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A GENERALIZATION OF THE HEISENBERG GROUP

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Abstract. In the former paper, author studied spectral synthesis on the Heisenberg group. This problem is closely connected with the finite-dimensional representations of the Heisenberg group on the space of continuous complex-valued functions. In this paper, we attempt to generalize the Heisenberg group over any commutative topological group. Starting with a basic commutative topological group we define a non-commutative topological group whose elements are triplets consisting of an element of the basic group, an exponential on the basic group, and a nonzero complex number which serves as a scaling factor. The group operation is a combination of the addition on the basic group, the multiplication of the exponentials, and the multiplication of complex nonzero numbers. Although there is no differentiability, our generalized Heisenberg group shares some basic properties with the classical one. In particular, we describe finite-dimensional representations of this group on the space of continuous functions, and we show that finite-dimensional translation invariant function spaces over this group consist of exponential polynomials.

1. The classical Heisenberg group

In this paper, $\mathbb{K}$ denotes either the set of real numbers $\mathbb{R}$, or the set of complex numbers $\mathbb{C}$, and $\mathbb{K}^*$ denotes the set of nonzero numbers, and $\mathbb{R}^+$ denotes the set of positive real numbers. For a given positive integer d, $L(\mathbb{K}, d)$ denotes the algebra of all $d \times d$ matrices with entries in $\mathbb{K}$, $GL(\mathbb{K}, d)$ denotes the multiplicative group of regular $d \times d$ matrices with entries in $\mathbb{K}$, and $D(\mathbb{K}^*, d)$ denotes the multiplicative group of $d \times d$ diagonal matrices with entries in $\mathbb{K}^*$.

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We recall that the three-dimensional Heisenberg group is defined on the set $H = \mathbb{R}\times\mathbb{R}\times\mathbb{R}$, by the following operation: for (x, y, t) and (u, v, s) in H we let

$$(x, y, t)\cdot(u, v, s) = (x + u, y + v, t + s + xv).$$

Then H is a group with identity $(0, 0, 0)$ and the inverse of (x, y, t) is $(-x, -y, -t + xy)$. This group is obviously noncommutative, the commutator of (x, y, t) and (u, v, s) is

$$(x, y, t)\cdot(u, v, s)\cdot(-x, -y, -t + xy)\cdot(-u, -v, -s + uv) = (0, 0, xv - uy).$$

Using the Euclidean topology on $\mathbb{R}\times\mathbb{R}\times\mathbb{R}$, the Heisenberg group H is a locally compact topological group – in fact, it is a Lie group.

If we identify (x, y, t) with the matrix

$$\begin{pmatrix}1 & x & t\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix}, \tag{1.1}$$

then we set up an isomorphism between H and the subgroup of $GL(\mathbb{R}, 3)$ consisting of all matrices of the given type. Indeed,

$$\begin{pmatrix}1 & x & t\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix}\cdot\begin{pmatrix}1 & u & s\\ 0 & 1 & v\\ 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}1 & x+y & t+s+xv\\ 0 & 1 & y+s\\ 0 & 0 & 1\end{pmatrix}.$$

We shall denote the Lie group of these matrices with H as well. The Lie algebra $\mathfrak{h}$ of H can be identified with the algebra of matrices of the form

$$\begin{pmatrix}0 & x & t\\ 0 & 0 & y\\ 0 & 0 & 0\end{pmatrix}. \tag{1.2}$$

This construction can easily be generalized to $\mathbb{R}^d$. Let $H_d = \mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}$, and for (x, y, t) and (u, v, s) in H_d, we define

$$(x, y, t)\cdot(u, v, s) = (x + y, u + v, t + s + x{\bullet}v),$$

where $\bullet$ stands for the inner product in $\mathbb{R}^d$. This product can also be represented by matrix multiplication. Let I_d denote the $d\times d$ unit matrix, and we identify (x, y, t) with the $(d + 2)\times(d + 2)$ matrix

$$\begin{pmatrix}1 & x & t\\ 0 & I_d & y\\ 0 & 0 & 1\end{pmatrix}.$$

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Here x is a row vector of type $1\times d$, and y is a column vector of type $d\times 1$. Then we have

$$\begin{pmatrix}1 & x & t\\ 0 & I_d & y\\ 0 & 0 & 1\end{pmatrix}\cdot\begin{pmatrix}1 & u & s\\ 0 & I_d & v\\ 0 & 0 & 1\end{pmatrix}=\begin{pmatrix}1 & x+u & t+s+x\bullet v\\ 0 & I_d & y+v\\ 0 & 0 & 1\end{pmatrix}.$$

The corresponding Lie algebra $\mathfrak{h}_d$ consists of all $(d+2)\times(d+2)$ matrices of the form

$$\begin{pmatrix}0 & x & t\\ 0 & 0_d & y\\ 0 & 0 & 1\end{pmatrix},$$

where x,y are in $\mathbb{R}^d$, t is in $\mathbb{R}$, and 0_d is the $d\times d$ zero matrix. The exponential mapping $\exp:\mathfrak{h}_d\to H_d$ is given by

$$\exp\begin{pmatrix}0 & x & t\\ 0 & 0_d & y\\ 0 & 0 & 0\end{pmatrix}=\begin{pmatrix}1 & x & t+\frac{1}{2}x\bullet y\\ 0 & 1_d & y\\ 0 & 0 & 1\end{pmatrix}.$$

It is well-known that the exponential map from $\mathfrak{h}_d$ onto H_d is bijective.

The Lie algebra $\mathfrak{h}_d$ has the basis

$$A_i=\begin{pmatrix}0 & e_i & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},\ B_i=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & e_i\\ 0 & 0 & 0\end{pmatrix},\ C=\begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} \tag{1.3}$$

$(i=1,2,\dots,d)$. The only nontrivial commutation relations are

$$[A_i,B_i]=C,\quad (i=1,2,\dots,d).$$

It follows that $\big[[X,Y],Z\big]=0$ for any three matrices X,Y,Z in $\mathfrak{h}_d$. By the Campbell–Baker–Hausdorff formula (see e.g. [7, Proposition 1.3.2], p.25), it follows that for any matrices X,Y in $\mathfrak{h}_d$ we have

$$e^Xe^Ye^{-X}e^{-Y}=e^{[X,Y]}, \tag{1.4}$$

and

$$e^Xe^Y=e^{X+Y+\frac{1}{2}[X,Y]}. \tag{1.5}$$

2. Generalization for commutative topological groups

There is an analogue of the Heisenberg group on locally compact Abelian groups, which appears implicitely in the paper of Mackey [5]. Generalizations on certain locally compact Abelian groups also appear in a paper of Weil [10]. The idea of those generalizations was exhibited explicitely in [2]. Our basic idea in the present generalization is that we focus on spectral synthesis on Abelian groups, where one considers non-unitary representations. Accordingly, we move on from characters to exponentials, and from unitary representations of $L^2(G)$ to non-unitary representations of $\mathcal{C}(G)$. In order to do this, we replace $G\times\widehat{G}\times\mathbb{T}$ by $G\times\operatorname{Hom}(G,\mathbb{C}^*)\times\mathbb{C}^*$, where G is not necessarily locally compact. The point is that in this setting we show that every finite-dimensional variety on the generalized Heisenberg

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group consists of exponential polynomials in the classical sense: linear combinations of products of exponential polynomials on G, $\mathrm{Hom}\,(G,\mathbb{C}^*)$ and $\mathbb{C}^*$.

Let G, H be commutative topological groups. The operation in G will be denoted by $+$, and for each $f : G \to H$ we write $\check{f}(x) = f(-x)$. The set of all continuous homomorphisms of G into H is denoted by $\mathrm{Hom}\,(G,H)$. This is a group, if the operation in $\mathrm{Hom}\,(G,H)$ is defined pointwise. In fact, it is a topological group, when equipped with the topology of compact convergence. If H is the multiplicative topological group of $\mathbb{C}^*$, then the elements of $\mathrm{Hom}\,(G,\mathbb{C}^*)$ are called *exponentials*. If H is the multiplicative topological group of $\mathbb{T}$, the complex unit circle, then the elements of $\widehat{G} = \mathrm{Hom}\,(G,\mathbb{T})$ are called *characters*. If H is the additive topological group of $\mathbb{C}$, then the elements of $\mathrm{Hom}\,(G,\mathbb{C})$ are called *additive functions*. If H is the additive topological group of $\mathbb{R}$, then the elements of $\mathrm{Hom}\,(G,\mathbb{R})$ are called *real characters*. In fact, $\mathrm{Hom}\,(G,\mathbb{C})$, resp. $\mathrm{Hom}\,(G,\mathbb{R})$ is a complex, resp. real linear space.

Given an exponential $m : G \to \mathbb{C}^*$, we have $m = m|\check{m}||m|$. Here $\chi = m|\check{m}|$ is a character of G, and $a = \ln|m|$ is a real character, further we have the equation $m = \chi \exp a$. This representation is unique, and it is called the *polar decomposition* of the exponential m. It shows that $\mathrm{Hom}\,(G,\mathbb{C}^*)$ is topologically isomorphic to $\widehat{G} \times \mathrm{Hom}\,(G,\mathbb{R})$.

Let $H_G = G \times \mathrm{Hom}\,(G,\mathbb{C}^*) \times \mathbb{C}^*$ equipped with the product topology. We define the multiplication on H_G as follows:

$$(x,m,u)\cdot(y,n,v) = (x+y, mn, uvn(x))$$

whenever x, y are in G, m, n are in $\mathrm{Hom}\,(G,\mathbb{C}^*)$, and u, v are in $\mathbb{C}^*$. The associativity of this operation can be checked easily. The element $(0,1,1)$ is the identity of this multiplication, where the first component 0 is the zero element of the group G, the second component 1 is the exponential identically 1 on G, and the third component is the number 1 in $\mathbb{C}^*$. The element $(-x, \check{m}, u^{-1}m(x))$ is the inverse of (x,m,u). It follows that H_G is a topological group.

A distinguished closed subgroup of H_G is $H_G^+ = G \times \mathrm{Hom}\,(G,\mathbb{R}^+) \times \mathbb{R}^+$. Indeed, $\mathrm{Hom}\,(G,\mathbb{R}^+)$ is a closed subgroup in $\mathrm{Hom}\,(G,\mathbb{C}^*)$, and $\mathbb{R}^+$ is a closed subgroup in $\mathbb{C}^*$, hence H_G^+ is topologically closed, and it is also closed with respect to the multiplication defined above. If u is a nonzero complex number, then we write it uniquely in the form $u = t \cdot e^{i\theta}$, where $t = |u|$ and θ is in $\mathbb{T}$. For θ we may choose the principal value of the argument $\theta = \mathrm{Arg}\,u$ in the interval $[-\pi, \pi[$. It turns out that the subgroup $H_{\mathbb{R}^d}^+$ is isomorphic to the d-dimensional Heisenberg group H_d. Indeed, if (x,m,u) is in $H_{\mathbb{R}^d}^+ = \mathbb{R}^d \times \mathrm{Hom}\,(\mathbb{R}^d,\mathbb{R}^+) \times \mathbb{R}^+$, then we can write $m(x) = \exp a(x) = \exp \mu{\bullet}x$, where μ is in $\mathbb{R}^d$. Further, we can write $u = |u| = \exp t$, where t is in $\mathbb{R}$. Finally, if we define for each (x,m,u) in $\mathrm{Hom}\,(\mathbb{R}^d,\mathbb{R}^+)$ the mapping

$$(x,m,u) \mapsto \Phi(x,m,u) = (x,\mu,t),$$

then $\Phi : H_{\mathbb{R}^d}^+ \to H_d$ is a continuous surjective isomorphism, as

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$$\Phi\big((x,m,u)(y,n,v)\big)= \Phi\big(x+y,mn,uvn(x)\big)$$
$$= (x+y,\mu+\nu,t+s+\nu{\bullet}x) = (x,\mu,t)\circ(y,\nu,s),$$

where $n(x) = \exp\nu{\bullet}x$, and $s = \ln v$, and $\circ$ is the multiplication in classical the Heisenberg group H_d.

The commutator subgroup of H_G is generated by the elements

$$(x,m,u)(y,n,v)(x,m,u)^{-1}(y,n,v)^{-1} = (x+y,mn,uvn(x))(-x,\check{m},u^{-1}m(x))(-y,\check{n},v^{-1}n(y))$$
$$= (x+y,mn,uvn(x))(-x-y,\check{m}\check{n},u^{-1}v^{-1}n(x)m(x)n(y))$$
$$= (0,1,n(x)m(x)n(y)n(x)\cdot\check{m}(x)\check{n}(x)\check{m}(y)\check{n}(y)) = (0,1,n(x)\check{m}(y)).$$

On the other hand,

$$(x,m,u)(0,1,v) = (x,m,uv) = (0,1,v)(x,m,u),$$

hence the elements $(0,1,v)$ are in the center of H_G. It follows that the second commutator subgroup is trivial, hence the group H_G is 2-nilpotent.

Let $\mathfrak{h}_G$ denote $G\times \mathrm{Hom}\,(G,\mathbb{R})\times\mathbb{R}$, which we identify with the set of all matrices of the form

$$X = \begin{pmatrix} 0 & x & t\\ 0 & 0 & a\\ 0 & 0 & 0\end{pmatrix}$$

with x in G, a in $\mathrm{Hom}\,(G,\mathbb{R})$, and t in $\mathbb{R}$. Then $\mathfrak{h}_G$ is a ring with respect to matrix addition and a formal multiplication of matrices, where the "product" of x in G and a in $\mathrm{Hom}\,(G,\mathbb{R})$ is interpreted as $a(x)$. As

$$X\cdot Y = \begin{pmatrix} 0 & x & t\\ 0 & 0 & a\\ 0 & 0 & 0\end{pmatrix}\cdot\begin{pmatrix} 0 & y & s\\ 0 & 0 & a\\ 0 & 0 & 0\end{pmatrix} = \begin{pmatrix} 0 & 0 & a(x)\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}.$$

It follows that the product of any three elements in this ring is zero, hence $\mathfrak{h}_G$ is a 2-nilpotent ring. We introduce Lie bracket in $\mathfrak{h}_G$ in the obvious way, by defining

$$[X,Y] = \left[\begin{pmatrix} 0 & x & t\\ 0 & 0 & a\\ 0 & 0 & 0\end{pmatrix},\begin{pmatrix} 0 & y & s\\ 0 & 0 & b\\ 0 & 0 & 0\end{pmatrix}\right] =$$

$$\begin{pmatrix} 0 & x & t\\ 0 & 0 & a\\ 0 & 0 & 0\end{pmatrix}\cdot\begin{pmatrix} 0 & y & s\\ 0 & 0 & b\\ 0 & 0 & 0\end{pmatrix} - \begin{pmatrix} 0 & y & s\\ 0 & 0 & b\\ 0 & 0 & 0\end{pmatrix}\cdot\begin{pmatrix} 0 & x & t\\ 0 & 0 & a\\ 0 & 0 & 0\end{pmatrix} =$$

$$\begin{pmatrix} 0 & 0 & b(x)-a(y)\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}.$$

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Accordingly, $\mathfrak{h}_G$ is a 2-nilpotent Lie algebra .

In the theory of Lie groups the exponential mapping is a basic tool. On $\mathfrak{h}_G$ we define the exponential mapping using the (finite) matrix Taylor series

$$\exp\begin{pmatrix}0 & x & t\\ 0 & 0 & a\\ 0 & 0 & 0\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} + \begin{pmatrix}0 & x & t\\ 0 & 0 & a\\ 0 & 0 & 0\end{pmatrix} + \frac{1}{2}\begin{pmatrix}0 & x & t\\ 0 & 0 & a\\ 0 & 0 & 0\end{pmatrix}^2,$$

all the other terms being zero, by 2-nilpotency. In other words

$$\exp\begin{pmatrix}0 & x & t\\ 0 & 0 & a\\ 0 & 0 & 0\end{pmatrix} = \begin{pmatrix}1 & x & t+\frac{1}{2}a(x)\\ 0 & 1 & a\\ 0 & 0 & 1\end{pmatrix}. \tag{2.1}$$

Unfortunately, exp does not map $\mathfrak{h}_G$ into H_G. However, it turns out that H_G^+, which is a subgroup of H_G, is isomorphic to the image of exp, which we denote by $\exp(\mathfrak{h}_G)$. We define $\Phi : H_G^+ \to \exp(\mathfrak{h}_G)$ as follows. We note that every element of H_G^+ can uniquely be written in the form $(x, \exp a, e^t)$ where t is in $\mathbb{R}$ and a is in $\operatorname{Hom}(G, \mathbb{R})$. Then we define

$$\Phi(x, \exp a, e^t) = \begin{pmatrix}1 & x & t\\ 0 & 1 & a\\ 0 & 0 & 1\end{pmatrix},$$

which is clearly in $\exp(\mathfrak{h}_G)$, by (2.1).

Proposition 2.1. *The mapping Φ defined above is an isomorphism of H_G^+ onto $\exp(\mathfrak{h}_G)$.*

Proof. We can compute as follows:

$$\Phi[(x, \exp a, e^t)\cdot(y, \exp b, e^s)] = \Phi(x+y, \exp a \exp b, e^t e^s \exp b(x)) =$$

$$\Phi(x+y, \exp(a+b), e^{t+s+b(x)}) = \begin{pmatrix}1 & x+y & t+s+b(x)\\ 0 & 1 & a+b\\ 0 & 0 & 1\end{pmatrix} =$$

$$\begin{pmatrix}1 & x & t\\ 0 & 1 & a\\ 0 & 0 & 1\end{pmatrix}\cdot\begin{pmatrix}1 & y & s\\ 0 & 1 & b\\ 0 & 0 & 1\end{pmatrix} = \Phi(x, \exp a, e^t)\Phi(y, \exp b, e^s).$$

This proves that Φ is a homomorphism, which is clearly one-to-one, hence it is an isomorphism. Finally, if

$$\exp\begin{pmatrix}0 & x & t\\ 0 & 0 & a\\ 0 & 0 & 0\end{pmatrix} = \begin{pmatrix}1 & x & t+\frac{1}{2}a(x)\\ 0 & 1 & a\\ 0 & 0 & 1\end{pmatrix}$$

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is an arbitrary element in $\exp(\mathfrak{h}_G)$, then we have

$$\begin{pmatrix} 1 & x & t+\frac{1}{2}a(x) \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix} = \Phi(x, \exp a, e^{t+\frac{1}{2}a(x)})$$

which shows that Φ is surjective, hence $\exp(\mathfrak{h}_G)$ is isomorphic to H_G^+. $\square$

The inverse of exp is the logarithm, which is defined on $\exp(\mathfrak{h}_G)$ by

$$\log \begin{pmatrix} 1 & x & t \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & x & t-\frac{1}{2}a(x) \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix}.$$

We have the following commutation relation:

$$\exp X \exp Y \exp(-X) \exp(-Y) = \exp[X,Y]. \tag{2.2}$$

Indeed, for

$$X = \begin{pmatrix} 0 & x & t \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & y & s \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}$$

we obtain

$$\begin{aligned} \exp X \exp Y &= \begin{pmatrix} 1 & x & t+\frac{1}{2}a(x) \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & y & s+\frac{1}{2}b(y) \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & x+y & t+s+\frac{1}{2}a(x)+\frac{1}{2}b(y)+b(x) \\ 0 & 1 & a+b \\ 0 & 0 & 1 \end{pmatrix}, \end{aligned}$$

and

$$\exp(-X)\exp(-Y) = \begin{pmatrix} 1 & -x & -t+\frac{1}{2}a(x) \\ 0 & 1 & -a \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -y & -s+\frac{1}{2}b(y) \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & -x-y & -t-s+\frac{1}{2}a(x)+\frac{1}{2}b(y)+b(x) \\ 0 & 1 & -a-b \\ 0 & 0 & 1 \end{pmatrix}.$$

Finally

$$\exp X \exp Y \exp(-X) \exp(-Y) =$$

$$\begin{pmatrix} 1 & x+y & t+s+\frac{1}{2}(a(x)+b(y))+b(x) \\ 0 & 1 & a+b \\ 0 & 0 & 1 \end{pmatrix}.$$

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$$\begin{pmatrix} 1 & -x-y & -t-s+\frac{1}{2}(a(x)+b(y))+b(x) \\ 0 & 1 & -a-b \\ 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 0 & b(x)-a(y) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \exp \begin{pmatrix} 0 & 0 & b(x)-a(y) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \exp[X,Y].$$

Also we have the Campbell–Baker–Hausdorff formula:

$$\exp X \exp Y = \exp(X+Y+\frac{1}{2}[X,Y]),$$

for each X, Y in $\mathfrak{h}_G$. Indeed, by the above computation

$$\exp X \exp Y = \begin{pmatrix} 1 & x & t+\frac{1}{2}a(x) \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & y & s+\frac{1}{2}b(y) \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & x+y & t+s+\frac{1}{2}a(x)+\frac{1}{2}b(y)+b(x) \\ 0 & 1 & a+b \\ 0 & 0 & 1 \end{pmatrix},$$

and

$$\exp(X+Y+\frac{1}{2}[X,Y]) =$$

$$\exp \begin{pmatrix} 0 & x+y & t+s+\frac{1}{2}b(x)-\frac{1}{2}a(y) \\ 0 & 0 & a+b \\ 0 & 0 & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & x+y & t+s+\frac{1}{2}b(x)-\frac{1}{2}a(y)+\frac{1}{2}a(x)+\frac{1}{2}a(y)+\frac{1}{2}b(x)+\frac{1}{2}b(y) \\ 0 & 1 & a+b \\ 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & x+y & t+s+\frac{1}{2}a(x)+\frac{1}{2}b(y)+b(x) \\ 0 & 1 & a+b \\ 0 & 0 & 1 \end{pmatrix},$$

which proves the statement.

3. finite-dimensional representations

Let $\mathcal{C}(G)$ denote the space of all continuous complex-valued functions on the topological group G. Equipped with the topology of uniform convergence on compact sets and with the pointwise operations (addition and multiplication by complex numbers) $\mathcal{C}(G)$ is a locally convex topological vector space. For each (x, m, u) in H_G, we let

$$T(x,m,u)\varphi = u\, m \cdot \tau_x \varphi,$$

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whenever φ is in $\mathcal{C}(G)$ and τ_x denotes the translation operator on $\mathcal{C}(G)$ corresponding to x. In more details:

$$T(x,m,u)\varphi(g) = u\,m(g)\cdot\tau_x\varphi(g) = u\,m(g)\cdot\varphi(g+x)$$

for each g in G. Clearly, $T(x,m,u):\mathcal{C}(G)\to\mathcal{C}(G)$ is a linear operator on $\mathcal{C}(G)$. Moreover, we have

$$\begin{aligned}
[T(x,m,u)\circ T(y,n,v)]\varphi &= T(x,m,u)\big(v\,n\cdot\tau_y\varphi\big)\\
&= uvm\cdot\tau_x(n\tau_y\varphi)\\
&= uvn(x)mn\,\tau_{x+y}\varphi\\
&= T(x+y,mn,uvn(x))\varphi\\
&= T[(x,m,u)\cdot(y,n,v)]\varphi.
\end{aligned}$$

In other words, $(x,m,u)\mapsto T(x,m,u)$ is a representation of H_G on $\mathcal{C}(G)$.

In [9] we studied and described finite-dimensional representations of the classical Heisenberg group from the point of view of matrix functional equations. Now we generalize those results for our present situation. We shall need some preliminary results.

Theorem 3.1. *Let G be a commutative topological group, and d a positive integer. Every continuous function $f:G\to GL(\mathbb{C},d)$ satisfying*

$$f(g+h)=f(g)f(h) \tag{3.1}$$

for each g,h in G, has the form

$$f(g)=\mathcal{M}(g)\exp\mathcal{A}(g), \tag{3.2}$$

where $\mathcal{M}:G\to D(\mathbb{C}^*,d)$ and $\mathcal{A}:G\to L(\mathbb{C},d)$ are continuous functions satisfying

$$\mathcal{M}(g+h)=\mathcal{M}(g)\mathcal{M}(h),\ \ \mathcal{A}(g+h)=\mathcal{A}(g)+\mathcal{A}(h),\ \ \mathcal{A}(g)\mathcal{A}(h)=\mathcal{A}(h)\mathcal{A}(g), \tag{3.3}$$

for each g,h in G.

This theorem follows from [6, Theorem 0.], or from the Jordan–Chevalley decomposition theorem of matrices (see e.g. [4, p.17]). From this result we derive the following particular cases.

Theorem 3.2. *Let $\rho:\mathbb{T}\to GL(\mathbb{C},d)$ be a continuous homomorphism. Then there exists a real diagonal matrix Λ such that*

$$\rho(e^{it})=\exp i\Lambda t$$

holds for each real number t.

Proof. By the theorem above, we have $\rho(e^{it})=\mathcal{M}(u)\exp\mathcal{A}(u)$ for u in $\mathbb{T}$, where $\mathcal{M}(uv)=\mathcal{M}(u)\mathcal{M}(v)$ and $\mathcal{A}(uv)=\mathcal{A}(u)+\mathcal{A}(v)$. We define the function $m:\mathbb{R}\to D(\mathbb{C}^*,d)$ by $m(t)=\mathcal{M}(e^{it})$, and $a:\mathbb{R}\to L(\mathbb{C},d)$ by $a(t)=\mathcal{A}(e^{it})$, then m,a are 2π-periodic. As the entries of a are periodic additive functions, they must be zero, hence $\mathcal{A}$ is zero, and $\exp\mathcal{A}(u)$ is the identity matrix, for each u. The

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matrix m is diagonal, hence its diagonal elements are exponential functions, having the form $e^{\lambda t}$ with λ in $\mathbb{C}$. By 2π-periodicity, λ must be purely imaginary. □

We can write: $\rho(u) = e^{i \operatorname{Arg} u \Lambda}$.

Theorem 3.3. *Let $\rho : \mathbb{C}^* \to GL(\mathbb{C}, d)$ be a continuous homomorphism. Then there exists a matrix Γ in $L(\mathbb{C}, d)$, and a matrix Λ in $L(\mathbb{R}, d)$, such that*

$$\rho(u) = \exp(\ln |u| \cdot \Gamma + i \operatorname{Arg} u \cdot \Lambda)$$

holds for each u in $\mathbb{C}^*$.

Proof. We have $u = |u| \cdot \exp(i \operatorname{Arg} u)$, hence, by the previous theorem

$$\rho(u) = \rho(|u|) \cdot \rho(\exp i \operatorname{Arg} u) = \rho(|u|) \cdot \exp(i \operatorname{Arg} u \cdot \Lambda),$$

where Λ is in $L(\mathbb{R}, d)$. On the other hand, we can write

$$|u| = \exp \ln |u|,$$

hence

$$\rho(|u|) = \rho(\exp \ln |u|),$$

and

$$\rho(|u|) \cdot \rho(|v|) = \rho(|u| \cdot |v|) = \rho(\exp(\ln |u| + \ln |v|)),$$

or

$$(\rho \circ \exp)(t + s) = (\rho \circ \exp)(t) \cdot (\rho \circ \exp)(s),$$

for each t, s in $\mathbb{R}$. Hence $\rho \circ \exp$ is a one parameter subgroup in $GL(\mathbb{C}, d)$, consequently, by [1, Theorem I, p. 139], it has the form

$$\rho(|u|) = \rho(\exp \ln |u|) = \exp(\ln |u| \cdot \Gamma),$$

where Γ is in $GL(\mathbb{C}, d)$. □

Using these results, we have the following theorem:

Theorem 3.4. *Let G be a commutative topological group, and d a positive integer. The continuous function $F : H_G \to GL(\mathbb{C}, d)$ satisfies*

$$F(x, m, u)F(y, n, v) = F(x + y, mn, uvn(x)) \tag{3.4}$$

for each x, y in G, m, n in $\operatorname{Hom}(G, \mathbb{C}^*)$ and u, v in $\mathbb{C}^*$ if and only if there are continuous homomorphisms $M : G \to D(\mathbb{C}^*, d)$, $\mathcal{M} : \operatorname{Hom}(G, \mathbb{C}^*) \to D(\mathbb{C}^*, d)$, $A : G \to L(\mathbb{C}, d)$, $\mathcal{A} : \operatorname{Hom}(G, \mathbb{C}^*) \to L(\mathbb{C}, d)$, and there is a Λ matrix in $L(\mathbb{R}, d)$ such that

$$i \operatorname{Arg} m(x)\Lambda = [A(x), \mathcal{A}(m)] \tag{3.5}$$

and

$$F(x, m, u) = \tag{3.6}$$

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$$M(x)\mathcal{M}(m)\exp\big[A(x)+\mathcal{A}(m)+\ln|u|\cdot\Gamma+i\Lambda\big(\mathrm{Arg}\,u-\frac{1}{2}\mathrm{Arg}\,m(x)\big)\big]$$

holds for each (x,m,u) *in* H_G.

We note that equation (3.5) expresses the canonical commutation relation for the position and the momentum.

Proof. First we prove the necessity. The function

$$x\mapsto F(x,1,1)$$

is a continuous homomorphism of the commutative topological group G into the multiplicative group $GL(n,\mathbb{C})$. By Theorem 3.1, it has the form

$$F(x,1,1)=M(x)\exp A(x), \tag{3.7}$$

where $M:G\to D(\mathbb{C}^*,d)$ and $A:G\to L(\mathbb{C},d)$ are continuous homomorphisms. By the same argument, we have

$$F(0,m,1)=\mathcal{M}(m)\exp\mathcal{A}(m), \tag{3.8}$$

where $\mathcal{M}:\mathrm{Hom}\,(G,\mathbb{C}^*)\to D(\mathbb{C}^*,d)$ and $\mathcal{A}:\mathrm{Hom}\,(G,\mathbb{C}^*)\to L(\mathbb{C},d)$ are continuous functions homomorphisms.

Finally, $u\mapsto F(0,0,u)$ satisfies

$$F(0,1,u)F(0,1,v)=F(0,1,uv)$$

for each u,v in $\mathbb{C}^*$, hence it has the form

$$F(0,1,u)=\exp(\ln|u|\cdot\Gamma+i\,\mathrm{Arg}\,u\cdot\Lambda)$$

with some matrices Γ in $GL(\mathbb{C},d)$ and Λ in $L(\mathbb{R},d)$, by Theorem 3.3.

From (3.4), using the Campbell–Baker–Hausdorff formula, we derive

$$F(x,m,u)=F(0,m,1)F(x,1,1)F(0,1,u)= \tag{3.9}$$

$$\mathcal{M}(m)M(x)\exp\mathcal{A}(m)\exp A(x)\exp(\ln|u|\cdot\Gamma+i\mathrm{Arg}\,u\,\Lambda)=$$

$$\mathcal{M}(m)M(x)\exp\Big[\mathcal{A}(m)+A(x)+\frac{1}{2}\big[\mathcal{A}(m),A(x)\big]+\ln|u|\cdot\Gamma+i\mathrm{Arg}\,u\,\Lambda\Big].$$

Substitution into (3.4) gives (3.5), and putting it into (3.9) we obtain (3.6). In fact, this computation shows that (3.5) is necessary and sufficient for (3.6), which completes the proof. $\square$

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4. **Varieties**

Given the commutative topological group G the space of all continuous complex-valued functions on H_G is the space $\mathcal{C}(H_G)$, which is equipped with the topology of compact convergence. Its dual can be identified with the space $\mathcal{M}_c(H_G)$ of all compactly supported complex Borel measures on H. The space $\mathcal{M}_c(H_G)$ is equipped with the convolution:

$$\int_{H_G} f\, d(\mu * \nu) =$$

$$\int_G \int_{\mathrm{Hom}\,(G,\mathbb{C}^*)} \int_{\mathbb{C}^*} f(x+y, mn, uvn(x))\, d\mu(x,m,u)\, d\nu(y,n,v)$$

whenever f is in $\mathcal{C}(H_G)$ and μ, ν are in $\mathcal{M}_c(H_G)$. With this convolution – together with the linear operations – $\mathcal{M}_c(H_G)$ is a topological algebra. The space $\mathcal{C}(H_G)$ turns into a left module over $\mathcal{M}_c(H_G)$ under the action

$$\mu * f(x,m,u) = \int_{H_G} f(x-y, m\check{n}, uv^{-1}n(y)\check{m}(y))\, d\mu(y,n,v)$$

corresponding to the left translation on H_G. Closed submodules of this module will be called *varieties*.

Proposition 4.1. *The closed subspace of $\mathcal{C}(H_G)$ is a variety, if and only if it is closed under left translation.*

Proof. Suppose that V is a variety in $\mathcal{C}(H)$, and f is in V, (y,n,v) is in H_G. If $\delta_{(y,n,v)^{-1}}$ denotes the point mass supported at the singleton $(y,n,v)^{-1}$, then we have

$$\delta_{(y,n,v)^{-1}} * f(x,m,u) = \int_{H_G} f(x-z, m\check{k}, uw^{-1}k(z)\check{m}(z))\, d\delta_{(y,n,v)^{-1}}(z,k,w) =$$

$$f(x+y, mn, uvm(y)) = f\big((y,n,v)\cdot(x,m,u)\big),$$

which is the left translation of f by (y,n,v). As V is a variety, the function $\delta_{(y,n,v)^{-1}} * f$ is in V, hence V is left translation invariant. The converse statement follows from the fact, that point masses span a weak*-dense subspace in $\mathcal{M}_c(H_G)$, hence if convolution with point mass from the left leaves V invariant, then the same holds for their finite linear combinations and their weak*-limits as well, which implies that V is a variety. □

As an illustration, we describe all one-dimensional varieties in $\mathcal{C}(H_G)$. If V is one-dimensional, then let f be a nonzero function in V. Then for each (y,n,v) in H_G there exists a complex number $\lambda(y,n,v)$ such that

$$f(x+y, mn, uvm(y)) = \lambda(y,n,v) f(x,m,u) \tag{4.1}$$

holds for each (x,m,u). Clearly, $f(0,1,1) \neq 0$, hence we have that $c\lambda = f$ for some nonzero complex number c. In particular, λ is continuous. It follows that λ maps H_G into $\mathbb{C}^*$ and

$$\lambda(x+y, mn, uvm(y)) = \lambda(y,n,v)\lambda(x,m,u) \tag{4.2}$$

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holds for each (x, m, u) and (y, n, v). Putting $y = 0$, $v = 1$, $m = 1$, we get

$$\lambda(x, n, u) = \lambda(0, n, 1)\lambda(x, 1, u).$$

On the other hand, from (4.2) with $y = 0$, $m = n = 1$ and $v = 1$ we infer

$$\lambda(x, 1, u) = \lambda(x, 1, 1)\lambda(0, 1, u)$$

which implies that

$$\lambda(x, m, u) = \lambda(x, 1, 1)\lambda(0, m, 1)\lambda(0, 1, u),$$

and here $x \mapsto \lambda(x, 1, 1)$ is an exponential on G, $m \mapsto \lambda(0, m, 1)$ is an exponential on $\mathrm{Hom}\,(G, \mathbb{C}^*)$, and $u \mapsto \lambda(0, 1, u)$ is an exponential on $\mathbb{C}^*$. Nevertheless, these exponentials are not arbitrary: in fact, we have that $\lambda(0, 1, m(x)) = 1$ for each x in G and exponential m on G. Indeed

$$\lambda(0, 1, m(x)) = \lambda(x, \check{m}, 1)\lambda(-x, m, 1) = \lambda(0, \check{m}, 1)\lambda(x, 1, 1)\lambda(-x, m, 1) =$$

$$\lambda(0, \check{m}, 1)\lambda(-x, m, 1)\lambda(x, 1, 1) = \lambda(-x, 1, 1)\lambda(x, 1, 1) = 1.$$

Finally, we arrive at $\lambda(x, y, t) = e^{\mu x + \nu y}$. It is easy to check that indeed, such functions span one-dimensional varieties in $\mathcal{C}(H)$ for any choice of complex numbers μ, ν, hence we have proved the following result:

Proposition 4.2. *A variety in $\mathcal{C}(H_G)$ is one-dimensional if and only if it is spanned by a function of the form $(x, m, u) \mapsto M(x)\mathcal{M}(m)\rho(u)$ with some exponentials $M : G \to \mathbb{C}^*$, $\mathcal{M} : \mathrm{Hom}\,(G, \mathbb{C}^*) \to \mathbb{C}^*$ and $\rho : \mathbb{C}^* \to \mathbb{C}^*$ satisfying $\rho(m(x)) = 1$ for each x in G and exponential $m : G \to \mathbb{C}^*$.*

The problem of describing finite-dimensional varieties over H_G is equivalent to the study offinite-dimensional representations of H_G. On commutative topological groups the representing functions of finite dimensional representations are the so-called *exponential polynomials*: they are those continuous complex-valued functions, which are the elements of the function algebra generated by continuous homomorphisms into the additive group of complex numbers, and into the multiplicative group of nonzero complex numbers. In fact, an exponential polynomial is a polynomial of such homomorphisms. The natural question arises also in our present non-commutative situation: is it true that the representing functions, that is the matrix elements of finite-dimensional representations of H_G are have the similar form? Can they be described using polynomials of continuous homomorphisms of G, $\mathrm{Hom}\,(G, \mathbb{C}^*)$ and $\mathbb{C}^*$?

In what follows we shall call a continuous complex-valued function on H_G an *exponential polynomial*, if it is a polynomial of continuous homomorphisms of G, $\mathrm{Hom}\,(G, \mathbb{C}^*)$ and $\mathbb{C}^*$ into the additive group of complex numbers and the multiplicative group of nonzero complex numbers.

Theorem 4.3. *Every finite-dimensional variety on the generalized Heisenberg group consists of exponential polynomials.*

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Proof. We recall the Jordan–Chevalley decomposition theorem (see e.g. [4, Section 4.2, p.17], which asserts that any endomorphism X of a finite-dimensional vector space can be decomposed as the sum of a semisimple X_s (diagonizable) and a nilpotent X_u endomorphism: $X = X_s \cdot X_n$, which commute.

If V is a finite-dimensional variety on H_G, then the corresponding finite-dimensional representation of H_G on V induces a continuous function $F : H_G \to GL(\mathbb{C}, d)$ which satisfies the functional equation (3.4) with the additional condition $F(0, 1, 1) = I_d$, the $d \times d$ identity matrix. We have seen in Theorem 3.4 that F has the following form:

$$F(x, m, u) = \big(\mathcal{M}(m) \exp \mathcal{A}(m)\big) \cdot \big(M(x) \exp A(x)\big) \cdot \rho(u), \tag{4.3}$$

where $\mathcal{M} : \mathrm{Hom}\,(G, \mathbb{C}^*) \to D(\mathbb{C}^*, d)$, $\mathcal{A} : \mathrm{Hom}\,(G, \mathbb{C}^*) \to L(\mathbb{C}, d)$, further $M : G \to D(\mathbb{C}^*, d)$, $A : G \to L(\mathbb{C}, d)$ and $\rho : \mathbb{C}^* \to L(\mathbb{C}, d)$ are continuous homomorphisms. To prove our statement it is enough to show that the matrix elements of the matrices $\mathcal{M}(m) \exp \mathcal{A}(m)$ and $M(x) \exp A(x)$ are exponential polynomials on the corresponding (commutative) groups, as the elements of the matrix $\rho(u)$ are obviously exponential polynomials on $\mathbb{C}^*$. First we consider $M(x) \exp A(x)$. The matrices $M(x)$ for each x in G are diagonal, and all diagonal elements are exponentials on G. The matrices $A(x)$ for x in G commute, by (3.3), and they can be written in the form $A(x) = S^{-1} A_s(x) S + A_n(x)$, where $S^{-1} A_s(x) S$ is diagonal and $A_n(x)$ is nilpotent, further $A_s(x)$ and $A_n(x)$ commute. Then we have

$$S^{-1} M(x) \exp A(x) S = M(x) S^{-1} \exp(A_s(x) + A_n(x)) S =$$
$$M(x) \exp S^{-1} A_s(x) S \cdot \exp S^{-1} A_n(x) S.$$

As $A_n(x)$ is nilpotent, so is $S^{-1} A_n(x) S$, and we have $(S^{-1} A_n(x) S)^N = 0$ for some N and for each x in G. It follows that $S^{-1} \exp A_n(x) S = \exp S^{-1} A_n S$ is a polynomial of the matrix elements of $S^{-1} A_n S$, which are additive functions. On the other hand, the matrix elements of the diagonal matrix $S^{-1} A_s(x) S$ are additive functions as well, hence the matrix elements of the diagonal matrix $S^{-1} \exp A_s(x) S$ are exponentials on G. We conclude that the matrix elements of $M(x) \exp A(x)$ are exponential polynomials on G.

We can apply the same argument for $\mathcal{M}(m) \exp \mathcal{A}(m)$, on the topological group $\mathrm{Hom}\,(G, \mathbb{C}^*)$. We arrive at the conclusion that all matrix elements of are exponential polynomials, and the theorem is proved. □

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