of order at most p^r, and there is an elementary abelian p-section A/B of G, such that $|A/B| = p^r$. We say that the group G has **finite section rank**, if $sr_p(G)$ is finite for all primes p. We let ω denotes the first infinite ordinal.

Our results are the following.

DOI: https://dx.doi.org/10.30504/JIMS.2023.400933.1123

Theorem A. *Let G be a locally generalized radical group with finite section rank. If G has no non-trivial G-perfect factors, then G satisfies the following conditions:*

(i) *for every prime p, there exists a positive integer s_p, such that the s_p-center of G, $\zeta_{s_p}(G)$, contains the Sylow p-subgroups of G;*

(ii) *the factor group $G/Tor(G)$ is nilpotent.*

In particular, G is hypercentral, and the hypercentral length of G is at most $\omega + k$, for some positive integer k.

Theorem B. *Let G be a group, and let A be a normal nilpotent subgroup of G, such that G/A is a locally finite group, whose Sylow p-subgroups are Chernikov for all prime p. If G has no nontrivial G-perfect factors, then G satisfies the following conditions:*

(i) *for every prime p there exists a positive integer s_p such that $\zeta_{s_p}(G)$ contains the Sylow p-subgroups of G;*

(ii) *the factor group $G/Tor(G)$ is nilpotent.*

In particular, G is hypercentral, and the hypercentral length of G is at most $\omega + k$ for some positive integer k.

Theorem B has some interesting corollaries.

Corollary 1.1. *Let G be a periodic group, and let A be a normal nilpotent subgroup of G, such that G/A is a locally finite group, whose Sylow p-subgroups are Chernikov for all primes p. If G has no nontrivial G-perfect factors, then G is Sylow-nilpotent.*

Corollary 1.2. *Let G be a group, and let A be a normal nilpotent subgroup of G such that G/A is finite. If G has no non-trivial G-perfect factors, then G is nilpotent.*

2. Generalized radical groups without non-trivial perfect factors.

In this section we investigate locally generalized radical groups G without non-trivial perfect factors. We start by studying finite factors and finitely generated abelian factors of groups with this property.

Lemma 2.1. *Let G be a group, and let S, K be normal subgroups of G such that $K \leq S$ and S/K is finite. If G has no non-trivial G-perfect factors, then S/K is contained in thenth term of the upper central series of G/K, for some positive integer n. In particular, S/K is nilpotent.*

Proof. Let $K = K_0 \leq K_1 \leq \cdots \leq K_{t-1} \leq K_t = S$ be a series of G-invariant subgroups whose factors are G-chief factors. Since $[K_j/K_{j-1}, G]$ is a G-invariant subgroup of K_j/K_{j-1}, the hypothesis $[K_j/K_{j-1}, G] \neq K_j/K_{j-1}$ implies that $[K_j, G] = K_{j-1}$, $1 \leq j \leq t$. Therefore the above series is G-central. □

Let G be a group and A be a normal abelian subgroup of G. Define the G-center of A by the following rule

$$\zeta_G(A) = \{a \in A \mid [a, g] = 1, \forall g \in G\},$$

DOI: https://dx.doi.org/10.30504/JIMS.2023.400933.1123

and the *upper G-central series* of A,

$$\langle 1\rangle = \zeta_{G,0}(A) \leq \zeta_{G,1}(A) \leq \cdots \leq \zeta_{G,\alpha}(A) \leq \zeta_{G,\alpha+1}(A) \leq \cdots \leq \zeta_{G,\gamma}(A),$$

writing: $\zeta_{G,1}(A) = \zeta_G(A)$, $\zeta_{G,\alpha+1}(A)/\zeta_{G,\alpha}(A) = \zeta_{G,1}(A/\zeta_{G,\alpha}(A))$, $\zeta_{G,\lambda}(A) = \bigcup_{\beta<\lambda} \zeta_{G,\beta}(A)$, for all limit ordinals, $\zeta_G(A/\zeta_{G,\gamma}(A)) = \{1\}$.

Notice that every term of this series is a $\mathbb{Z}G$-submodule.

We say that A is G-hypercentral, if the last term $\zeta_{G,\gamma}(A)$ of this series coincides with A. We say that A is G-nilpotent, if $A = \zeta_{G,n}(A)$ for some positive integer n.

Lemma 2.2. *Let G be a group , and let S, K be normal subgroups of G, such that $K \leq S$ and S/K is a finitely generated torsion-free abelian group. If G has no non-trivial G-perfect factors, in the nth term of the upper central series of G/K, for some positive integer n. In particular, S/K is nilpotent.*

Proof. Let $r = r_0(S/K)$. For every prime p, the factor $(S/K)/(S/K)^p$ is an elementary abelian p-group of order p^r. Write $S_p/K = (S/K)^p$. By Lemma 2.1, there exists a positive integer t, such that $S/S_p \leq \zeta_t(G/S_p)$. Since S/S_p has order p^r, so $t \leq r$. Therefore $[S,\ _rG] \leq S_p$, for all prime p. Therefore $[S,\ _rG] \leq \bigcap_{p\in P} S_p$, where P is the set of all primes. Since S/K is a free abelian group, then $\bigcap_{p\in P} S_p/K = \bigcap_{p\in P}(S/K)^p$ is trivial. Thus $[S,\ _rG] \leq K$. It follows that $S/K \leq \zeta_r(G/K)$, as required. □

Corollary 2.3. *Let G be a group, and let S, K be normal subgroups of G, such that $K \leq S$ and S/K is a finitely generated abelian group. If G has no non-trivial G-perfect factors, then S/K is contained in the nth term of the upper central series of G/K, for some positive integer n. In particular, S/K is nilpotent.*

Now interesting informationconcerning some particular normal subgroups , follows.

Corollary 2.4. *Let G be a group, and let K be a normal subgroup of G, having a finite series of G-invariant subgroups, whose factors are either finite groups or finitely generated abelian groups. If G has no non-trivial G-perfect factors, then K is contained in the nth term of the upper central series of G, for some positive integer n. In particular, K is G-nilpotent.*

Corollary 2.5. *Let G be a group, and let K be a normal polycyclic-by-finite subgroup of G. If G has no non-trivial G-perfect factors, then K is contained in the nth term of the upper central series of G, for some positive integer n. In particular, K is G-nilpotent.*

Now, a useful lemma.

Lemma 2.6. *Let G be a group, and let A be a normal abelian subgroup of G, such that A is finitely generated as $\mathbb{Z}G$-module and $G/C_G(A)$ is a finitely generated nilpotent group. If G has no non-trivial G-perfect factors, then A is contained in the nth term of the upper central series of G, for some positive integer n.*

DOI: https://dx.doi.org/10.30504/JIMS.2023.400933.1123

Proof. Let M be a maximal $\mathbb{Z}G$-submodule of A. Since the factor A/M is a G-chief factor, the hypothesis $[A/M, G] \neq A/M$ implies that $[A, G] = M$. Then the factor A/M is G-central. Therefore A is G-nilpotent (see [7, Corollary 13.3]). $\square$

Using the previous results, we can prove that finitely generated soluble-by-finite groups are nilpotent, if they have no non-trivial G-perfect factors.

Corollary 2.7. *Let G be a finitely generated soluble-by-finite group. If G has no non-trivial G-perfect factors, then G is nilpotent.*

Proof. By Lemma 2.1 every finite factor group of G is nilpotent. Hence G is soluble. Let A be a normal abelian subgroup of G such that G/A is nilpotent. Since a finitely generated nilpotent group is finitely presented, A is finitely generated as $\mathbb{Z}G$-submodule (see, for example, [1, Proposition 8.1.12]). Then Lemma 2.6 implies that $A \leq \zeta_r(G)$, for some positive integer r. It follows that G is nilpotent. The result follows by induction. $\square$

In the next three lemmas, we study normal torsion-free subgroups of finite 0-rank in a group without non-trivial G-perfect factors. Here we say that a group G has finite 0-rank ($r_0(G) = r$), if G has an ascending series whose factors are either infinite cyclic or periodic and the number of infinite cyclic factors is exactly r.

Lemma 2.8. *Let G be a group, and let A be a normal abelian torsion-free subgroup of G, such that $r_0(A)$ is finite and $G/C_G(A)$ is a finitely generated nilpotent group. If G has no non-trivial G-perfect factors, then A is contained in the nth term of the upper central series of G, for some positive integer n.*

Proof. Since A has finite 0-rank, there exists a finitely generated subgroup B of A, such that A/B is periodic. Let $D = B^G$. Write $S = G/C_G(A)$ and consider A as a $\mathbb{Z}S$-module. Then D is a finitely generated $\mathbb{Z}S$-submodule of A, containing a free abelian subgroup C such that D/C is periodic and the set $\Pi(D/C)$ is finite (see, for example, [7, Corollary 1.8]). Denote by π the set of all primes p such that $p \notin \Pi(D/C)$. If $p \in \pi$, then C/C^p is the Sylow p-subgroup of D/C^p. Then we have $D/C^p = C/C^p \times E/C^p$, where E/C^p is the Sylow p'-subgroup of D/C^p. This direct decomposition shows that $(D/C^p)^p = E/C^p$. On the other hand, $(D/C^p)^p = (D^pC^p)/C^p = D^p/C^p$, so that $D^p/C^p = E/C^p$. It follows that $C \cap D^p = C^p$. Let $r = r_0(A)$. For every prime p the factor D/D^p is an elementary abelian p-group of order p^r. By Lemma 2.1 there exists a positive integer t such that $D/D^p \leq \zeta_t(G/D^p)$ Since D/D^p has order at most p^r, we have $t \leq r$, thus $[D, {}_rG] \leq D^p$. This holds for every prime $p \in \pi$, therefore $[D, {}_rG] \leq \bigcap_{p\in\pi} D^p$. We have $(\bigcap_{p\in\pi} D^p) \cap C = \bigcap_{p\in\pi}(D^p \cap C) = \bigcap_{p\in\pi} C^p$. Since C is a free abelian group and the set π is infinite, then $\bigcap_{p\in\pi} C^p = \langle 1\rangle$. Thus
$\bigcap_{p\in\pi} D^p \simeq (\bigcap_{p\in\pi} D^p)/((\bigcap_{p\in\pi} D^p) \cap C) \simeq (\bigcap_{p\in\pi} D^p)C/C$.

Notice that D/C is periodic, so that $\bigcap_{p\in\pi} D^p$ is periodic. On the other hand, the subgroup D is torsion-free. It follows that $\bigcap_{p\in\pi} D^p = \langle 1\rangle$. Thus $[D, {}_rG] = \langle 1\rangle$, therefore $D \leq \zeta_r(G)$. Since the factor A/D is periodic, it is now easy to prove that $A \leq \zeta_r(G)$. $\square$

DOI: https://dx.doi.org/10.30504/JIMS.2023.400933.1123

Let G be a group and A be an abelian torsion-free normal subgroup of G. Then A is called G-rationally irreducible, if the factor A/B is periodic for every non-trivial G-invariant subgroup B. Notice that A is G-rationally irreducible, if and only the $\mathbb{Q}G$-module $A \bigotimes_{\mathbb{Z}} \mathbb{Q}$ is simple.

Lemma 2.9. *Let G be a group, and let A be a normal abelian torsion-free subgroup of G such that $r_0(A)$ is finite, A is G-rationally irreducible and $G/C_G(A)$ is a locally nilpotent group of finite 0-rank. If G has no nontrivial G-perfect factors, then A is contained in the nth term of the upper central series of G, for some positive integer n.*

Proof. Since A has finite 0-rank, there exists a finitely generated subgroup B of A such that A/B is periodic. Let $D = B^G$. Write $S = G/C_G(A)$, and consider A as a $\mathbb{Z}S$-module. Then D is a finitely generated $\mathbb{Z}S$-submodule of A. There exists a set π of primes such that $D \neq D^p$ for each $p \in \pi$ and the set $P \setminus \pi$ is finite where P is the set of all primes (see, for example, [7, Theorem 1.15]). Let $r = r_0(A)$. For every prime p the factor D/D^p is an elementary abelian p-group of order at most p^r. By Lemma 2.1 there exists a positive integer t such that $D/D^p \leq \zeta_t(G/D^p)$. Since D/D^p has order at most p^r, we have $t \leq r$. Therefore $[D, {}_rG] \leq D^p$. This is true for every prime $p \in \pi$, thus $[D, {}_rG] \leq \bigcap_{p\in\pi} D^p$. Now, the subgroup $C = \bigcap_{p\in\pi} D^p$ is G-invariant. Suppose that it is not trivial, then A/C is periodic, hence D/C is also periodic. Moreover, the choice of C implies that the set $\Pi(D/C)$ is infinite. On the other hand, D/C is a finitely generated $\mathbb{Z}G$-module, then the set $\Pi(D/C)$ is finite. This contradiction shows that $\bigcap_{p\in\pi} D^p = \langle 1\rangle$. Thus $[D, {}_rG] = \langle 1\rangle$, that is $D \leq \zeta_r(G)$. Since the factor A/D is periodic, it is now easy to prove that $A \leq \zeta_r(G)$. □

Corollary 2.10. *Let G be a group, A be a normal abelian torsion-free subgroup of G such that $r_0(A)$ is finite and $G/C_G(A)$ is a locally nilpotent group of finite 0-rank. If G has no nontrivial G-perfect factors, then A is contained in in the nth term of the upper central series of G, for some positive integer n.*

Proof. Since A has finite 0-rank, A has a series $\langle 1\rangle = A_0 \leq A_1 \leq \cdots \leq A_{t-1} \leq A_t = A$ of G-invariant pure subgroups of A whose factors are G-rationally irreducible. The result follows using induction and Lemma 2.9. □

Now we can prove part (ii) of Theorem A. Recall that, if G is a group, then $Tor(G)$ denotes the largest normal periodic subgroup of G.

Corollary 2.11. *Let G be a locally generalized radical group, with finite 0-rank. If G has no nontrivial G-perfect factors, then the factor group $G/Tor(G)$ is nilpotent.*

Proof. Without loss of generality we may suppose that $Tor(G) = \langle 1\rangle$. Then G has a series $\langle 1\rangle \leq L \leq K \leq G$ of normal subgroups, such that L is a torsion-free nilpotent subgroup, K/L is a finitely generated torsion-free abelian group, G/K is a finite group (see, for example, [1, Theorem 2.4.13]). By Lemma 2.1, G/K is nilpotent. By Lemma 2.2, there exists a positive integer t such that $K/L \leq \zeta_t(G/L)$. Thus G/L is nilpotent. Let $\langle 1\rangle = C_0 \leq C_1 \leq \cdots \leq C_{n-1} \leq C_n = L$ be the upper central series of L. Then $C_1, \dots, C_n$ are G -invariant. Moreover, the factors C_j/C_{j-1} are torsion-free,

$1 \leq j \leq n$ (see, for example, [1, Corollary 1.2.9]). Then Corollary 2.10 implies that there exists a positive integer s, such that $L/C_{n-1} \leq \zeta_s(G/C_{n-1})$. Therefore G/C_{n-1} is nilpotent. Using repeatedly Corollary 2.10, we obtain that G is nilpotent. □

Now we study locally finite normal subgroups in a group with no non-trivial perfect factors. We start with the following definition.

Let G be a group, and let A be an infinite normal subgroup of G. We say that A is called G-quasifinite, if A does not contain proper infinite G-invariant subgroups. Obviously, every G-quasifinite normal subgroup satisfies the minimal condition on G-invariant subgroups (the condition Min-G). Moreover, either A contains a proper G-invariant subgroup C such that the factor A/C is a G-chief factor, or A is generated by the union of its proper finite G-invariant subgroups. Conversely, suppose that A satisfies the minimal condition on G-invariant subgroups, and consider the family

$\mathcal{M} = \{B \mid , B$ is an infinite G-invariant subgroup of A $\}$.

Then, ordered by inclusion, the family $\mathcal{M}$ has a minimal element C, and obviously C, is a G-quasifinite normal subgroup of G.

Lemma 2.12. *Let G be a group, and let K be a normal G-quasifinite subgroup of G. If G has no non-trivial G-perfect factors, then K is a Prüfer p-subgroup, for some prime p. Moreover, $K \leq \zeta(G)$.*

Proof. Put $S = [K, G]$, then S is finite. The inclusion $K/S \leq \zeta(G/S)$ together with the fact that K/S is G-quasifinite, imply that K/S is a Prüfer p-subgroup for some prime p. Moreover, K is the union of its proper finite G-invariant subgroups. If H is a finite G-invariant subgroup of K, then its centralizer $C_G(H)$ has finite index in G. Obviously, K does not contain proper G-invariant subgroups of finite index. Hence $C_G(H) \cap K = K$, then $H \leq \zeta(K)$. This is true for each finite G-invariant subgroup of K, hence $K = \zeta(K)$. Therefore K is abelian and then K is a Prüfer p-subgroup. Put $C = C_G(S)$. Since S is finite, so G/C is finite. Let g be an arbitrary element of C. For each element $a \in K$, we have $g^{-1}ag = ab$, where $b \in S$. Then $g^{-2}ag^2 = g^{-1}(g^{-1}ag)g = g^{-1}(ab)g = (g^{-1}ag)(g^{-1}bg) = abb = ab^2$, and, by induction, we obtain that $g^{-n}ag^n = ab^n$ for every positive integer n. Hence, if we write $|S| = k$, we get that $g^k \in C_G(K)$. Therefore the factor $C/C_C(K)$ is bounded. Since G/C is finite, we obtain that $G/C_G(K)$ is bounded. By Lemma 2.1 there exists a positive integer n
such that $S \leq \zeta_n(G)$. Then $[S, {}_nG] = \langle 1 \rangle$, hence $[K, {}_{n+1}G] = \langle 1 \rangle$. Since K is a divisible subgroup and $G/C_G(K)$ is periodic, it follows that $[K, G] = \langle 1 \rangle$ (see, for example, [9, Lemma 3.13]), i.e. $K \leq \zeta(G)$. □

Lemma 2.13. *Let G be a group, and let K be a normal Chernikov subgroup of G. If G has no non-trivial G-perfect factors, then there exists a positive integer n such that $K \leq \zeta_n(G)$. In particular, K is G-nilpotent. Moreover, if the factor group $G/C_G(K)$ is periodic, then the divisible part of K is contained in $\zeta(G)$.*

Proof. The group K satisfies the minimal condition on subgroups. Let D be the divisible part of K. Then K has a finite series $C = C_1 \leq C_2 \leq \cdots \leq C_n = D \leq K$ of G-invariant subgroups, such that the factors C_{j+1}/C_j are G-quasifinite, $1 \leq j \leq n$, while the factor K/D is finite. By Lemma 2.12,

DOI: https://dx.doi.org/10.30504/JIMS.2023.400933.1123

every factor C_{j+1}/C_j is G-central. Then $D \leq \zeta_n(G)$. Since K/D is finite, by Lemma 2.1, there exists a positive integer t such that $K/D \leq \zeta_t(G/D)$. Therefore $K \leq \zeta_{n+t}(G)$.

Finally, suppose that $G/C_G(K)$ is periodic. Since $[D, {}_nG] = \langle 1\rangle$, we obtain $[D, G] = \langle 1\rangle$ (see, for example, [9, Lemma 3.13]). □

Let G be a group and H be a normal subgroup of G. Write $H = \gamma_{G,1}(H)$, and, recursively, $\gamma_{G,\alpha+1(H)} = [\gamma_{G,\alpha}(H), G]$, for all ordinals α and $\gamma_{G,\beta}(G) = \bigcap_{\lambda<\beta} \gamma_{G,\lambda}(H)$, for a limit ordinal β. The series

$$H = \gamma_{G,1}(H) \geq \cdots \geq \gamma_{G,\alpha}(H) \geq \gamma_{G,\alpha+1}(H) \geq \cdots \geq \gamma_{G,\delta}(H)$$

is called the **lower G-central series** of H. The terms of this series are G-invariant subgroups of H. The last term $\gamma_{G,\delta}(H)$ is called the **lower G-hypocenter** of H, and we have $\gamma_{G,\delta}(H) = [\gamma_{G,\delta}(H), G]$. If $H = G$, we obtain the lower central series of G.

Notice that if the group G has no non-trivial G-perfect factors, then the lower G-hypocenter of every normal subgroup of G is trivial.

Lemma 2.14. *Let G be a group and H be a locally finite normal subgroup of G. If G has no non-trivial G-perfect factors, then H is locally nilpotent. Moreover, if $C/C_G(H)$ is a locally finite subgroup of $G/C_G(H)$, then $C/C_G(H)$ is a locally nilpotent $\Pi(H)$-group.*

Proof. Consider the lower G-central series of H, as we noticed, the lower G-hypocenter of H is trivial, and $H = \gamma_{G,1}(H) \geq \cdots \geq \gamma_{G,\alpha}(H) \geq \gamma_{G,\alpha+1}(H) \geq \cdots \geq \gamma_{G,\delta}(H) = \langle 1\rangle$.

We will use induction on δ. If δ is finite, then the result is obvious. Suppose that $\delta = \omega$ is the first infinite ordinal. Let S be an arbitrary finite subgroup of H. Then there exists a positive integer t such that $S \cap \gamma_{G,t}(H) = \langle 1\rangle$, and so $S \simeq S/(S \cap \gamma_{G,t}(H)) \simeq (S\gamma_{G,t}(H))/\gamma_{G,t}(H)$, and these isomorphisms show that S is nilpotent. Suppose that $H/\gamma_{G,\alpha}(H)$ is locally nilpotent for all ordinals $\alpha < \delta$. If δ is not a limit ordinal, then $\gamma_{G,\delta-1}(H)$ is contained in the center of G . Then S is central-by-nilpotent, thus S is nilpotent. Suppose now that δ is a limit ordinal. Then there exists an ordinal $\beta < \delta$ such that $S \cap \gamma_{G,\beta}(H) = \langle 1\rangle$. Therefore we have $S \simeq S/(S \cap \gamma_{G,\beta}(H)) \simeq (S\gamma_{G,\beta}(H))/(\gamma_{G,\beta}(H))$, and these isomorphisms show that S is nilpotent. Hence H is locally nilpotent.

Since H is locally nilpotent, thus $H = Dr_{p\in\Pi(H)} H_p$ where H_p is the Sylow p-subgroup of H. Let $D/C_G(H_p)$ be an arbitrary finite subgroup of $C/C_G(H_p)$. Then $D/C_D(H_p/\gamma_{G,n}(H_p))$ is nilpotent for each positive integer n, by Kaluzhnin theorem (see, for example, [1, Theorem 1.2.22]). Moreover it is easy to see that $D/C_D(H_p/\gamma_{G,n}(H_p))$ is a p-group. Since $D/C_G(H_p)$ is finite, there exists a positive integer t such that $C_D(H_p/\gamma_{G,t}(H_p)) = C_D(H_p/\gamma_{G,\omega}(H_p))$. Therefore $D/C_D(H_p/\gamma_{G,\omega}(H_p))$ is a finite p-group. Suppose that $C_D(H_p/\gamma_{G,\omega}(H_p)) \neq \langle 1\rangle$. Let α be the greatest ordinal such that $C_D(H_p/\gamma_{G,\omega}(H_p)) = C_D(H_p/\gamma_{G,\alpha}(H_p))$. Put $D_1 = C_D(H_p/\gamma_{G,\alpha}(H_p))$. Using again Kaluzhnin theorem, we obtain that $D_1/C_{D_1}(H_p/\gamma_{G,\alpha+n}(H_p))$ is a p-group, for each positive integer n. Since $D_1/C_G(H_p)$ is finite, there exists a positive integer k such that $C_{D_1}(H_p/\gamma_{G,\alpha+k}(H_p)) = C_{D_1}(H_p/\gamma_{G,\alpha+\omega}(H_p))$. Then $D/C_{D_1}(H_p/\gamma_{G,\alpha+\omega}(H_p))$ is a p-group. Since $C_{D_1}(H_p/\gamma_{G,\alpha+\omega}(H_p))$ is not trivial, we could repeat the previous arguments. After finitely many steps, we obtain that

DOI: https://dx.doi.org/10.30504/JIMS.2023.400933.1123

$D/C_G(H_p)$ is a finite p-group. Then $C/C_G(H_p)$ is a p-group. Since $C_G(H) = \bigcap_{p\in\Pi(H)} C_G(H_p)$, ws obtain that $C/C_G(H)$ is isomorphic to a subgroup of $Cr_{p\in\Pi(H)}C/C_G(H_p)$. Since $C/C_G(H)$ is locally finite and $C/C_G(H_p)$ is a p-group for each $p \in \Pi(H)$, we obtain that $C/C_G(H)$ is a locally nilpotent $\Pi(H)$-group. □

Now we can prove Theorem A.

Theorem A. *Let G be a locally generalized radical groupwith finite section rank. If G has no non-trivial G-perfect factors, then the following hold:*

(i) *for every prime p there exists a number s_p such that every Sylow p-subgroup of G is contained in $\zeta_{s_p}(G)$;*
(ii) *the factor-group $G/Tor(G)$ is nilpotent.*

In particular, G is hypercentral, and the hypercentral length of G is at most $\omega + k$, for some positive integer k.

Proof. Let $T = Tor(G)$. Since T is locally finite, Lemma 2.14 shows that T is locally nilpotent. Let $p \in \Pi(T)$ and S_p be the Sylow p-subgroup of T. Then S_p is a Chernikov subgroup (see, for example [1, Theorem 4.2.1]). Then Lemma 2.13 shows that there exists a positive integer n, such that $S_p \leq \zeta_n(G)$. Therefore $T \leq \zeta_{\omega(G)}$. By Corollary 2.11 G/T is nilpotent. Hence there exists a positive integer k such that $G = \zeta_{\omega+k}(G)$, as required. □

From Theorem A the following corollary follows easily.

Corollary 2.15. *Let G be a locally generalized radical group, having finite section rank, and let the set $\Pi(G)$ is finite. If G has no nontrivial G-perfect factors, then G is nilpotent.*

3. Abelian-by-(locally finite) groups with no non-trivial perfect factors.

In this section, we prove Theorem B. We begin with some probably well-known results, that are interesting in their own right.

Lemma 3.1. *Let $\mathcal{X}$ be a class of groups, closed by taking subgroups and finite direct products of subgroups. Let G be a group containing a normal abelian subgroup A, such that $G/C_G(A)$ is finite. If A contains a subgroup B, with $A/B \in \mathcal{X}$, then A contains a G-invariant subgroup C such that $C \leq B$ and $A/C \in \mathcal{X}$.*

Proof. For each element $g \in G$ we have $A/B^g = A^g/B^g \simeq A/B$, hence $A/B^g \in \mathcal{X}$. Since the subgroup $C_G(A)$ has finite index in G, the set $\{B^g \mid g \in G\}$ is finite. Write $\{B^g \mid g \in G\} = \{B_1, \cdots, B_n\}$, and $C = B_1 \cap \cdots \cap B_n$. Using Remaks theorem we obtain an embedding $A/C \leq A/B_1 \times \cdots \times A/B_n$. Since $A/B_j \in \mathcal{X}$, for every $j \in \{1, \cdots, n\}$ we obtain that $A/C \in \mathcal{X}$, as required. □

Lemma 3.2. *Let A be an abelian p-group, where p is a prime. If A is not bounded, then A contains a subgroup B such that A/B is a divisible Chernikov group.*

DOI: https://dx.doi.org/10.30504/JIMS.2023.400933.1123

Proof. Suppose first that A is the direct product of cyclic subgroups. Since A is not bounded, then A contains a subgroup C such that $A/C = Dr_{n\in\mathbb{N}}\langle d_n\rangle$, where the element d_n has order p^n , $n \in \mathbb{N}$. Consider the subgroup $B/C = \langle d_n d_{n+1}^p | n \in \mathbb{N}\rangle$. Then the factor group A/B is a Prüfer p-group, as required.

Suppose now that A can not be decomposed into a direct product of cyclic subgroups. In this case , using for example [2, Theorem 32.3], we obtain that A contains a subgroup D satisfying the following conditions: D is a direct product of cyclic subgroups; D is a pure subgroup of A (that is $D^n = D \cap A^n$, for every $n \in \mathbb{N}$); A/D is a divisible group. Therefore, A/D is a direct product of Prüfer p-subgroups (see, for example, [2, Theorem 23.1]), hence A/D has a subgroup B/D such that A/B is a Prüfer p-group, as required. □

Lemma 3.3. *Let G be a group, p be a prime, and suppose that G contains a normal abelian p-subgroup A such that $G/C_G(A)$ is finite. If A contains a G-invariant divisible Chernikov subgroup D, then A contains a G-invariant subgroup S such that $A = SD$ and the intersection $S \cap D$ is finite.*

Proof. Since D is divisible, there exists a subgroup B of A such that $A = D \times B$ (see, for example, [2, Theorem 21.2]). Then if $n = |G/C_G(A)|$, using for example Theorem 5.9 of [8], we obtain a G-invariant subgroup C of A such that $(D \cap C)^n = \langle 1\rangle$ and $A^n \le DC$. In particular, the intersection $D \cap C$ is finite. Hence, $DC/C \simeq D/(D \cap C) \simeq D$. In particular, DC/C is a divisible subgroup of A/C, therefore there exists a subgroup E/C such that $A/C = (DC/C) \times E/C$. Since the factor A/DC is bounded, so E/C is bounded and $(E/C)^n = \langle 1\rangle$. Put $n = p^k$, so we have $E/C \le \Omega_k(A/C)$. Also consider $S/C = \Omega_k(A/C)$, then the intersection $(S/C) \cap (DC/C)$ is finite and $A/C = (DC/C)(S/C)$. It follows that $A = SD$. Since both $D \cap C$ and $(S/C) \cap (DC/C)$ are finite, thus $S \cap D$ is finite, as required. □

Lemma 3.4. *Let G be a group, and let A be a normal elementary abelian p-subgroup of G such that $G/C_G(A)$ is a finite p-group, where p is a prime. Then there exists a positive integer n such that $A \le \zeta_n(G)$.*

Proof. Let $|G/C_G(A)| = p^n$. We use induction on n. Suppose first that $n = 1$. Then the factor group $G/C_G(A)$ is cyclic of order p, namely $G = C_G(A)\langle g\rangle$. For every $a \in A$ consider the subgroup $\langle a\rangle^{\langle g\rangle} = \langle a, a^g, \cdots, a^{g^{p-1}}\rangle$. Then $\langle a\rangle^{\langle g\rangle}$ is normal in G and of order at most p^p. Moreover $g^p \in \zeta(\langle a, g\rangle)$ and $\langle a, g\rangle/\langle g^p\rangle$ is a finite p-group. Then $\langle a, g\rangle$ is nilpotent of class $\le p$, hence $a \in \zeta_p(\langle a, g\rangle) \le \zeta_p(G)$. Therefore $A \le \zeta_p(G)$.

Suppose now that $n > 1$. Let $z \in \zeta(G/C_G(A))$ be an element of order p. Arguing as in the first part of the proof on the group $C_G(A)\langle z\rangle$, we obtain a series of G-invariant subgroups of A, $\langle 1\rangle = A_0 \le A_1 \le \cdots \le A_{t-1} \le A_t = A$, such that $A_1 = C_A(z)$, $A_{j+1}/A_j = C_{A/A_j}(z)$, $1 \le j \le t-1$, where $t \le p$. Since $z \in C_G(A_{j+1}/A_j)$, we have $|G/C_G(A_{j+1}/A_j)| < |G/C_G(A)|$, for each j, $0 \le j \le t-1$. Thus we can apply for each factor A_{j+1}/A_j the induction hypothesis and the result follows. □

DOI: https://dx.doi.org/10.30504/JIMS.2023.400933.1123

Corollary 3.5. *Let G be a group, and let A be a normal bounded abelian p-subgroup of G, such that $G/C_G(A)$ is a finite p-group, where p is a prime. Then there exists a positive integer n, such that $A \leq \zeta_n(G)$.*

Proof. Since A is bounded, so $A = \Omega_t(A)$, for some positive integer t. We use induction on t. If $t = 1$, then A is an elementary abelian p-subgroup, and the result follows from Lemma 3.4. Suppose now that $t > 1$. Set $B = \Omega_1(A)$, then B is an elementary abelian p-subgroup and $C_G(A) \leq C_G(B)$, so that $G/C_G(B)$ is a finite p-group. By Lemma 3.4 there exists a positive integer k such that $B \leq \zeta_k(G)$. If A is not contained in $\zeta_k(G)$, then $A\zeta_k(G)/\zeta_k(G) = \Omega_s(A\zeta_k(G)/\zeta_k(G))$, where $s < t$. By the induction hypothesis, $A\zeta_k(G)/\zeta_((G) \leq \zeta_{k+m}(G)/\zeta_k(G)$, for some positive integer m. Thus $A \leq \zeta_{k+m}(G)$, as required. □

Now we study normal abelian p-subgroups A of a group G with no non-trivial G-perfect factors, when $G/C_G(A)$ is finite.

Lemma 3.6. *Let G be a group, and let A be a normal abelian subgroup of G such that $G/C_G(A)$ is finite. Suppose that A contains a subgroup B such that A/B is a divisible Chernikov group. If G has no non-trivial G-perfect factors, then $A/[G, A]$ is not bounded.*

Proof. We have $C = Core_G(B) = \bigcap_{g\in G} B^g$. So C is a G-invariant subgroup of G and, by Lemma 3.1, A/C is an abelian Chernikov group. Therefore A contains a G-invariant subgroup H such that $C \leq H$, H/C is finite and A/H is a divisible Chernikov group. Without loss of generality we may suppose that, A/C is a divisible Chernikov group. By Lemma 2.13 we have $A/C \leq \zeta(G/C)$, thus $[G, A] \leq C$. Since A/C is a divisible Chernikov group, so $A/[G, A]$ is not bounded, as required. □

Proposition 3.7. *Let G be a group and suppose that G contains a normal abelian p-subgroup A such that $G/C_G(A)$ is finite, where p is a prime. If G has no non-trivial G-perfect factors, then the subgroup $[G, A]$ is bounded.*

Proof. If the subgroup A is bounded, the result holds. Assume that A is not bounded, then by Lemma 3.2, there exists a subgroup B of A such that A/B is a divisible Chernikov group. Then Lemma 3.6 implies that $A/[G, A]$ is not bounded. Suppose that the subgroup $D = [G, A]$ is not bounded. Then, by Lemma 3.2, there exists a subgroup C of D such that D/C is a divisible Chernikov group. Put $E = Core_G(C) = \bigcap_{g\in G} C^g$. Then E is G-invariant and D/E is a Chernikov group, by Lemma 3.1. Therefore D contains a G-invariant subgroup H such that $E \leq H$, H/E is finite and D/H is a divisible Chernikov group. Without loss of generality we can suppose that D/E is a divisible Chernikov group. We have $[G/E, A/E] = [G, A]E/E = DE/E = D/E$. Hence $[G/E, A/E]$ is a divisible Chernikov group. By Lemma 3.3, A/E contains a G-invariant subgroup S/E such that $A/E = (D/E)(S/E)$ and the intersection $(D/E) \cap (S/E)$ is finite. Then $A/S \simeq (A/E)/(S/E) = (D/E)(S/E)/(S/E) \simeq (D/E)/((D/E) \cap (S/E)) \simeq D/E$ is a divisible Chernikov group. Furthermore, $A/S = (DS)/S = [G, A]S/S = [G/S, A/S]$. We obtain a contradiction, since A/S is not G-perfect. This contradiction shows that the subgroup $[G, A]$ is bounded. □

DOI: https://dx.doi.org/10.30504/JIMS.2023.400933.1123

Corollary 3.8. *Let G be a group, and let A be a be normal abelian p-subgroup of G such that $G/C_G(A)$ is a Chernikov p-group, where p is a prime. If G has no non-trivial G-perfect factors, then the subgroup $[G, A]$ is bounded.*

Proof. Let $D/C_G(A)$ be the divisible part of $G/C_G(A)$. Then, by Lemma 2.13, $D/C_G(A) \leq \zeta(G/C_G(A))$ and $G/C_G(A)$ is nilpotent. Since $[G, A] \neq A$, then $[D, A] \neq A$. Suppose that $C = [D, A] \neq \langle 1\rangle$. Then again $[D, C] = E \neq C$. The factor D/E is nilpotent and D/A is divisible, then $A/E \leq \zeta(D/E)$ (see, for example [5]). But in this case $[D, A] \leq E$, and we obtain a contradiction. This contradiction shows that $[D, A] = \langle 1\rangle$, thus $D \leq C_G(A)$. In particular, $G/C_G(A)$ is finite. The result follows from Proposition 3.7. □

We continue our investigation of normal abelian subgroups A of a group G with no non-trivial G-perfect factors, assuming that $G/C_G(A)$ is a locally finite group whose Sylow p-subgroups are Chernikov groups, for every prime p.

Lemma 3.9. *Let G be a group and let A be a normal abelian p-subgroup, where p is a prime, such that $G/C_G(A)$ is a locally finite group whose Sylow p-subgroups are Cernikov. If G has no non-trivial G-perfect factors, then $G/C_G(A)$ is a finite p-group.*

Proof. By Lemma 2.14 the factor group $G/C_G(A)$ is a locally nilpotent p-group. In particular, it is a Chernikov group. Let D/A be the divisible part of G/A. Since $[G, A] \neq A$, then $[D, A] \neq A$. Suppose that $C = [D, A] \neq \langle 1\rangle$. Then again $[D, C] = E \neq C$. The factor D/E is nilpotent and D/A is divisible, then $A/E \leq \zeta(D/E)$ (see, for example, [5]). But in this case $[D, A] \leq E$, and we obtain a contradiction. This contradiction shows that $[D, A] = \langle 1\rangle$, thus $D \leq C_G(A)$. In particular, $G/C_G(A)$ is finite. □

Corollary 3.10. *Let G be a group and let A be a normal abelian p-subgroup of G, where p is a prime, such that $G/C_G(A)$ is a locally finite group whose Sylow p-subgroups are Chernikov. If G has no non-trivial G-perfect factors, then there exists a positive integer n such that $A \leq \zeta_n(G)$.*

Proof. By Lemma 3.9, $G/C_G(A)$ is a finite p-group. By Corollary 3.8, the subgroup $D = [G, A]$ is bounded. Then Corollary 3.5 implies that there exists a positive integer t such that $D \leq \zeta_t(G)$. The choice of D implies that $A\zeta_t(G)/\zeta_t(G) \leq \zeta(G/\zeta_t(G))$, hence $A \leq \zeta_{t+1}(G)$, as required. □

Corollary 3.11. *Let G be a group and let A be a normal nilpotent p-subgroup of G, where p is a prime, such that $G/C_G(A)$ is a locally finite group whose Sylow p-subgroups are Cernikov. If G has no non-trivial G-perfect factors, then there exists a positive integer n such that $A \leq \zeta_n(G)$.*

Proof. Let $\langle 1\rangle = A_0 \leq A_1 \leq \cdots \leq A_{n-1} \leq A_n = A$ be the upper central series of A. We use induction on n. If $n = 1$, then the result follows from Corollary 3.10. Let $n > 1$ and suppose that $A_{n-1} \leq \zeta_m(G)$. Then A/A_{n-1} is a normal abelian p-subgroup of the factor group G/A_{n-1}. Using again Corollary 3.10, there exists a positive integer k such that $A/A_{n-1} \leq \zeta_k(G/A_{n-1})$, Then $A \leq \zeta_{m+k}(G)$, as required. □

DOI: https://dx.doi.org/10.30504/JIMS.2023.400933.1123

Corollary 3.12. *Let G be a group and let A be a normal nilpotent p-subgroup of G, where p is a prime, such that G/A is a Chernikov group. If G has no non-trivial G-perfect factors, then G is nilpotent*

Proof. By Corollary 3.11, there exists a positive integer t such that $A \leq \zeta_t(G)$. By Lemma 2.13 the factor group G/A is nilpotent. It follows that G itself is nilpotent. □

Proposition 3.13. *Let G be a group and let A be a normal periodic nilpotent subgroup of G, such that G/A is a locally finite group whose Sylow p-subgroups are Chernikov for all primes p. If G has no non-trivial G-perfect factors, then G is Sylow-nilpotent.*

Proof. The group G is locally finite, thus, by Lemma 2.14, G is locally nilpotent. Corollary 3.12 shows that every Sylow p-subgroup of G is nilpotent. □

Now we study torsion-free normal abelian subgroups of a group with no non-trivial G-perfect factors.

Proposition 3.14. *Let G be a group and let A be a normal abelian torsion-free subgroup of G, such that $G/C_G(A)$ is a locally finite group whose Sylow p-subgroups are Chernikov, for all primes p. If G has no non-trivial G-perfect factors, then $A \leq \zeta(G)$.*

Proof. Lemma 2.14 implies that $G/C_G(A)$ is locally nilpotent. Then $G/C_G(A) = Dr_{p\in\pi} S_p/A$ where $S_p/C_G(A)$ is the Sylow p-subgroup of $G/C_G(A)$ and $\pi = \pi(G/C_G(A))$. Let $D_p/C_G(A)$ be divisible part of $S_p/C_G(A)$, and $Q_p/C_G(A) = Dr_{q\neq p} S_q/C_G(A)$.

Let M be an arbitrary finite subset of A and let $C =< M >^G$. Then C is a finitely generated $\mathbb{Z}H$-module, where $H = G/C_G(A)$.

We claim that there exists a positive integer t such that $C \leq \zeta_t(G)$.

If C has finite 0-rank, this follows from Corollary 2.10. Suppose that C has infinite 0-rank. There exists a set μ of primes such that, $\Pi \setminus \mu$ is finite and $C \neq C^p$ for every $p \in \mu$ (see, for example [7, Theorem 1.15]). Let $p \in \mu$. Then, by Lemma 2.14, $G/C_G(C/C^p)$ is a p-group, that is $C_G(C/C^p) \geq Q_p$. In particular, it is a Chernikov group. Therefore Lemma 3.9 implies that $G/C_G(C/C^p)$ is finite, that is $C_G(C/C^p) \geq D_pQ_p$. Hence the group C/C^p is finite. Write $C_1 = C^p, C_2 = C_1^p, C_{n+1} = C_n^p$, $n \in \mathbb{N}$. Since C is abelian and torsion-free, the map $c \in C \longmapsto c^p \in C$ is a monomorphism. Therefore the factors C/C_1 and C_1/C_2 are isomorphic. In particular, C_1/C_2 is finite, thus C/C_2 is finite. By induction, all factors C/C_n are finite, $n \in \mathbb{N}$. It follows that $C_G(C/C_n) \geq D_pQ_p$, for every $n \in \mathbb{N}$. Put $E = \bigcap_{n\in\mathbb{N}} C_n$, then $C_G(C/E) \geq D_pQ_p$, so that $C_G(C/E)$ has finite index. Since $C \neq C_1$, we obtain that $C_n \neq C_{n+1}$, for all $n \in \mathbb{N}$. Therefore the group C/E is infinite. Let T/E be the periodic part of C/E. Since T/E is a pure subgroup of C/E, we have $(T/E)^p = (T/E) \cap (C/E)^p$. Then $(T/E)/(T/E)^p = (T/E)/((T/E) \cap C/E)^p) \simeq (T/E)(C/E)^p/(C/E)^p \leq (C/E)/(C/E)^p = (C/E)/(C^pE/E) \simeq C/C^pE$.
Hence the factor $(T/E)/(T/E)^p$ is finite, and $T/E = B/E \times P/E$, where B/E is finite and P/E is divisible (see, for example, [6, Lemma 3]). Since C/E is residually finite, we obtain that $T/E = B/E$ is finite. Thus C/T is an infinite torsion-free group. Since $G/C_G(C/T)$ is finite, we obtain that C/T

DOI: https://dx.doi.org/10.30504/JIMS.2023.400933.1123

is a finitely generated abelian group. It follows that T has infinite 0-rank, in particular, the subgroup T is not trivial.

Assume that T is divisible. Consider $[T, G] = T_1 \neq T$. Suppose that the factor T/T_1 is not periodic and let T_2/T_1 be the periodic part of T/T_1. Then T/T_2 is a divisible torsion-free group. Let $g \in C_G(C/T)$. Then, for every $c \in C$, $(gT_2)^{-1}cT_2gT_2 = cT_2zT_2$, where $zT_2 \in T/T_2$. Notice that the factor group $G/C_G(A)$ is periodic, therefore there exists a positive integer k such that $g^k \in C_G(A)$. We have $cT_2 = (g^{-k}cg^k)T_2 = (gT_2)^{-k}cT_2(gT_2)^k = cT_2(zT_2)^k$. Thus $(zT_2)^k = T_2$. Then $zT_2 = T_2$, since T/T_2 is torsion free. Therefore $g \in C_G(C/T_2)$, hence $C_G(C/T) \leq C_G(C/T_2)$. In particular, $C_G(C/T_2)$ has finite index. Then C/T_2 is a finitely generated abelian group, since C is finitely generated as $\mathbb{Z}G$-module. Since the group T/T_2 is divisible, we obtain a contradiction. This contradiction shows that the group T/T_1 is periodic. Suppose that there exists a prime q such that $T_1^q = T_3 \neq T_1$. By Lemma 3.9 and Corollary 3.5, there exists a positive integer s such that $T_1/T_3 \leq \zeta_s(G/T_3)$. Since the factor T/T_1 is G-central, there exists a positive integer t such that $[T/T_3, {}_tG] = \langle 1 \rangle$. Since T/T_3 is divisible and G/A is periodic, we obtain that $[T/T_3, G] = \langle 1 \rangle$ (see, for example, [9, Lemma 3.13]), hence $[G, T] \leq T_3 \neq T_1 = [G, T]$, and we obtain a contradiction. This contradiction proves that $T_1^q = T_1$, for all primes q, hence the subgroup T_1 is also divisible. Suppose that $T_1 \neq \langle 1 \rangle$. Then $[T_1, G] = T_4 \neq T_1$. Assume first that the factor T_1/T_4 is not periodic and let T_5/T_4 be the periodic part of T_1/T_4. Then T_1/T_5 is a divisible torsion-free group. Being divisible, the subgroup T_1/T_5 has a complement in T/T_5, hence $T/T_5 = T_1/T_5 \times T_6/T_5$ (see, for example, [2, Theorem 21.2]). The isomorphism $T_6/T_5 \simeq T/T_1$ shows that T_6/T_5 is a periodic divisible subgroup. Then T_6/T_5 is the periodic part of T/T_5 and, in particular, T_6/T_5 is G-invariant. Let x be an element of G, b an element of T_6. Since the factor T/T_1 is G-central, $[xT_5, bT_5] \in T_1/T_5$. Since T_6/T_5 is G-invariant, we have that $[xT_5, bT_5] \in T_6/T_5$, then $[xT_5, bT_5] \in (T_1/T_5) \cap (T_6/T_5) = \langle 1 \rangle$. Hence $T_6/T_5 \leq \zeta(G/T_5)$. It follows that $T/T_5 \leq \zeta(G/T_5)$. Then $[G, T] \leq T_5 \neq T_1 = [G, T]$, a contradiction. This contradiction shows that the factor T_1/T_4 is periodic. The factors T/T_1 and T_1/T_4 are G-central, therefore $[T/T_4,{}_2 G] = \langle 1 \rangle$. Since T/T_4 is divisible and G/A is periodic we obtain that $[T/T_4, G] = \langle 1 \rangle$ (see, for example [9, Lemma 3.13]), thus $[G, T] \leq T_4 \neq T_1 = [G, T]$, and we obtain a contradiction. This contradiction proves that $T_1 = \langle 1 \rangle$. Hence $T \leq \zeta(G)$ and then the subgroup T is torsion-free, a contradiction.

This contradiction shows that the subgroup T is not divisible.

Therefore, there exists a prime q such that $T \neq T^q$. Obviously T/T^q is the periodic part of C/T^q and, since T/T^q is bounded, there exists a subgroup R/T^q such that $C/T^q = T/T^q \times R/T^q$ (see, for example, [2, Theorem 27.5]). The isomorphism $R/T^q \simeq C/T$ shows that R/T^q is a free abelian group of finite 0-rank. Then $R/T^q \neq (R/T^q)^q$. We have $(C/T^q)^q = (R/T^q)^q$, moreover $(C/T^q)^q = C^qT^q/T^q = C^q/T^q$. We have $(C/T^q)/(C/T^q)^q = (C/T^q)/(C^q/T^q) \simeq C/C^q$ and the equality $(C/T^q)^q = (R/T^q)^q$ implies that $|C/C^q| = |T/T^q||(R/T^q)/(R/T^q)^q| = |T/T^q||(C/T)/(C/T)^q|$. Then $T \leq C^p$ implies $C/C^p \simeq (C/T)/(C^p/T)$. In particular, $q \neq p$, moreover $T = T^p$. Write $U_1 = T^q, U_2 = U_1^q, U_{n+1} = U_n^q$, $r_n = q^n, n \in \mathbb{N}$. Now, T/U_n is the periodic part of C_n/U_n, and, since T/U_n is bounded, there exists a subgroup R_n/U_n such that $C_n/U_n = T/U_n \times R_n/U_n$ (see, for example Theorem 27.5 of [2]). Then $R_n \cap T = U_n$, $n \in \mathbb{N}$. The isomorphism $R_n/U_n \simeq C_n/T$ shows that R_n/U_n is a free abelian group of

DOI: https://dx.doi.org/10.30504/JIMS.2023.400933.1123

finite 0-rank. Then $R_n/U_n \neq (R_n/U_n)^{r_n}$. We have $(C_n/U_n)^{r_n} = (R_n/U_n)^{r_n}$. On the other hand, the inclusion $T \leq C_n$ implies that $U_n = T^{r_n} \leq C_n^{r_n}$, so that $(C_n/U_n)^{r_n} = (C_n^{r_n}U_n)/U_n = C_n^{r_n}/U_n$. Write $V_n = C_n^{r_n}$, $n \in \mathbb{N}$. Then C/V_n is a direct product of finitely many cyclic groups, having order either p^n or q^n and $V_n \cap T = U_n$, $n \in \mathbb{N}$. Write $W = \bigcap_{n\in\mathbb{N}} V_n$. Then $C_G(C/W)$ has finite index. Therefore C/W is a finitely generated abelian group. We have $W \cap T = (\bigcap_{n\in\mathbb{N}} V_n) \cap T = \bigcap_{n\in\mathbb{N}}(V_n \cap T) = \bigcap_{n\in\mathbb{N}} U_n = Y$. Obviously $U_n \neq U_{n+1}$, for each $n \in \mathbb{N}$, hence T/Y is infinite. Thus $TW/W \simeq T/(W \cap T) = T/Y$ is a non periodic finitely generated abelian group. Then $(TW/W)^p \neq TW/W$. On the other hand we have $T^p = T$, hence $(TW/W)^p = TW/W$, and we obtain a contradiction. This contradiction proves that $T = \langle 1 \rangle$.

Therefore the subgroup C is finitely generated, and by Lemma 2.2, there exists a positive integer t such that $C \leq \zeta_t(G)$, as required.

Suppose that C is not contained in $\zeta(G)$, and $Z_1 = C \cap \zeta(G), Z_2 = C \cap \zeta_2(G)$. Then the subgroups Z_1, Z_2 are pure in C. Let $y \in Z_2$ such that $y \notin Z_1$. Then there exists an element g such that $y^g \neq y$. We have $y^g = yz$ for some $z \in Z_1$. Since $G/C_G(A)$ is locally finite, there exists a positive integer m such that $g^m \in C_G(A)$. DIFdel Then,So $y = g^{-m}yg^m = yz^m$, hence $z^m = 1$, and so $z = 1$, since C is torsion-free, and we obtain a contradiction. This contradiction proves the inclusion $C \leq \zeta(G)$. The choice of C implies that $A \leq \zeta(G)$. □

Corollary 3.15. *Let G be a group and A be a normal nilpotent torsion-free subgroup such that $G/C_G(A)$ is a locally finite group whose Sylow p-subgroup are Chernikov, for all primes p. If G has no non-trivial G-perfect factors, then there exists a positive integer t such that $A \leq \zeta_t(G)$.*

Now we can prove Theorem B.

Theorem B. *Let G be a group and A be a normal nilpotent subgroup of G such that G/A is a locally finite group whose Sylow p-subgroups are Chernikov for all prime p. If G has no non-trivial G-perfect factors, then G satisfies the following conditions:*

(i) *for every prime p there exists a positive integer s_p such that $\zeta_{s_p}(G)$ contains the Sylow p-subgroups of G;*
(ii) *the factor group $G/Tor(G)$ is nilpotent.*

In particular, G is hypercentral, and the hypercentral length of G is at most $\omega + k$ for some positive integer k.

Proof. Let $T = Tor(A)$, then $T = Dr_{p\in\Pi(T)}T_p$, where T_p is the Sylow p -subgroup of T, $p \in \Pi(T)$. By Lemma 2.14, the factor group G/A is locally nilpotent, moreover it is hypercentral. By Corollary 3.11, for each prime p there exists a positive integer m_p such that $T_p \leq \zeta_{m_p}(G)$. Thus $T \leq \zeta_\omega(G)$. By Corollary 3.15, $A/T \leq \zeta_t(G/T)$, for a suitable positive number t. Since G/A is hypercentral, then G is hypercentral. Then the subset R of all elements of G having finite order is a characteristic subgroup of G. Furthermore, for every prime p the Sylow p-subgroup of R/T is a Chernikov group, and there exists a positive integer n_p such that it is contained in $\zeta_{n_p}(G/T)$, by Lemma 2.13. Then, for every

DOI: https://dx.doi.org/10.30504/JIMS.2023.400933.1123

prime p, there exists a positive integer s_p such that the Sylow p-subgroup of R is contained in $\zeta_{s_p}(G)$. Finally, G/R is nilpotent, since AR/R is nilpotent and $(G/R)/(AR/R)$ is periodic.

□

Acknowledgements

This work supported by the "National Group for Algebraic and Geometric Structures, and their Applications" (GNSAGA - INdAM), Italy, and the National Foundation of Ukraine (Grant No.2020.02/0066).

The first author is grateful to the Department of Mathematics of the University of Salerno for hospitality and support, during this research.

Also, the first author is also grateful to Isaac Newton Institute for Mathematical Sciences and to the University of Edinburgh for the support provided in the frame of LMS Solidarity Supplementary Grant Program.

Some of these references are not used in the text.

References

[1] M.R. Dixon, L.A. Kurdachenko, I.Ya. Subbotin, Ranks of groups. The tools, characteristics and restrictions, *Wiley*, New York (2017).

[2] L. Fuchs, Infinite abelian groups, Vol. 1, *Academic Press*, New York (1970).

[3] R.I. Grigorchuk, To the Burnside's problem on periodic groups, *Funk. Analis* **14** (1980), no. 1, 53-54.

[4] H. Heineken, A. Mohamed, A group with trivial centre satisfying the normalizer condition, *J.Algebra* **10** (1968), no. 3, 368-376.

[5] L.A. Kurdachenko, Locally nilpotent groups with the weak minimal condition for normal subgroups, *Sibirsk. Mat. Zh.* **25** (1984), no. 4, 99-106.

[6] L.A. Kurdachenko, Nonperiodic FC-groups and related classes of locally normal groups, *Sibirsk. Mat. J.* **27** (1986), no. 2, 227-236.

[7] L.A. Kurdachenko, J. Otal, I.Ya. Subbotin, Groups with prescribed quotient groups and associated module theory, *World Scientific*, New Jersey (2002).

[8] L.A. Kurdachenko, J. Otal and I. Ya. Subbotin, Artinian modules over group rings, *Frontiers in Mathematics, Birkhäuser*, Basel (2007).

[9] D.J.S. Robinson, Finiteness Conditions and Generalized Soluble Groups, Part 1 and 2, *Springer-Verlag*, Berlin, Heidelberg, New York (1972).

[10] D.J.S. Robinson, A course in the theory of groups, *Springer-Verlag*, New York (1982).

Leonid A. Kurdachenko

Department of Algebra and Geometry, School of Mathematics and Mechanics, Gagarin prospect 72, Dnipro 10, 49010 Ukraine. Email: lkurdachenko@i.ua

Patrizia Longobardi

Dipartimento di Matematica, Università di Salerno, via Giovanni Paolo II, 132, 84084 Fisciano (Salerno), Italy. Email: plongobardi@unisa.it

Mercede Maj

Dipartimento di Matematica, Università di Salerno, via Giovanni Paolo II, 132, 84084 Fisciano (Salerno), Italy. Email: mmaj@unisa.it

DOI: https://dx.doi.org/10.30504/JIMS.2023.400933.1123