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ON THE MULTIPLIERS OF THE FIGÀ-TALAMANCA HERZ ALGEBRA

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Dedicated to Professor A. T.-M. Lau

Abstract. Let G be a locally compact group and $p, q \in \mathbb{R}$ with $p > 1$, $p \neq 2$, and q between 2 and p (if $p < 2$ then $p < q < 2$, if $p > 2$ then $2 < q < p$). The main result of the paper is that $A_q(G)$ multiplies $A_p(G)$, more precisely we show that the Banach algebra $A_p(G)$ is a Banach module on $A_q(G)$.

Let G be a locally compact group, in [1] an entirely self-contained proof of the following result was given (section 8.3 Theorem 8 p. 158): for every $1 < p < \infty$, $A_p(G)$ is a Banach module over $A_2(G)$, i.e. for all $u \in A_2(G)$ and $v \in A_p(G)$, we have $uv \in A_p(G)$, and $\|uv\|_{A_p} \leq \|u\|_{A_2}\|v\|_{A_p}$. Now let $p, q \in \mathbb{R}$, with $p > 1$, $p \neq 2$, and q between 2 and p. We show (Theorem 1.10) that for every $u \in A_q(G)$, and $v \in A_p(G)$, we have $uv \in A_p(G)$ and $\|uv\|_{A_p} \leq \|u\|_{A_q}\|v\|_{A_p}$. We recall some definitions. For φ a map of G into $\mathbb{C}$, we set $\check{\varphi}(x) = \varphi(x^{-1})$, ${}_a\varphi(x) = \varphi(ax)$, and $\varphi_a(x) = \varphi(xa)$, $a, x \in G$. For $1 < p < \infty$, $L^p(G)$ is a L^p-space, concerning a left Haar measure on G. Let $\mathcal{L}(L^p(G))$, be the Banach algebra of all bounded linear operators on $L^p(G)$, the operator norm of an operator T is denoted by $|\!|\!|T|\!|\!|_p$. An element T of $\mathcal{L}(L^p(G))$, is said to be a p-convolution operator (written $T \in CV_p(G)$) if $T({}_a\varphi) = {}_aT(\varphi)$, for every $a \in G$, and $\varphi \in L^p(G)$. We denote the set of all continuous complex valued functions on G with compact support, by $C_{00}(G)$. Let μ be a Radon measure on G, we put $\check{\mu}(\varphi) = \mu(\check{\varphi})$. Suppose that μ is a bounded Radon measure on G ($\mu \in M^1(G)$), then there is a unique $S \in \mathcal{L}(L^p(G))$, such that

$$S[\varphi] = \left[\varphi * \left(\Delta_G^{1/p'}\check{\mu}\right)\right]$$

for every $\varphi \in C_{00}(G)$.

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(What is p', in the above definition? This symbol is repeated throughout the paper and is probably the dual scalar of p Please explain it once.)

The operator S is denoted by $\lambda_G^p(\mu)$.We have $\lambda_G^p(\mu) \in CV_p(G)$, and $\left\|\left|\lambda_G^p(\mu)\right|\right\|_p \leq \|\mu\|$. We also have $\lambda_G^p(\delta_a)\varphi = \varphi_a \Delta_G(a)^{1/p}$. In [1], the author proved that, for every $1 < p < \infty$ and for every amenable locally compact group G, the following deep inequality holds: $\left\|\left|\lambda_G^2(\mu)\right|\right\|_2 \leq \left\|\left|\lambda_G^p(\mu)\right|\right\|_p$, for all $\mu \in M^1(G)$. Even for finite abelian groups, this result is not trivial.

In this paper, we show (Corollary 2.2) that for every amenable locally compact group G, for $p, q \in \mathbb{R}$ with $p > 1$, $p \neq 2$, and q between 2 and p, we have $\left\|\left|\lambda_G^q(\mu)\right|\right\|_q \leq \left\|\left|\lambda_G^p(\mu)\right|\right\|_p$ for every $\mu \in M^1(G)$.

Let G be a locally compact group, and $1 < p < \infty$. We denote by $\mathcal{A}_p(G)$ the set of all pairs $((k_n)_{n=1}^\infty, (l_n)_{n=1}^\infty)$ where $(k_n)_{n=1}^\infty$ is a sequence of $\mathcal{L}^p(G)$ and $(l_n)_{n=1}^\infty$ is a sequence of $\mathcal{L}^{p'}(G)$ with

$$\sum_{n=1}^{\infty} N_p(k_n) N_{p'}(l_n) < \infty.$$

The topology on $\mathcal{L}(L^p(G))$, associated to the family of seminorms

$$T \mapsto \left| \sum_{n=1}^{\infty} < T[k_n], [l_n] > \right|$$

with $((k_n)_{n=1}^\infty, (l_n)_{n=1}^\infty) \in \mathcal{A}_p(G)$, is called the ultraweak topology. The ultraweak closure of $\lambda_G^p(M^1(G))$ in $\mathcal{L}(L^p(G))$ is denoted by $PM_p(G)$, and elements of $PM_p(G)$, are called p-pseudomeasures. The space $PM_p(G)$ can be identified as the dual of a Banach algebra $A_p(G)$ of continuous functions on G.

We denote by $A_p(G)$, the set:

$$\Big\{ u : G \to \mathbb{C} : \text{there is } ((k_n)_{n=1}^\infty, (l_n)_{n=1}^\infty)) \in \mathcal{A}_p(G) \text{ such that}$$

$$u(x) = \sum_{n=1}^{\infty} \left(\overline{k_n} * \check{l_n}\right)(x) \text{ for every } x \in G \Big\}.$$

For $u \in A_p(G)$, we put

$$\|u\|_{A_p} := \inf \left\{ \sum_{n=1}^{\infty} N_p(k_n) N_{p'}(l_n) : ((k_n)_{n=1}^\infty, (k_n)_{n=1}^\infty) \in \mathcal{A}_p(G) \right.$$

$$\left. \text{such that } u = \sum_{n=1}^{\infty} \overline{k_n} * \check{l_n} \right\}.$$

Under the pointwise multiplication, the set $A_p(G)$ is a Banach algebra, called the Figà-Talamanca Herz of G. We have $A_p(G) \subset C_0(G)$ and for every $u \in A_p(G)$, the following inequality holds: $\|u\|_\infty \leq \|u\|_{A_p}$. Moreover the space $A_p(G) \cap C_{00}(G)$, is dense in $A_p(G)$.

For every $T \in PM_p(G)$, we put

$$\Psi_G^p(T)(u) = \sum_{n=1}^{\infty} \overline{< T[\tau_p k_n], [\tau_{p'} l_n] >}$$

for every $u \in A_p(G)$ and for every $((k_n)_{n=1}^\infty, (k_n)_{n=1}^\infty) \in \mathcal{A}_p(G)$, such that

$$u = \sum_{n=1}^{\infty} \overline{k_n} * \check{l_n},$$

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where $\tau_p\varphi(x) = \varphi(x^{-1})\Delta_G(x^{-1})^{1/p}$. Then Ψ_G^p is a conjugate linear isometry of $PM_p(G)$ onto $A_p(G)'$ with the following properties:

(1) $\Psi_G^p(\lambda_G^p(\tilde{\mu})) = \mu$, for every $\mu \in M^1(G)$, where $\tilde{\varphi}(x) = \overline{\varphi(x^{-1})}$ and $\tilde{\mu}(\varphi) = \overline{\mu(\tilde{\varphi})}$.

(2) Ψ_G^p is a homeomorphism of $PM_p(G)$ with the ultraweak topology, onto $A_p(G)'$, with the weak topology $\sigma({A_p}', A_p)$. We will use the fact (see [1, Chapter 5]) that $CV_p(G)$ carries a natural structure of left normed $A_p(G)$-module $(u,T) \mapsto uT$. We shortly recall the definition of uT: for $k \in \mathcal{L}^p(G)$, $l \in \mathcal{L}^{p'}(G)$ and $T \in CV_p(G)$ there is a unique linear bounded operator of $L^p(G)$, denoted $(\overline{k} * \check{l})T$, such that

$$\left\langle \left((\overline{k} * \check{l})T\right)[\varphi], [\psi] \right\rangle = \int_G \left\langle T[{}_{t^{-1}}(\check{k})\varphi], [{}_{t^{-1}}(\check{l})\psi] \right\rangle dt$$

for every $\varphi, \psi \in C_{00}(G)$. Then, for $u \in A_p(G)$, we put $uT = \sum_{n=1}^{\infty} (\overline{k_n} * \check{l_n})T$ for every $((k_n)_{n=1}^{\infty}, (l_n)_{n=1}^{\infty}) \in \mathcal{A}_p(G)$ such that $u = \sum_{n=1}^{\infty} \overline{k_n} * \check{l_n}$.

1. The general case

In this paragraph, G is a arbitrary locally compact group, G is not assumed to be amenable.

Theorem 1.1. *Let G be a locally compact group, $1 < p < \infty, u \in A_p(G), T \in CV_p(G)$ and $\varphi \in L^p(G) \cap L^2(G)$.*

(1) $\tau_p(uT)\tau_p\varphi \in L^2(G)$,

(2) $\left\|\tau_p(uT)\tau_p\varphi\right\|_2 \le \|u\|_{A_p} \left\|T\right\|_p \|\varphi\|_2$.

Proof. See [1] Section 8.3 Theorem 1 p. 156. □

Proposition 1.2. *Let G be a locally compact group, $p, q \in \mathbb{R}$ with $p > 1, p \neq 2$, and q between p and 2. We denote by $\mathcal{S}$ the set $\{[r] : r \in \mathcal{L}^1(G), r$ is a step function$\}$. Then for every $1 < p < \infty$, $u \in A_p(G)$, $T \in CV_p(G)$, and all $\varphi \in \mathcal{S}$:*

(1) $\tau_p(uT)\tau_p\varphi \in L^q(G)$,

(2) $\left\|\tau_p(uT)\tau_p\varphi\right\|_q \le \|u\|_{A_p} \left\|T\right\|_p \|\varphi\|_q$.

Proof. Case(I) We prove the statement, if $p < 2$.

Put

$$t = \frac{2(q-p)}{q(2-p)}.$$

We have $0 < t < 1$, and

$$\frac{1}{q} = \frac{1-t}{p} + \frac{t}{2}.$$

According to Riesz-Thorin and Theorem 1.1 $\tau_p(uT)\tau_p\varphi \in L^q(G)$, and

$$\left\|\tau_p(uT)\tau_p\varphi\right\|_q \le \left\|uT\right\|_p^{1-t} \left(\|u\|_{A_p} \left\|T\right\|_p\right)^t \|\varphi\|_q$$

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but

$$\left\| uT \right\|_p^{1-t} \left(\|u\|_{A_p} \left\| T \right\|_p \right)^t \|\varphi\|_q \leq \|u\|_{A_p}^{1-t} \left\| T \right\|_p^{1-t} \|u\|_{A_p}^t \left\| T \right\|_p^t \|\varphi\|_q = \|u\|_{A_p} \left\| T \right\|_p \|\varphi\|_q .$$

Case(II) We prove the statement, if $p > 2$.

Put

$$t = \frac{(2-q)p}{(2-p)q}.$$

We have $0 < t < 1$ and

$$\frac{1}{q} = \frac{1-t}{2} + \frac{t}{p}.$$

As in (I) $\tau_p(uT)\tau_p\varphi \in L^q(G)$ and

$$\left\| \tau_p(uT)\tau_p\varphi \right\|_q \leq \left(\|u\|_{A_p} \left\| T \right\|_p \right)^{1-t} \left\| uT \right\|_p^t \|\varphi\|_q$$

and consequently $\left\| \tau_p(uT)\tau_p\varphi \right\|_q \leq \|u\|_{A_p} \left\| T \right\|_p \|\varphi\|_q$.

□

Theorem 1.3. *Let G be a locally compact group, $p, q \in \mathbb{R}$ with $p > 1$, $p \neq 2$ and q between p and 2. Then for every $u \in A_p(G)$, for every $T \in CV_p(G)$ and every $\varphi \in L^p(G) \cap L^q(G)$ we have:*

(1) $\tau_p(uT)\tau_p\varphi \in L^q(G)$,
(2) $\left\| \tau_p(uT)\tau_p\varphi \right\|_q \leq \|u\|_{A_p} \left\| T \right\|_p \|\varphi\|_q$.

Definition 1.4. *Let G be a locally compact group, $p, q \in \mathbb{R}$ with $p > 1$, $p \neq 2$ and q between p and 2. We denote by $E_{p,q}$, the set of all $T \in CV_p(G)$, such that :*

(1) $\tau_p T \tau_p \varphi \in L^q(G)$, *for every* $\varphi \in L^p(G) \cap L^q(G)$,
(2) *There is* $C > 0$, *such that* $\left\| \tau_p T \tau_p \varphi \right\|_q \leq C \|\varphi\|_q$, *for every* $\varphi \in L^p(G) \cap L^q(G)$.

Proposition 1.5. *Let G be a locally compact group, $p, q \in \mathbb{R}$ with $p > 1, p \neq 2$ and q between p and 2. The set $E_{p,q}$ is a subalgebra of $CV_p(G)$.*

Proof. Let $S, T \in E_{p,q}$, C, C' the constants of Definition 1.4, and $\varphi \in L^p(G) \cap L^q(G)$. From $\tau_p T \tau_p \varphi \in L^p(G) \cap L^q(G)$ it follows $(\tau_p S \tau_p)(\tau_p T \tau_p \varphi) \in L^p(G) \cap L^q(G)$, and consequently $\tau_p S T \tau_p \varphi \in L^p(G) \cap L^q(G)$. Moreover

$$\|\tau_p S T \tau_p \varphi\|_q = \|\tau_p S \tau_p (\tau_p T \tau_p \varphi)\|_q \leq C \|\tau_p T \tau_p \varphi\|_q \leq C C' \|\varphi\|_q$$

this implies that $ST \in E_{p,q}$. □

Proposition 1.6. *Let G be a locally compact group, $p, q \in \mathbb{R}$ with $p > 1, p \neq 2$ and q between p and 2. For every $\mu \in M^1(G)$, we have $\lambda_G^p(\mu) \in E_{p,q}$, and for every $\varphi \in L^p(G) \cap L^q(G)$, $\tau_p \lambda_G^p(\mu) \tau_p \varphi = \rho_G^q(\mu)\varphi$.*

Proof. For every $\varphi \in C_{00}(G)$ we have $\tau_p\big(\tau_p\varphi * \Delta_G^{1/p'}\check{\mu}\big) = \varphi * \mu$. Let $\varphi \in L^p(G) \cap L^q(G)$ we obtain $\tau_p \lambda_G^p(\mu) \tau_p \varphi = \rho_G^q(\mu)\varphi$. But $\rho_G^q(\mu)\varphi$ belongs to $L^q(G)$ and therefore $\tau_p \lambda_G^p(\mu) \tau_p \varphi \in L^q(G)$. Finally the inequality $\|\rho_G^q(\mu)\varphi\|_q \leq \|\mu\| \|\varphi\|_q$ implies $\|\tau_p \lambda_G^p(\mu) \tau_p \varphi\|_q \leq \|\mu\| \|\varphi\|_q$.

□

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Proposition 1.7. *Let G be a locally compact group, $p, q \in \mathbb{R}$ with $p > 1, p \neq 2$ and q between p and 2. For every $T \in E_{p,q}$, there is a unique $S \in \mathcal{L}(L^q(G))$, such that $S\varphi = \tau_p T \tau_p \varphi$, for every $\varphi \in L^p(G) \cap L^q(G)$. We have $\tau_q S \tau_q \in CV_q(G)$.*

Definition 1.8. *Let G be a locally compact group, $p, q \in \mathbb{R}$ with $p > 1, p \neq 2$ and q between p and 2, $T \in E_{p,q}$, and S as in Proposition 1.7. We define $\alpha_{p,q}(T) = \tau_q S \tau_q$.*

Theorem 1.9. *Let G be a locally compact group, $p, q \in \mathbb{R}$ with $p > 1, p \neq 2$ and q between p and 2. Then:*

(1) *$\alpha_{p,q}$ is a monomorphism of the algebra $E_{p,q}$ into $CV_q(G)$.*
(2) *$\alpha_{p,q}(\lambda_G^p(\mu)) = \lambda_G^q(\mu)$ for every $\mu \in M^1(G)$.*
(3) *for every $u \in A_p(G)$ and for every $T \in CV_p(G)$ we have $uT \in E_{p,q}$ and $\|\alpha_{p,q}(uT)\|_q \leq \|u\|_{A_p} \|T\|_p$.*
(4) *for every $T \in PM_p(G) \cap E_{p,q}$ we have $\alpha_{p,q}(T) \in PM_q(G)$.*

Proof. (1) Let $T, T' \in E_{p,q}$ $C, C' > 0$ $S, S' \in \mathcal{L}(L^q(G))$ such that $\|\tau_p T \tau_p \varphi\|_q \leq C\|\varphi\|_q$, $\|\tau_p T' \tau_p \varphi\|_q \leq C'\|\varphi\|_q$, $S\varphi = \tau_p T \tau_p \varphi$, and $S'\varphi = \tau_p T' \tau_p \varphi$ for every $\varphi \in L^p(G) \cap L^q(G)$. We have $\alpha_{p,q}(T) = \tau_q S \tau_q$ and $\alpha_{p,q}(T') = \tau_q S' \tau_q$.

(I)$_1$ $\alpha_{p,q}(T + T') = \alpha_{p,q}(T) + \alpha_{p,q}(T')$.

There is $S'' \in \mathcal{L}(L^q(G))$ such that $\tau_p(T + T')\tau_p\varphi = S''\varphi$ for every $\varphi \in L^p(G) \cap L^q(G)$. We have $\alpha_{p,q}(T+T') = \tau_q S'' \tau_q$. For $\varphi \in L^p(G) \cap L^q(G)$ we get $\tau_p(T+T')\tau_p\varphi = \tau_p(T\tau_p\varphi + T'\tau_p\varphi) = \tau_p T \tau_p \varphi + \tau_p T' \tau_p \varphi = S\varphi + S'\varphi = S''\varphi$. Consequently $S + S' = S''$.Thus $\alpha_{p,q}(T) + \alpha_{p,q}(T') = \alpha_{p,q}(T + T')$.

(I)$_2$ $\alpha_{p,q}(\gamma T) = \gamma\, \alpha_{p,q}(T)$ for $\gamma \in \mathbb{C}$ and $\alpha_{p,q}(TT') = \alpha_{p,q}(T)\alpha_{p,q}(T')$.

The proof of (I)$_2$, is similar to the one of (I)$_1$.

(I)$_3$ If $\alpha_{p,q}(T) = 0$ then $T = 0$.

We have $\tau_q S \tau_q = 0$, this implies $S = 0$. For every $\varphi \in L^p(G) \cap L^q(G)$ $S\varphi = \tau_p T \tau_p \varphi$, consequently $\tau_p T \tau_p \varphi = 0$, hence $T\tau_p\varphi = 0$. Consider $r \in C_{00}(G)$, we have $T[r] = T\tau_p\tau_p[r]$, taking into account that $\tau_p[r] \in L^p(G) \cap L^q(G)$ we get $T\tau_p\tau_p[r] = 0$ i.e $T[r] = 0$ and finally $T = 0$.

(2) According to Proposition 1.6 $\lambda_G^p(\mu) \in E_{p,q}$. Let $S \in \mathcal{L}(L^q(G))$ such that $S\varphi = \tau_p \lambda_G^p(\mu) \tau_p \varphi$ for every $\varphi \in L^p(G) \cap L^q(G)$, we have $\alpha_{p,q}(\lambda_G^p(\mu)) = \tau_q S \tau_q$. By Proposition 3 for every $\varphi \in L^p(G) \cap L^q(G)$ we have $\tau_p \lambda_G^p(\mu) \tau_p \varphi = \rho_G^q(\mu)\varphi$. Consequently $S\varphi = \rho_G^q(\mu)\varphi$ every $\varphi \in L^p(G) \cap L^q(G)$, this implies $S = \rho_G^q(\mu)$. We obtain $\tau_q S \tau_q = \tau_q \rho_G^q(\mu) \tau_q$. It is straightforward to verify that $\tau_q \rho_G^q(\mu) \tau_q = \lambda_G^q(\mu)$, we finally conclude that $\alpha_{p,q}(\lambda_G^p(\mu)) = \lambda_G^q(\mu)$.

(3) According to Theorem 1.3 $uT \in E_{p,q}$. Let $S \in \mathcal{L}(L^q(G))$ such that $S\varphi = \tau_p \lambda_G^p(\mu) \tau_p \varphi$ for every $\varphi \in L^p(G) \cap L^q(G)$. By Theorem 1.3 again for every $\varphi \in L^p(G) \cap L^q(G)$ $\|S\varphi\|_q \leq \|u\|_{A_p} \|T\|_p \|\varphi\|_q$. Consequently, for every $\varphi \in L^q(G)$ we have $\|S\varphi\|_q \leq \|u\|_{A_p} \|T\|_p \|\varphi\|_q$. It follows that for every $\varphi \in L^q(G)$ we have $\|\tau_q S \tau_q \varphi\|_q = \|S\tau_q\varphi\|_q \leq u\|_{A_p} \|T\|_p \|\tau_q\varphi\|_q = \|u\|_{A_p} \|T\|_p \|\varphi\|_q$, thus $\|\tau_q S \tau_q\|_q \leq \|T\|_p \|u\|_{A_p}$. We finally have $\|\alpha_{p,q}(T)\|_q = \|\tau_q S \tau_q\|_q \leq \|T\|_p \|u\|_{A_p}$.

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(4) Let $\big((k_n)_{n=1}^\infty, (l_n)_{n=1}^\infty\big) \in \mathcal{A}_q(G)$ such that $k_n, l_n \in C_{00}(G)$ for every $n \geq 1$ and such that

$$\sum_{n=1}^\infty \overline{k_n} * \check{l_n} = 0.$$

We have $\big\langle \alpha_{p,q}(T)[\tau_q k_n], [\tau_{q'} l_n]\big\rangle_{L^q,L^{q'}} = \big\langle T[\tau_p k_n], [\tau_{p'} l_n]\big\rangle_{L^p,L^{p'}}$ for every $n \geq 1$. By [1, Lemma 5 Section 4.1 page 48]

$$\sum_{n=1}^\infty \big\langle T[\tau_p k_n], [\tau_{p'} l_n]\big\rangle_{L^p,L^{p'}} = 0$$

and therefore

$$\sum_{n=1}^\infty \big\langle \alpha_{p,q}(T)[\tau_q k_n], [\tau_{q'} l_n]\big\rangle_{L^q,L^{q'}} = 0.$$

The Corollary 8 of [1] Section 4.1 page 52 implies that $\alpha_{p,q}(T) \in PM_q(G)$.

□

The following theorem is the main result of the paper.

Theorem 1.10. *Let G be a locally compact group, $p, q \in \mathbb{R}$ with $p > 1, p \neq 2$ and q between p and 2. Let $u \in A_q(G)$ and $v \in A_p(G)$. Then:*

(1) *$uv \in A_p(G)$, and $\|uv\|_{A_p} \leq \|u\|_{A_q}\|u\|_{A_p}$, i.e $A_p(G)$ is a Banach module on the Banach algebra $A_q(G)$,*

(2) *for every $T \in PM_p(G)$ we have*

$$\big\langle uv, T\big\rangle_{A_p,PM_p} = \big\langle u, \alpha_{p,q}(vT)\big\rangle_{A_q,PM_q}$$

(according to Theorem 1.9 $\alpha_{p,q}(vT) \in PM_q(G)$).

Proof. (I)

We define the linear map ω (depending on u) of $A_p(G)'$ into $A_p(G)''$.

There is $\big((k_n)_{n=1}^\infty, (l_n)_{n=1}^\infty\big) \in \mathcal{A}_q(G)$ such that

$$u = \sum_{n=1}^\infty \overline{k_n} * \check{l_n}.$$

Let F be an arbitrary element of $A_p(G)'$. We put

$$\omega(F) = \sum_{n=1}^\infty \overline{\big\langle \alpha_{p,q}(v(\Psi_G^p)^{-1}(F))[\tau_q k_n], [\tau_{q'} l_n]\big\rangle}_{L^q,L^{q'}}.$$

It is easy to verify that for $\gamma \in \mathbb{C}$ $\omega(\gamma F) = \gamma\omega(F)$ and that $\omega(F + F') = \omega(F) + \omega(F')$ for every $F, F' \in A_p(G)'$. Moreover

$$|\omega(F)| \leq \sum_{n=1}^\infty \big|\!\big|\!\big|\alpha_{p,q}(v(\Psi_G^p)^{-1}(F))\big|\!\big|\!\big|_q N_q(k_n) N_{q'}(l_n)$$

according to Theorem 1.10 part 3, the last expression is not larger than

$$\sum_{n=1}^\infty \|v\|_{A_p} \big|\!\big|\!\big|(\Phi_G^p)^{-1}(F)\big|\!\big|\!\big|_p N_q(k_n) N_{q'}(l_n)$$

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$$= \|v\|_{A_p}\|F\|_{A'_p}\sum_{n=1}^{\infty}N_q(k_n)N_{q'}(l_n)$$

this implies that $\omega(F) \in A_p(G)''$.

(II) Let F be an element of $A_p(G)'$ and $(F_i)_{i\in I}$ a net of $A_p(G)'$, such that $\lim F_i = F$, with the topology $\sigma(A'_p, A_p)$. We assume the existence of $C > 0$, with $\|F_i\|_{A'_p} \leq C$, for every i. Then $\lim \omega(F_i) = \omega(F)$.

By [1, Theorem 6 Section 4.1 page 49] $\lim(\Psi^p_G)^{-1}(F_i) = (\Psi^p_G)^{-1}(F)$, with the ultraweak topology on $\mathcal{L}(L^p(G))$. We also have $\left|\!\left|\!\left|(\Psi^p_G)^{-1}(F_i)\right|\!\right|\!\right|_p \leq C$ for every $i \in I$.

$(\mathrm{II})_1$ Let $(S_i)_{i\in I}$ be a net of $PM_p(G)$ and $S \in PM_p(G)$. Suppose that $\lim S_i = S$, with the ultraweak topology on $\mathcal{L}(L^p(G))$. Then for every $v \in A_p(G)$, we have $\lim vS_i = vS$ with the ultraweak topology on $\mathcal{L}(L^p(G))$.

For every $a \in A_p(G)$ we have $\lim < a, S_i >_{A_p,PM_p} = < a, S >_{A_p,PM_p}$. In particular $\lim < av, S_i >_{A_p,PM_p} = < av, S >_{A_p,PM_p}$ hence $\lim < a, vS_i >_{A_p,PM_p} = < a, vS >_{A_p,PM_p}$ and consequently $\lim vS_i = vS$ for the ultraweak topology on $\mathcal{L}(L^p(G))$.

$(\mathrm{II})_2$By $(\mathrm{II})_1$, $\lim v(\Psi^p_G)^{-1}(F_i) = v(\Psi^p_G)^{-1}(F)$ for the ultraweak topology on $\mathcal{L}(L^p(G))$. Moreover for every $i \in I$,

$$\left|\!\left|\!\left|v(\Psi^p_G)^{-1}(F_i)\right|\!\right|\!\right|_p \leq \|v\|_{A_p}\left|\!\left|\!\left|(\Psi^p_G)^{-1}(F_i)\right|\!\right|\!\right|_p \leq \|v\|_{A_p}C.$$

Let $r, s \in C_{00}(G)$. We have

$$\lim\left\langle v(\Psi^p_G)^{-1}(F_i)[\tau_p r], [\tau_{p'}s]\right\rangle_{L^p,L^{p'}} = \left\langle v(\Psi^p_G)^{-1}(F)[\tau_p r], [\tau_{p'}s]\right\rangle_{L^p,L^{p'}}.$$

Taking into account that, $[r] \in L^p(G) \cap L^q(G)$, and Theorem 1.3, we have

$$\tau_q\alpha_{p,q}\big(v(\Psi^p_G)^{-1}(F)\big)\tau_q[r] = \tau_p v(\Psi^p_G)^{-1}(F)\tau_p[r]$$

and

$$\tau_q\alpha_{p,q}\big(v(\Psi^p_G)^{-1}(F_i)\big)\tau_q[r] = \tau_p v(\Psi^p_G)^{-1}(F_i)\tau_p[r].$$

Consequently,

$$\left\langle v(\Psi^p_G)^{-1}(F)[\tau_p r], [\tau_{p'}s]\right\rangle_{L^p,L^{p'}} = \left\langle \alpha_{p,q}\big(v(\Psi^p_G)^{-1}(F)\big)[\tau_q r], [\tau_{q'}s]\right\rangle_{L^q,L^{q'}}$$

and

$$\left\langle v(\Psi^p_G)^{-1}(F_i)[\tau_p r], [\tau_{p'}s]\right\rangle_{L^p,L^{p'}} = \left\langle \alpha_{p,q}\big(v(\Psi^p_G)^{-1}(F_i)\big)[\tau_q r], [\tau_{q'}s]\right\rangle_{L^q,L^{q'}}.$$

But for every $i \in I$,

$$\left|\!\left|\!\left|\alpha_{p,q}\big(v(\Psi^p_G)^{-1}(F_i)\big)\right|\!\right|\!\right|_q \leq \|v\|_{A_p}\left|\!\left|\!\left|(\Psi^p_G)^{-1}(F_i)\right|\!\right|\!\right|_p \leq \|v\|_{A_p}C.$$

It follows that,

$$\lim\alpha_{p,q}\big(v(\Psi^p_G)^{-1}(F_i)\big) = \alpha_{p,q}\big(v(\Psi^p_G)^{-1}(F)\big)$$

ultraweakly on $\mathcal{L}(L^q(G))$. In particular

$$\lim\sum_{n=1}^{\infty}\left\langle \alpha_{p,q}\big(v(\Psi^p_G)^{-1}(F_i)\big)[\tau_q k_n], [\tau_{q'}l_n]\right\rangle_{L^q,L^{q'}}$$

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$$= \sum_{n=1}^{\infty} \big\langle \alpha_{p,q}\big(v(\Psi_G^p)^{-1}(F)\big)[\tau_q k_n], [\tau_{q'} l_n]\big\rangle_{L^q, L^{q'}}$$

i.e $\lim \omega(F_i) = \omega(F)$.

(III)There is $w \in A_p(G)$ such that $\omega(\Psi_G^p(T)) =< w, T >_{A_p, PM_p}$, for every $T \in PM_p(G)$.

According to [1, Theorem 6 Section 4.2 page 54], there is $w \in A_p(G)$ with $\omega(F) = F(w)$ for every $F \in A_p(G)'$. For every $T \in PM_p(G)$, we get $\omega(\Psi_G^p(T)) = \Psi_G^p(T)(w) =< w, T >_{A_p, PM_p}$.

(IV) For every $T \in PM_p(G)$, we have

$$< w, T >_{A_p, PM_p} =< u, \alpha_{p,q}(vT) >_{A_q, PM_q} .$$

We have

$$< w, T >_{A_p, PM_p} = \omega(\Psi_G^p(T))$$

$$= \sum_{n=1}^{\infty} \overline{\big\langle \alpha_{p,q}\big(v(\Psi_G^p)^{-1}(\Psi_G^p(T))\big)[\tau_q k_n], [\tau_{q'} l_n]\big\rangle}_{L^q, L^{q'}}$$

$$= \sum_{n=1}^{\infty} \overline{\big\langle \alpha_{p,q}(vT)[\tau_q k_n], [\tau_{q'} l_n]\big\rangle}_{L^q, L^{q'}} =< u, \alpha_{p,q}(vT) >_{A_q, PM_q} .$$

(V) For every $\mu \in M^1(G)$ we have

$$< w, \lambda_G^p(\mu) >_{A_p, PM_p} =< u, \alpha_{p,q}(v\lambda_G^p(\mu)) >_{A_q, PM_q}$$

but $v\lambda_G^p(\mu) = \lambda_G^p(\tilde{v}\mu)$ and $\alpha_{p,q}(\lambda_G^p(\tilde{v}\mu)) = \lambda_G^q(\tilde{v}\mu)$. From $< w, \lambda_G^p(\mu) >_{A_p, PM_p} = \Psi_G^p(\lambda_G^p(\mu))(w) = \tilde{\mu}(w)$ and

$$\big\langle u, \lambda_G^q(\tilde{v}\mu)\big\rangle_{A_q, PM_q} = \tilde{\mu}(uv),$$

it follows $\tilde{\mu}(uv) = \tilde{\mu}(w)$. This implies $w = uv$ and therefore $uv \in A_p(G)$. We get moreover

$$\big\langle uv, T\big\rangle_{A_p, PM_p} = \big\langle u, \alpha_{p,q}(vT)\big\rangle_{A_q, PM_q}$$

for every $T \in PM_p(G)$. We obtain the following estimate

$$\big|\big\langle uv, T\big\rangle_{A_p, PM_p}\big| = \big|\big\langle u, \alpha_{p,q}(vT)\big\rangle_{A_q, PM_q}\big| \le \|u\|_{A_q} \, \|\!|\alpha_{p,q}(vT)|\!\|_q$$

$$\le \|u\|_{A_q} \|v\|_{A_p} \, \|\!|T|\!\|_p .$$

We finally conclude that $\|uv\|_{A_p} \le \|u\|_{A_q} \|v\|_{A_p}$.

□

Remark 1.11. *For a related result see Theorem B of [3]. Our approach is based on properties of $A_p(G)$, studied in [1].*

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2. Amenable groups

We apply the results of the previous section, to the case of amenable groups.

Theorem 2.1. *Let G be an amenable locally compact group, $p, q \in \mathbb{R}$ with $p > 1, p \neq 2$ and q between 2 and p. Then for every $T \in CV_p(G)$, we have $T \in E_{p,q}$ and $\|\alpha_{p,q}(T)\|_q \leq \|T\|_p$.*

Proof. For every $\varphi \in L^p(G) \cap L^q(G)$, we have $\tau_p T \tau_p \varphi \in L^q(G)$ and $\|\tau_p T \tau_p \varphi\|_q \leq \|T\|_p \|\varphi\|_q$.

It suffices to verify (I) for $[\varphi]$ with $\varphi \in C_{00}(G)$. Let $\psi \in C_{00}(G)$ with $N_{q'}(\psi) \leq 1$ and $\varepsilon > 0$. By [1, Lemma 1 Section 5.4 page 80], there is $k, l \in C_{00}(G)$ with $k \geq 0$, $l \geq 0$, $N_p(k) = N_{p'}(l) = \int_G k(t)l(t)dt = 1$ and such that

$$\left|\left\langle \big((k * \check{l})T\big)[\tau_p \varphi], [\tau_{p'} \psi] \right\rangle_{L^p, L^{p'}} - \left\langle T[\tau_p \varphi], [\tau_{p'} \psi] \right\rangle_{L^p, L^{p'}}\right| < \varepsilon.$$

But

$$\left\langle \big((k * \check{l})T\big)[\tau_p \varphi], [\tau_{p'} \psi] \right\rangle_{L^p, L^{p'}} = \left\langle \tau_p \big((k * \check{l})T\big) \tau_p[\varphi], [\psi] \right\rangle_{L^p, L^{p'}}$$

and according to Theorem 1.3 $\tau_p \big((k * \check{l})T\big)\tau_p[\varphi]$, belongs to $L^q(G)$ this implies that,

$$\left\langle \tau_p \big((k * \check{l})T\big) \tau_p[\varphi], [\psi] \right\rangle_{L^p, L^{p'}} = \left\langle \tau_p \big((k * \check{l})T\big) \tau_p[\varphi], [\psi] \right\rangle_{L^q, L^{q'}}$$

and therefore,

$$\left|\left\langle \big((k * \check{l})T\big)[\tau_p \varphi], [\tau_{p'} \psi] \right\rangle_{L^p, L^{p'}}\right| \leq \|\tau_p \big((k * \check{l})T\big) \tau_p[\varphi]\|_q \|[\psi]\|_{q'}.$$

Using Theorem 1.3 again, we have

$$\|\tau_p \big((k * \check{l})T\big) \tau_p[\varphi]\|_q \leq \|k * \check{l}\|_{A_p} \|T\|_p \|[\varphi]\|_q \leq \|T\|_p \|[\varphi]\|_q,$$

and we obtain,

$$\left|\left\langle \big((k * \check{l})T\big)[\tau_p \varphi], [\tau_{p'} \psi] \right\rangle_{L^p, L^{p'}}\right| \leq \|T\|_p \|[\varphi]\|_q \|\psi\|_{q'} \leq \|T\|_p \|[\varphi]\|_q$$

and therefore,

$$\left|\left\langle T[\tau_p \varphi].[\tau_{p'} \psi] \right\rangle_{L^p, L^{p'}}\right| < \varepsilon + \|T\|_p \|[\varphi]\|_q.$$

Thus

$$\left|\left\langle T[\tau_p \varphi].[\tau_{p'} \psi] \right\rangle_{L^p, L^{p'}}\right| \leq \|T\|_p \|[\varphi]\|_q,$$

and

$$\left|\left\langle \tau_p T \tau_p[\varphi].[\psi] \right\rangle_{L^p, L^{p'}}\right| \leq \|T\|_p \|[\varphi]\|_q$$

and finally $\tau_p T \tau_p[\varphi] \in L^q(G)$ with $\|\tau_p T \tau_p[\varphi]\|_q \leq \|T\|_p \|[\varphi]\|_q$.

Now we conclude that, $\|\alpha_{p,q}(T)\|_q \leq \|T\|_p$.

There is a unique $S \in \mathcal{L}(L^q(G))$ such that $S\varphi = \tau_p T \tau_p \varphi$ for every $\varphi \in L^p(G) \cap L^q(G)$. By (I) $\|S\|_q \leq \|T\|_p$, consequently $\|\alpha_{p,q}(T)\|_q = \|\tau_q S \tau_q\|_q = \|S\|_q \leq \|T\|_p$. □

Corollary 2.2. *Let G be an amenable locally compact group, $p, q \in \mathbb{R}$ with $p > 1, p \neq 2$ and q between 2 and p. Then:*

(1) *$\alpha_{p,q}$ is a contractive Banach algebra monomorphism of $CV_p(G)$ into $CV_q(G)$,*
(2) *for every $\mu \in M^1(G)$ we have $\left\|\lambda_G^q(\mu)\right\|_q \leq \left\|\lambda_G^p(\mu)\right\|_p$.*

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Proof. It suffices to verify 2. By Theorem 2.1

$$\left\|\!\left|\alpha_{p,q}(\lambda_G^p(\mu))\right|\!\right\|_q \leq \left\|\!\left|\lambda_G^p(\mu)\right|\!\right\|_p$$

but by Theorem 1.9 $\alpha_{p,q}(\lambda_G^p(\mu)) = \lambda_G^q(\mu)$. □

Remark 2.3. *(See [4, Corollary p. 512]) For another approach, for the unimodular case see [2].*

We obtain other properties of $\alpha_{p,q}$ in the next theorem.

Theorem 2.4. *Let G be an amenable locally compact group, $p, q \in \mathbb{R}$ with $p > 1, p \neq 2$ and q between 2 and p. Then:*

(1) *for every $u \in A_q(G)$, we have $u \in A_p(G)$ and $\|u\|_{A_p} \leq \|u\|_{A_q}$,*
(2) *for every $u \in A_q(G)$, and for every $T \in PM_p(G)$, we have*

$$\left\langle u, T \right\rangle_{A_p, PM_p} = \left\langle u, \alpha_{p,q}(T) \right\rangle_{A_q, PM_q},$$

(3) *for every $u \in A_q(G)$, and for every $T \in CV_p(G)$, we have*

$$\alpha_{p,q}(uT) = u\alpha_{p,q}(T).$$

Proof. Let u be a function of $A_q(G)$.

(1) Definition of a linear map ω (depending on u) of $A_p(G)'$ into $A_p(G)''$.

There is $\left((k_n)_{n=1}^{\infty}, (l_n)_{n=1}^{\infty}\right) \in \mathcal{A}_q(G)$, such that

$$u = \sum_{n=1}^{\infty} \overline{k_n} * \check{l_n}.$$

For every $F \in A_p(G)'$, we put

$$\omega(F) = \sum_{n=1}^{\infty} \overline{\left\langle \alpha_{p,q}\left((\Psi_G^p)^{-1}(F)\right)[\tau_q k_n], [\tau_{q'} l_n] \right\rangle_{L^q, L^{q'}}}.$$

For every $\gamma \in \mathbb{C}$, and $F, F' \in A_p(G)'$, we have $\omega(\gamma F) = \gamma\omega(F)$ and $\omega(F + F') = \omega(F) + \omega(F')$. We now show that $\omega(F) \in A_p(G)''$. Observe at first that

$$|\omega(F)| \leq \sum_{n=1}^{\infty} \left|\left\langle \alpha_{p,q}\left((\Psi_G^p)^{-1}(F)\right)[\tau_q k_n], [\tau_{q'} l_n] \right\rangle_{L^q, L^{q'}}\right|,$$

taking into account Theorem 1.9, we have for every $n \geq 1$

$$\left|\left\langle \alpha_{p,q}\left((\Psi_G^p)^{-1}(F)\right)[\tau_q k_n], [\tau_{q'} l_n] \right\rangle_{L^q, L^{q'}}\right|$$

$$\leq \left\|\!\left|(\Psi_G^p)^{-1}(F)\right|\!\right\|_p N_q(k_n) N_{q'}(l_n) = \|F\|_{A_p'} N_q(k_n) N_{q'}(l_n)$$

hence

$$|\omega(F)| \leq \|F\|_{A_p'} \sum_{n=1}^{\infty} N_q(k_n) N_{q'}(l_n).$$

This implies that $\omega(F) \in A_p(G)''$.

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(2) Let F be an element of $A_p(G)'$ and $(F_i)_{i\in I}$ a net of $A_p(G)'$ such that $\lim F_i = F$ for the topology $\sigma(A'_p, A''_p)$. We assume the existence of $C>0$ such that $\|F_i\|_{A'_p} \leq C$ for every $i \in I$. Then $\lim \omega(F_i) = \omega(F)$.

We have $\lim(\Psi^p_G)^{-1}(F_i) = (\Psi^p_G)^{-1}(F)$, ultraweakly in $\mathcal{L}(L^p(G))$. We also have $\left\|\left|(\Psi^p_G)^{-1}(F_i)\right|\right\|_p \leq C$ for every $i \in I$.

Let $r, s \in C_{00}(G)$. We have

$$\lim \left\langle (\Psi^p_G)^{-1}(F_i)[\tau_p r], [\tau_{p'} s]\right\rangle_{L^p,L^{p'}} = \left\langle (\Psi^p_G)^{-1}(F)[\tau_p r], [\tau_{p'} s]\right\rangle_{L^p,L^{p'}}.$$

Taking into account that, $[r] \in L^p(G) \cap L^q(G)$ and that $(\Psi^p_G)^{-1}(F_i), (\Psi^p_G)^{-1}(F_i) \in E_{p,q}$, we get

$$\tau_q \alpha_{p,q}\big((\Psi^p_G)^{-1}(F_i)\big)\tau_q[r] = \tau_p (\Psi^p_G)^{-1}(F_i)\tau_p[r]$$

and

$$\tau_q \alpha_{p,q}\big((\Psi^p_G)^{-1}(F)\big)\tau_q[r] = \tau_p (\Psi^p_G)^{-1}(F)\tau_p[r].$$

This implies that,

$$\left\langle (\Psi^p_G)^{-1}(F)[\tau_p r], [\tau_{p'} s]\right\rangle_{L^p,L^{p'}} = \left\langle \tau_q \alpha_{p,q}\big((\Psi^p_G)^{-1}(F)\big)\tau_q[r], [s]\right\rangle_{L^q,L^{q'}}$$

and for every $i \in I$,

$$\left\langle (\Psi^p_G)^{-1}(F_i)[\tau_p r], [\tau_{p'} s]\right\rangle_{L^p,L^{p'}} = \left\langle \tau_q \alpha_{p,q}\big((\Psi^p_G)^{-1}(F_i)\big)\tau_q[r], [s]\right\rangle_{L^q,L^{q'}}.$$

We have therefore,

$$\begin{aligned}&\lim \left\langle \alpha_{p,q}\big((\Psi^p_G)^{-1}(F_i)\big)[\tau_q r], [\tau_{q'} s]\right\rangle_{L^q,L^{q'}}\\ &= \left\langle \alpha_{p,q}\big((\Psi^p_G)^{-1}(F)\big)[\tau_q r], [\tau_{q'} s]\right\rangle_{L^q,L^{q'}}.\end{aligned}$$

But for every $i \in I$,

$$\left\|\left|\alpha_{p,q}\big((\Psi^p_G)^{-1}(F_i)\big)\right|\right\|_q \leq \left\|\left|(\Psi^p_G)^{-1}(F_i)\right|\right\|_p \leq C,$$

we obtain that

$$\lim \alpha_{p,q}\big((\Psi^p_G)^{-1}(F_i)\big) = \alpha_{p,q}\big((\Psi^p_G)^{-1}(F_i)\big)$$

ultraweakly on $\mathcal{L}(L^q(G))$. This finally implies that,

$$\begin{aligned}&\lim \sum_{n=1}^{\infty} \left\langle \alpha_{p,q}\big((\Psi^p_G)^{-1}(F_i)\big)[\tau_q k_n], [\tau_{q'} l_n]\right\rangle_{L^q,L^{q'}}\\ &= \sum_{n=1}^{\infty} \left\langle \alpha_{p,q}\big((\Psi^p_G)^{-1}(F)\big)[\tau_q k_n], [\tau_{q'} l_n]\right\rangle_{L^q,L^{q'}}\end{aligned}$$

i.e. $\lim \omega(F_i) = \omega(F)$.

(3)As in the proof of Theorem 1.10 (see step (III)), there is $v \in A_p(G)$ such that $\omega(F) = F(v)$ for every $F \in A_p(G)'$. For every $T \in CV_p(G)$ we have

$$\omega(\Psi^p_G(T)) = \Psi^p_G(T)(v) = \left\langle v, T\right\rangle_{A_p,PM_p}$$

$$= \sum_{n=1}^{\infty} \overline{\left\langle \alpha_{p,q}\big((\Psi^p_G)^{-1}(\Psi^p_G(T))\big)[\tau_q k_n], [\tau_{q'} l_n]\right\rangle}_{L^q,L^{q'}}$$

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$$= \sum_{n=1}^{\infty} \overline{\left\langle \alpha_{p,q}(T)[\tau_q k_n], [\tau_{q'} l_n] \right\rangle}_{L^q, L^{q'}} = \left\langle u, \alpha_{p,q}(T) \right\rangle_{A_q, PM_q}.$$

In particular for every $\mu \in M^1(G)$, we have

$$\left\langle v, \lambda_G^p(\mu) \right\rangle_{A_p, PM_p} = \left\langle u, \alpha_{p,q}(\lambda_G^p(\mu)) \right\rangle_{A_q, PM_q} = \left\langle u, \lambda_G^q(\mu) \right\rangle_{A_q, PM_q}.$$

This implies $\tilde{\mu}(v) = \tilde{\mu}(u)$ and therefore $u = v$, we obtain that $u \in A_p(G)$. We also obtain for every $u \in A_q(G)$, and for every $T \in CV_p(G)$,

$$\left\langle u, T \right\rangle_{A_p, PM_p} = \left\langle u, \alpha_{p,q}(T) \right\rangle_{A_q, PM_q}.$$

This implies the following estimate:

$$\begin{aligned} \left| \left\langle u, T \right\rangle_{A_p, PM_p} \right| = \left| \left\langle u, \alpha_{p,q}(T) \right\rangle_{A_q, PM_q} \right| &\leq \|u\|_{A_q} \, \|\alpha_{p,q}(T)\|_q \\ &\leq \|u\|_{A_q} \, \|T\|_p \,, \end{aligned}$$

thus $\|u\|_{A_p} \leq \|u\|_{A_q}$.

It remains to verify that for every $u \in A_q(G)$, and for every $T \in CV_p(G)$ we have,

$$\alpha_{p,q}(uT) = u\,\alpha_{p,q}(T).$$

Consider an arbitrary $v \in A_q(G)$. We have,

$$\left\langle v, \alpha_{p,q}(uT) \right\rangle_{A_q, PM_q} = \left\langle v, uT \right\rangle_{A_p, PM_p}.$$

But

$$\begin{aligned} \left\langle v, uT \right\rangle_{A_p, PM_p} = \left\langle vu, T \right\rangle_{A_p, PM_p} &= \left\langle vu, \alpha_{p,q}(T) \right\rangle_{A_q, PM_q} \\ &= \left\langle v, u\,\alpha_{p,q}(T) \right\rangle_{A_q, PM_q}. \end{aligned}$$

We get,

$$\left\langle v, \alpha_{p,q}(uT) \right\rangle_{A_q, PM_q} = \left\langle v, u\,\alpha_{p,q}(T) \right\rangle_{A_q, PM_q}$$

and finally,

$$\alpha_{p,q}(uT) = u\,\alpha_{p,q}(T).$$

□

References

[1] A. Derighetti, Convolution Operators on Groups, Lecture Notes of the Unione Matematica Italiana 11, Springer-Verlag Berlin Heidelberg, 2011.

[2] A. Derighetti, Survey on the Figà-Talamanca Herz algebra, Σ *Mathematics* **2019**, *7(8) 660; https://doi.org/10.3390/math7080660*.

[3] *C. Herz, The theory of p-spaces with an application to convolution operators,* Trans. Am. Math. **154** (1971), 69–82.

[4] C. Herz and N. Rivière, Estimates for translation-invariant operators on spaces with mixed norms, *Stud. Math.* **XLIV** (1972), 511–515

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