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TWO CLASSES OF J-OPERATORS

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Dedicated to Professor A. T.-M. Lau

Abstract. We define two classes $\mathfrak{A}$ and $\mathfrak{B}$ in the space $\mathcal{B}(\mathcal{H})$ of operators acting on a Hilbert space on the basis of J-order relation and spectra, and discuss various properties related to these classes.

1. **Introduction**

Let J be a non-trivial selfadjoint involution in $\mathcal{B}(\mathcal{H})$, the space of bounded linear operators on a Hibert space $\mathcal{H}$, that is,

$$J^* = J, \ J^2 = I \ \text{ and } \ J \neq I,$$

where I denotes the identity operator. Such selfadjoint involutions are closely connected to the theory of indefinite inner product spaces; see [1,3,4]. When both A, B are J-*selfadjoint*, that is, JA and JB are selfadjoint,

$$JA = A^*J \quad \text{and} \quad JB = B^*J,$$

we will use the notation

$$A \geq^J B \ \ for \ \ JA \geq JB.$$

Here, for selfadjoint operators X and Y, the order relation $X \geq Y$ means, as usual, that $X - Y$ is positive semi-definite.

In this connection, the following fact is used without explicit mention (see [2] and [6, Lemma 2.2]):

$$A \geq^J B \quad \Longrightarrow \quad CAC \ \geq^J \ CBC \qquad \text{for all } \ J-\text{selfadjoint } C.$$

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The *J-adjoint* $A^\sharp$ of A is defined as

$$A^\sharp \; = \; JA^*J.$$

Then clearly

$$\sigma(A^\sharp) \; = \; \sigma(A^*) \; = \; \overline{\sigma(A)}$$

where $\sigma(A)$ denotes the set of spectra of A and $\overline{\sigma(A)}$ does its complex conjugation.

With a J-control for a J-selfadjoint A, we understand something like

$$J \cdot I \; = \; J \geq JA \quad \text{or} \quad JA \geq 0 = J \cdot 0$$

Under such a J-control, it is immediate to see

$$\sigma(A) \; \subset \; (-\infty, \infty).$$

Our question is whether, with some additional J-control for A^2, we can press $\sigma(A)$ into $[0, \infty)$.

In this paper, we introduce two sub-classes $\mathfrak{A}$ and $\mathfrak{B}$ in the space $\mathcal{B}(\mathcal{H})$ on the basis of J-order relation and spectra as

$$A \in \mathfrak{A} \quad \overset{\text{Def}}{\Longleftrightarrow} \quad I \geq^J A \;\; \textit{and} \quad \sigma(A) \subset [0, \infty)$$

and

$$A \in \mathfrak{B} \quad \overset{\text{Def}}{\Longleftrightarrow} \quad A \geq^J 0 \;\; \textit{and} \;\; \sigma(A) \subset [0, \infty).$$

and discuss various properties related to those classes.

For instance, it is remarkable to see that

$$A, B \in \mathfrak{A} \quad \Longrightarrow \quad ABA \in \mathfrak{A},$$

and the corresponding result for $\mathfrak{B}$.

Throughout the paper, we use the following basic fact without any mention;

$$X, Y \; \in \; \mathcal{B}(\mathcal{H}) \quad \Longrightarrow \quad \sigma(XY) \; \subset \; \sigma(YX) \cup \{0\}.$$

2. Basic facts

Our first result reads as follows.

Theorem 2.1.

$$I \geq^J A \quad \Longrightarrow \quad \sigma(A) \; \subset \; (-\infty, \infty)$$

and

$$\min \, \sigma(A) \; \leq \; 1 \; \leq \; \max \, \sigma(A)$$

where, for instance,

$$\min \, \sigma(A) \; \overset{\text{Def}}{=} \; \min\{\lambda; \; \lambda \in \sigma(A)\}.$$

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Proof. Let $K := J - JA \geq 0$, so that

$$JK \;=\; I - A \quad \text{and} \quad \sigma(JK) \;=\; 1 - \sigma(A).$$

Therefore, since $K^{1/2}JK^{1/2}$ is selfadjoint,

$$\begin{aligned}\sigma(A) &= 1 - \sigma(JK) \\ &= 1 - \sigma(K^{1/2}JK^{1/2}) \;\subset\; (-\infty, \infty).\end{aligned}$$

Suppose next, by contradition, that

$$\min \sigma(A) \;>\; 1.$$

Then

$$\max \sigma(JK) \;=\; 1 - \min \sigma(A) \;<\; 0.$$

This implies that $K \geq 0$ is invertible. Then

$$\max \sigma(JK) \;=\; \max \sigma(K^{1/2}JK^{1/2}) \;<\; 0,$$

hence

$$K^{1/2}JK^{1/2} \;\leq\; 0$$

which leads to a contradiction $J < 0$.

In a similar way, $\max \sigma(A) < 1$ leads to a contradiction that $J > 0$. □

3. Class $\mathfrak{A}$

Let us introduce a class $\mathfrak{A}$ in $\mathcal{B}(\mathcal{H})$ as follows:

$$A \in \mathfrak{A} \quad \Longleftrightarrow \quad I \geq^J A \;\; \textit{and} \quad \sigma(A) \subset [0, \infty).$$

The spectral requirement $\sigma(A) \subset [0, \infty)$ in the definition of the class $\mathfrak{A}$ can be replaced by an operator inequality.

Theorem 3.1.

$$A \in \mathfrak{A} \quad \Longleftrightarrow \quad I \geq^J A \;\; \textit{and} \;\; A \geq^J A^2.$$

Proof. Let us write as usual:

$$K := J(I - A) \;\geq\; 0.$$

Since

$$A = I - JK$$

it is seen that

$$\sigma(A) \subset [0, \infty) \quad \Longleftrightarrow \quad \sigma(JK) \leq 1.$$

Since $K \geq 0$, this is further reduced to the requirement

$$\sigma(K^{1/2}JK^{1/2}) \;\leq\; 1 \quad \text{or} \quad K^{1/2}JK^{1/2} \;\leq\; I$$

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which is equivalent, by $K \geq 0$, to

$$KJK \leq K \quad \text{or} \quad K(I - JK) \geq 0.$$

Finally this last inequality means that

$$J(I - A)A \geq 0 \quad \text{or} \quad A \geq^J A^2.$$

□

Corollary 3.2.

$$A \in \mathfrak{A} \quad \Longleftrightarrow \quad A^* \in \mathfrak{A} \quad \Longleftrightarrow \quad A^\sharp \in \mathfrak{A}.$$

Proof. It follows from the facts:

$$I \ \geq^J \ A \quad \Longleftrightarrow \quad I \ \geq^J \ A^* \quad \Longleftrightarrow \quad I \ \geq^J \ A^\sharp$$

and

$$\sigma(A) \cap (-\infty, \infty) \ = \ \sigma(A^*) \cap (-\infty, \infty) \ = \ \sigma(A^\sharp) \cap (-\infty, \infty).$$

□

It follows from Theorem 3.1 also that

$$A \in \mathfrak{A} \quad \Longrightarrow \quad A^n \in \mathfrak{A} \ \ n = 2, 3, \ldots.$$

But this is a consequence of Theorem 3.5 below.

An element of $\mathfrak{A}$ can be invertible, for instance, $A = I$.

Here is a special position for invertible $A \in \mathfrak{A}$. The essence of the following result was established by Hassi and Nordstroem [5] and stated in the present form by Sano [6].

Corollary 3.3.

$$I \geq^J A \quad \textit{and} \quad \sigma(A) \subset (0, \infty) \quad \Longleftrightarrow \quad A \ \ \textit{invertible}, \quad A^{-1} \ \geq^J I \ \geq^J A.$$

Proof. First suppose that

$$I \geq^J A \quad \text{and} \quad \sigma(A) \subset (0, \infty).$$

Then by Theorem 3.1 $A \geq^J A^2$. Since A^{-1} is J-selfadjoint, this implies

$$A^{-1} \ = \ A^{-1}AA^{-1} \ \geq^J A^{-1}A^2A^{-1} \ = \ I.$$

Therefore $A^{-1} \geq^J I$.

Conversely

$$A^{-1} \geq^J I \quad \Longrightarrow \quad A = A \cdot A^{-1} \cdot A \ \geq^J AIA = A^2,$$

so by Theorem 3.1 $\sigma(A) \subset [0, \infty)$. Finally since A is invertible, $\sigma(A) \subset (0, \infty)$. □

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An operator $A \in \mathcal{B}(\mathcal{H})$ is called a *J-contraction* if $I \geq^J A^\sharp A$ or equivalently $J \geq A^* JA$. It is called a *J-bicontraction* if both A and $A^\sharp$ are J-contractions, that is,

$$I \geq^J A^\sharp A \quad \textit{and} \quad I \geq^J AA^\sharp,$$

or equivalently

$$J \geq A^* JA \quad \textit{and} \quad J \geq AJA^*.$$

Corollary 3.4. *For every J-bicontraction A, both $A^\sharp A$ and $AA^\sharp$ belong to the class $\mathfrak{A}$, that is,*

$$I \geq^J A^\sharp A, \ \ I \geq^J AA^\sharp \quad \Longrightarrow \quad \sigma(A^\sharp A) \ \subset \ [0, \infty).$$

Proof. On the basis of Theorem 3.1, it should be shown, for instance, that

$$A^\sharp A \ \geq^J \ (A^\sharp A) \cdot (A^\sharp A),$$

or equivalently

$$A^* JA \ - \ A^* JAJA^* JA \ \geq \ 0.$$

But this is guaranteed as follows:

$$A^*\{J - JAJA^*J\}A \ = \ A^*\{J - JAA^\sharp\}A \ \geq \ 0.$$

□

Corollary 2.1.4 *Let A be invertible with $\sigma(A^\sharp A) \subset [0, \infty)$. Then*

$$I \geq^J A^\sharp A \quad \Longleftrightarrow \quad I \geq^J AA^\sharp.$$

Proof. First observe that

$$\sigma(A^\sharp A) \ = \ \sigma(AA^\sharp).$$

Therefore it should be shown that

$$I \geq^J A^\sharp A \quad \text{and} \quad \sigma(A^\sharp A) \ \subset \ [0, \infty) \quad \Longrightarrow \quad I \geq^J AA^\sharp.$$

In other words:

$$I \geq^J (A^\sharp A) \geq^J (A^\sharp A)^2 \quad \Longrightarrow \quad I \geq^J AA^\sharp.$$

To see this, obverse the following

$$0 \ \ \leq \ \ J\{A^\sharp A \ - \ (A^\sharp A)^2\} \ = \ A^* JA - J \cdot JA^* JA \cdot JA^* JA$$

$$= \ \ A^* J\{J - AJA^*\}JA \ = \ A^*\{J - JAA^\sharp\}A.$$

Since A is invertible by assumption, this implies $I \geq^J AA^\sharp$. □

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Theorem 3.5.

$$A,\ B \in \mathfrak{A} \quad \Longrightarrow \quad ABA \in \mathfrak{A}.$$

Proof. Since A is J-selfadjoint

$$I \geq^J B \quad \Longrightarrow \quad A^2 \geq^J ABA.$$

Since $I \geq^J A \geq^J A^2$ by Theorem 3.1, this implies $I \geq^J ABA$.

Further since $B \geq^J B^2$ by Theorem 3.1

$$ABA \geq^J AB^2A \ \geq^J ABA^2BA \ = \ (ABA)^2$$

hence $ABA \in \mathfrak{A}$ again by Theorem 3.1. □

For a proof of the next theorem we need an analytic tool, *the square root.*

Lemma 3.6. *Let C be a J-selfadjoint operator with $\sigma(C) \subset (0, \infty)$. Then its square root $C^{1/2}$ is defined according to the Riesz-Dunford functional calculus:*

$$C^{1/2} := \frac{1}{2\pi i} \int_\Gamma \sqrt{\zeta}(\zeta I - C)^{-1} d\zeta$$

where Γ is a rectifiable contour in $\mathbb{C}^+$ (the open-right half-plane), surrounding $\sigma(C)$ in positiive direction.

The square-root $C^{1/2}$ is J-selfadjoint, and commutes with all X which commutes with C, and

$$(C^{1/2})^2 \ = \ C.$$

Theorem 3.7. $A,\ B \in \mathfrak{A},\ AB = BA \quad \Longrightarrow \quad AB \in \mathfrak{A}.$

Proof. Let, for each $\epsilon > 0$,

$$A_\epsilon := A + \epsilon I.$$

Since $\sigma(A_\epsilon) \ \subset \ (0, \infty)$, we can consider its square root $(A_\epsilon)^{1/2}$.

Then since $A_\epsilon^{1/2}B = BA_\epsilon^{1/2}$ and $I \geq^J B$

$$JA_\epsilon B \ = \ JA_\epsilon^{1/2}BA_\epsilon^{1/2} \ = \ (A_\epsilon^{1/2})^* JBA_\epsilon^{1/2}$$

$$\leq \ (A_\epsilon^{1/2})^* J(A_\epsilon)^{1/2} \ = \ JA_\epsilon.$$

Letting $\epsilon \to 0$, we can see

$$JAB \leq JA \leq J.$$

Finally the commutativity $AB = BA$ implies $\sigma(AB) \subset [0, \infty)$. □

The class $\mathfrak{A}$ is not bounded in norm.

In fact,

$$A_\lambda := \ \lambda(I - J) \quad \text{for all} \quad \lambda \geq 1$$

belongs to the class $\mathfrak{A}$.

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Theorem 3.8. *The class $\mathfrak{A}$ is not convex.*

Proof. Let us consider $\mathbb{M}_2$ with $J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

Let

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 & \sqrt{2} \\ -\sqrt{2} & 2 \end{bmatrix}.$$

Then

$$J(I-A) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \geq 0$$

and

$$J(I-B) = \begin{bmatrix} 2 & -\sqrt{2} \\ -\sqrt{2} & 1 \end{bmatrix} \geq 0.$$

Clearly since

$$\sigma(A) = \sigma(B) = \{0, 1\}$$

both A and B belong to the class $\mathfrak{A}$, and

$$\frac{A+B}{2} = \frac{1}{2} \begin{bmatrix} -1 & \sqrt{2} \\ -\sqrt{2} & 3 \end{bmatrix}.$$

Since

$$\det\Big(\frac{A+B}{2}\Big) = \frac{1}{4}(-3+2) < 0$$

The matrix $\frac{A+B}{2}$ does not belong to the class $\mathfrak{A}$, so that $\mathfrak{A}$ is not convex. □

4. Class $\mathfrak{B}$

Let us introduce another class $\mathfrak{B}$ in $\mathcal{B}(\mathcal{H})$ as follows:

$$A \in \mathfrak{B} \quad \overset{\text{Def}}{\iff} \quad A \geq^J 0 \ \ \textit{and} \ \ \sigma(A) \subset [0, \infty).$$

The spectral requirement, $\sigma(A) \subset [0, \infty)$ in the definition of the class $\mathfrak{B}$, can be replaced by an operator inequality.

Theorem 4.1.

$$A \in \mathfrak{B} \quad \iff \quad A \geq^J 0 \ \ \textit{and} \ \ A^2 \geq^J 0.$$

Proof. Let $K := JA \geq 0$. Then as usual

$$\begin{aligned} \sigma(A) \subset [0, \infty) &\iff \sigma(JK) \subset [0, \infty) \\ &\iff \sigma(K^{1/2}JK^{1/2}) \subset [0, \infty) \\ &\iff K^{1/2}JK^{1/2} \geq 0 \\ &\iff KJK \geq 0 \iff JA^2 \geq 0. \end{aligned}$$

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□

It follows from this theorem that

$$A \in \mathfrak{B} \quad \Longrightarrow \quad A^n \ \in \mathfrak{B} \ \ n = 2, 3, \ldots .$$

But this statement is also included in Theorem 4.3 and Theorem 4.4 below.

Neither $A \in \mathfrak{B}$ is invertible. In fact, if $A \in \mathfrak{B}$ is invertible, then

$$JA^2 \geq 0 \implies J \geq 0,$$

which is a contradicion.

The following is a counter-part of Corollary 3.3.

Corollary 4.2.

$$A^\sharp A \geq^J 0, \quad AA^\sharp \geq^J 0 \quad \Longrightarrow \quad \sigma(A^\sharp A),\ \sigma(AA^\sharp)\ \subset\ [0, \infty).$$

Proof. By Theorem 4.1 we have to show that

$$A^\sharp A \geq^J 0,\ AA^\sharp \geq^J 0 \quad \Longrightarrow \quad (A^\sharp A)^2 \geq^J 0.$$

First notice that

$$A^\sharp A \geq^J 0 \quad \Longleftrightarrow \quad A^* JA \geq 0$$

and

$$AA^\sharp \geq^J 0 \quad \Longleftrightarrow \quad AJA^* \geq 0.$$

Now

$$J(A^\sharp A)^2 \ = \ A^* JAJA^* JA \ \geq \ A^* J \cdot 0 \cdot JA \ = \ 0$$

so that $(A^\sharp A)^2 \geq^J 0.$ □

The following theorem is a counter part of Theorem 3.5.

Theorem 4.3.

$$A \in \mathfrak{B},\ B \geq^J 0 \implies ABA \in \mathfrak{B}.$$

Proof. Since A is J-selfadjoint

$$B \geq^J 0 \quad \Longrightarrow \quad ABA \ \geq^J \ A \cdot 0 \cdot A = 0,$$

and

$$(ABA)^2 \ = \ ABA^2BA \ \geq^J \ AB \cdot 0 \cdot BA \ = 0.$$

Therefore by Theorem 4.1 $ABA \in \mathfrak{B}$. □

The following theorem is a counter part of Thereom 3.7.

Theorem 4.4.

$$A \in \mathfrak{B}, \quad B \geq^J 0,\ AB = BA \quad \Longrightarrow \quad AB \in \mathfrak{B}.$$

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Proof. Let, for each $\epsilon > 0$,

$$A_\epsilon := A + \epsilon I.$$

Then A_ϵ is J-selfadjoint and $\sigma(A_\epsilon) \subset (0, \infty)$. Therefore we can consider its square root $(A_\epsilon)^{1/2}$, which is J-selfadjoint and commutes with B. Now

$$\begin{aligned} JA_\epsilon B &= J(A_\epsilon)^{1/2}B(A_\epsilon)^{1/2} \\ &= ((A_\epsilon)^{1/2})^* JB(A_\epsilon)^{1/2} \geq ((A_\epsilon)^{1/2})^* \cdot 0 \cdot (A_\epsilon)^{1/2} = 0 \end{aligned}$$

Letting $\epsilon \to 0$ we can conclude

$$JAB \geq 0, \text{ that is, } AB \geq^J 0.$$

Finally by commutativity

$$J(AB)^2 = JBA^2B = B^*JA^2B \geq 0.$$

Therefore by Theorem 4.1 $AB \in \mathfrak{B}$. □

The class $\mathfrak{B}$ is not bouned in norm. In fact,

$$\lambda(I + J) \in \mathfrak{B} \quad \text{for all} \quad \lambda > 0.$$

Theorem 4.5. *The class $\mathfrak{B}$ is not convex.*

Proof. Let us consider $\mathbb{M}_2$ with

$$J := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Let

$$A = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot [1, -1] = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}.$$

Then

$$JA = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \geq 0, \quad \sigma(A) = \{0, 0\}.$$

Let

$$B = \begin{bmatrix} 1 \\ -1 \end{bmatrix} [1, 1] = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}.$$

Then

$$JB = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \geq 0, \quad \sigma(B) = \{0, 0\}.$$

Therefore $A, B \in \mathfrak{B}$.

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Since

$$\frac{A+B}{2} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \sigma\Big(\frac{A+B}{2}\Big) = \{1, -1\},$$

$\frac{A+B}{2}$ does not belong to $\mathfrak{B}$. Therefore the class $\mathfrak{B}$ is not convex. □

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