



## DICHOTOMY BETWEEN OPERATORS ACTING ON FINITE AND INFINITE DIMENSIONAL HILBERT SPACES

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ABSTRACT. In this expository article, we give several examples showing how drastically different can be the behavior of operators acting on finite versus infinite dimensional Hilbert spaces. This essay is written as in such a friendly-reader to show that the situation in the infinite dimensional setting is trickier than the finite one.

### 1. Introduction

The notion of Hilbert space is a generalization of that of the Euclidean space  $\mathbb{R}^2$ , that is a vector space equipped with a scalar product  $\langle \cdot, \cdot \rangle$ . The norm is defined by

$$\|x\| = \langle x, x \rangle^{1/2}.$$

This notion was first introduced by David Hilbert in the setting of integral equations and named by others after him. A Hilbert space is a Banach space (i.e., a vector space equipped with a complete norm  $\|\cdot\|$ ), whose norm satisfies the parallelogram law  $\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$ . The space  $C[0, 1]$  of continuous linear functions on the interval  $[0, 1]$  endowed with the sup-norm  $\|f\| = \sup\{|f(t)| : t \in [0, 1]\}$  is a Banach space whose norm cannot be deduced from an inner product space since it does not satisfy the parallelogram law for  $f(t) = 1$  and  $g(t) = t$ .

Finite dimensional Hilbert spaces are isomorphic to the space  $\mathbb{C}^n$  endowed with the inner product  $\langle (x_i), (y_i) \rangle = \sum_{i=1}^n x_i \bar{y}_i$ .

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The infinite dimensional analogue of  $\mathbb{C}^n$  is the (separable) Hilbert space  $\ell_2 = \ell_2(\mathbb{N})$  of all complex sequences  $(x_n)$  satisfying  $\sum_{n=1}^{\infty} |x_n|^2 < \infty$  under pointwise operations  $\alpha(x_n) + (y_n) = (\alpha x_n + y_n)$  and the inner product

$$\langle (x_n), (y_n) \rangle = \sum_{n=1}^{\infty} x_n \bar{y}_n.$$

The standard orthonormal basis  $\{e_j : j = 1, 2, \dots\}$  of  $\ell_2$  is the direct analogue of the one of  $\mathbb{C}^n$ . Similarly, one can impose a Hilbert space structure on the linear space  $\ell_2(\mathbb{Z})$  consisting of all two-sided sequences of the form  $(\dots, x_{-2}, x_{-1}, x_0, x_1, x_2, \dots)$  such that  $\sum_{n=-\infty}^{\infty} |x_n|^2 < \infty$ .

Let  $(\mathcal{H}, \langle \cdot, \cdot \rangle)$  be a Hilbert space. By  $\mathbb{B}(\mathcal{H})$  we denote the algebra of all continuous linear operators on  $\mathcal{H}$  equipped with the pointwise-defined operations of addition and multiplication by scalars, while the multiplication is defined as the composition of operators. A linear operator  $A : \mathcal{H} \rightarrow \mathcal{H}$  is called bounded if  $\|Ax\| \leq M\|x\|$  for some  $M \geq 0$  and all  $x \in \mathcal{H}$ ; if this is the case,  $\|A\| := \sup\{\|Ax\| : \|x\| = 1\} < \infty$  is called the operator norm. The continuity of a linear operator is equivalent to its boundedness in virtue of  $\|Ax - Ay\| \leq \|A\| \|x - y\|$ . For every operator  $A \in \mathbb{B}(\mathcal{H})$ , there exists a unique operator  $A^* \in \mathbb{B}(\mathcal{H})$ , called the adjoint operator of  $A$ , such that  $\langle Ax, y \rangle = \langle x, A^*y \rangle$  for all  $x, y \in \mathcal{H}$ . Throughout the paper, a capital letter means a continuous linear operator in  $\mathbb{B}(\mathcal{H})$ , in particular,  $I$  denotes the identity operator. When a capital letter denotes a matrix, we explicitly state it.

The set  $\sigma(A) := \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not invertible in } \mathbb{B}(\mathcal{H})\}$  is called the spectrum of  $A$ , which is both nonempty and compact. The numerical range of  $A$  is defined and denoted by  $W(A) = \{\langle Ax, x \rangle : \|x\| = 1\}$ .

An operator  $A$  is called normal if  $A^*A = AA^*$ . It is self-adjoint (or hermitian) if  $A^* = A$ , or equivalently  $W(A) \subseteq \mathbb{R}$ . It is said to be positive (positive semidefinite) if  $W(A) \subseteq [0, \infty)$ ; the set of all positive semi-definite operators is denoted by  $\mathbb{B}(\mathcal{H})_+$ . An operator  $A$  is idempotent if  $A^2 = A$ . An orthogonal projection is a self-adjoint idempotent.

The Löwner order on the set  $\mathbb{B}(\mathcal{H})_h$  of self-adjoint operators is defined by  $A \leq B \iff B - A \in \mathbb{B}(\mathcal{H})_+$ .

There are many assertions in (finite dimensional) linear algebra that do not hold in an infinite dimensional Hilbert space; even less is true for general Banach spaces than Hilbert spaces.

First of all, let us explain that by the dimension of a linear space (in the algebraic sense) we understand the cardinality of any of its linear (or Hamel) bases, i.e., maximal linearly independent sets. In Banach spaces of finite dimension, such as  $\mathbb{C}^n$ , the closed unit ball is compact, all subspaces are (topologically) closed, and all norms on the space are equivalent. None of these statements is true anymore in Banach spaces of infinite dimension, such as  $\ell_2$ .

If  $\mathcal{H}$  is a vector space endowed with a scalar product and is finite dimensional, then  $\mathcal{H}$  automatically becomes a Hilbert space. But if  $\mathcal{H}$  is infinite-dimensional, this is not always true. In fact, if

$\dim(\mathcal{H}) = \aleph_0$  (the cardinality of  $\mathbb{N}$ ), then the Baire category theorem prevents  $\mathcal{H}$  to be Hilbert.

A Hilbert space  $\mathcal{H}$ , besides Hamel bases, also possesses the so-called Hilbert (or orthonormal) bases, that is, maximal families of orthogonal norm-one vectors (two vectors are orthogonal if their inner product is zero). If the dimension of  $\mathcal{H}$  is finite, then the Gram–Schmidt process allows us to produce a Hilbert basis from a linear basis, and the cardinalities of these bases are the same. If the dimension of  $\mathcal{H}$  is infinite, then the cardinality of a Hilbert basis for  $\mathcal{H}$  is strictly smaller than the cardinality of a linear basis for  $\mathcal{H}$ ; see [11].

If  $A$  is a continuous linear operator on  $\ell^2$  or  $\mathbb{C}^n$ , then it admits a matrix representation, i.e. an infinite (resp., finite) matrix whose  $(i, j)$ -entry is  $\langle Ae_j, e_i \rangle$  for all pairs  $i, j$ , and the action of  $A$  is described by the usual matrix product (evidently, a change of orthonormal basis results in a different matrix representation, and each can be endowed with some norm; see [5] for a study of variation of matrix norms as the basis varies). The converse is true for  $\mathbb{M}_n$  in the sense that an arbitrary matrix  $A \in \mathbb{M}_n$  corresponds to the linear mapping on  $\mathbb{C}^n$  defined by  $[z_1, \dots, z_n]^t \mapsto A[z_1, \dots, z_n]^t$  via a matrix product. A similar assertion is not valid for infinite matrices: not any matrix corresponds to a continuous linear operator. In principle, all information about an operator acting on a finite dimensional Hilbert space can be systematically obtained from its matrix representation; the latter in the infinite dimensional case is not useful.

As Halmos indicated [11, Chapter 5], if  $\sum_i \sum_j |\lambda_{ij}|^2 < \infty$ , then there is an operator (matrix, resp.)  $A \in \mathbb{B}(\ell^2)$  such that  $\lambda_{ij} = \langle Ae_j, e_i \rangle$ . Of course, this condition is not necessary. For example, it is not satisfied even in the simplest case of the identity operator.

Thus, we can identify  $\mathbb{B}(\mathbb{C}^n)$  with the space  $\mathbb{M}_n$  of all  $n \times n$  complex matrices in the canonical way. In this case, if  $A = [a_{ij}] \in \mathbb{M}_n$ , then  $A^* = [\overline{a_{ji}}]$ . In addition,  $\sigma(A)$  is exactly the set of eigenvalues of  $A$ , since  $A$  is invertible if and only if it is one-to-one.

Now we present several examples to demonstrate some differences between the properties of operators on finite dimensional Hilbert spaces and those on infinite dimensional ones. It is worthy to say that there are several tricks with matrices, in particular  $2 \times 2$  ones, which help researchers to establish results concerning operators that could not be treated easily; see e.g. [2, 15].

- A linear operator  $A \in \mathbb{B}(\mathbb{C}^n)$  is injective (one-to-one) if and only if it is surjective. This is not the case for linear operators on infinite dimensional Hilbert spaces. For example, the right (unilateral) shift operator  $A : \ell_2 \rightarrow \ell_2$  defined by

$$A(x_1, x_2, \dots) = (0, x_1, x_2, \dots)$$

is injective but not surjective. In addition,

$$A^*(x_1, x_2, \dots) = (x_2, x_3, \dots),$$

which is called left (backward) shift operator, is surjective but not injective.

From another point of view, we can describe the situation above by stating that a matrix  $A$  is an isometry (i.e.  $\|Ax\| = \|x\|$  for all  $x$ ) if and only if it is unitary. In the framework

of infinite dimensional Hilbert spaces this is not valid, since the right shift operator  $A$  is an isometry ( $A^*A = I$ ) but not unitary ( $AA^* \neq I$ ); see also [7].

Still, there is another direction to look at this from: We observe that the right shift operator  $A$  has a left inverse but not a right inverse whilst a square matrix having a left inverse will automatically have a right inverse.

- Every matrix has an eigenvalue while the right shift operator  $A$  has no eigenvalues since  $Ax = \lambda x$  implies that  $x = 0$ . This shows that the spectrum of an operator may have no eigenvalue but still is nonempty. It is worthy to mention that the lack of eigenvalues for normal operators is replaced by the spectral theorem.
- By the rank-nullity theorem,  $\dim \ker(A) = \dim \ker(A^*)$  for any square matrix  $A$ . This is not true in an infinite dimensional Hilbert space, in general. For example, if  $A$  is the right shift operator on  $\ell^2$ , then  $\dim \ker(A) = 0 \neq 1 = \dim \ker(A^*)$ .
- Every matrix has a finite number of eigenvalues while an operator may have infinitely (even uncountably) many eigenvalues. For example, every  $\lambda$  in the open unit disk of the complex plane is an eigenvalue of the left shift operator [17, Example 2.3.2]. On the other hand, the right shift operator has no eigenvalues.
- Unlike the finite dimensional case in which the trace of each matrix is a complex number, the trace of an arbitrary operator  $A \in \mathbb{B}(\ell_2)$  defined by  $\text{tr}(A) = \sum_{j=1}^{\infty} \langle Ae_j, e_j \rangle$  may be infinite (or even non-existing). For example, for the diagonal operator

$$A(x_1, x_2, x_3, \dots) = (x_1, \frac{1}{2}x_2, \frac{1}{3}x_3, \dots)$$

on  $\ell_2$ , we have  $\text{tr}(A) = \sum_{j=1}^{\infty} \frac{1}{j} = \infty$ . By the way, Grothendieck [9] has an example of an operator on a Banach Space where the trace is not the sum of the eigenvalues.

- The spectrum of a matrix  $A$  is contained in its numerical range, and the latter set is closed. Generally, neither statement is true for operators. For example, if  $A$  is the diagonal operator  $\text{diag}(1, 1/2, 1/3, \dots)$  (see the item above), then  $\sigma(A) = \{1/n : n \in \mathbb{N}\} \cup \{0\} \not\subseteq (0, 1] = W(A)$  and  $W(A)$  is not a closed subset of the complex plane; cf. [11, Problem 212]. However,  $\sigma(A)$  is a subset of the closure of  $W(A)$  for every operator  $A$ .
- Two operators  $T$  and  $S$  are similar if  $T = W^{-1}SW$  for some invertible operator  $W$ . They are asymptotically similar if there exist sequences  $(W_n)$  and  $(V_n)$  of invertible operators such that  $S = \lim_n W_n^{-1}TW_n$  and  $T = \lim_n V_n^{-1}SV_n$ . In the finite dimensional case, these two notions coincide but that is not the case in the infinite dimensional realm; cf. [12, Theorem 2.1].

- It is known that the numerical range of any operator  $A$  satisfying  $A^n = I$  cannot be a disk in the finite dimensional setting; cf. [14]. However, the authors of [10] construct an operator  $A$  acting on an infinite dimensional Hilbert space such that  $T^3 = I$  and  $W(A)$  is an open disk centered at the origin.
- If an invertible matrix  $A$  is such that  $\|A^k\|$ ,  $k = \pm 1, \pm 2, \dots$  is constant, then  $A$  is unitary. This is not so in the infinite-dimensional Hilbert spaces. Indeed, it is shown in [6] that for each  $\varepsilon > 0$ , there exists a nonunitary invertible operator  $A$  on  $\ell_2(\mathbb{Z}) \oplus \ell_2(\mathbb{Z})$  such that  $\|A^k\| = 1 + \varepsilon$  for all  $k \geq 1$ .
- The determinant of a matrix is equal to the product of its eigenvalues counted with their multiplicities. Evidently, this definition does not carry over to ‘all’ operators acting on infinite dimensional Hilbert spaces.

An extension of the notion of determinant is the Fredholm determinant, which is defined for operators of the form  $I + A$ , as an extension of  $\det(I + A) = \exp(\text{tr}(\log(I + A)))$ , where  $A$  is a trace class operator, that is, an operator on a Hilbert space  $\mathcal{H}$  such that  $\sum_{e \in \mathcal{E}} \langle |A|e, e \rangle < \infty$ , where  $\mathcal{E}$  is an arbitrary orthonormal basis and  $|A|$  stands for the positive square root of  $A^*A$ . Indeed, for operators in  $I +$  trace class the determinant is the product of eigenvalues (this is usually stated in terms of the trace being the sum of eigenvalues for trace class operators and called Lidskii’s theorem); see [19].

- It is easily observed from

$$\|Ax\| \leq \|A\| \|x\| \quad \text{and} \quad \|Ax\| = \left\| \sum_{j=1}^n \langle x, e_j \rangle Ae_j \right\| \leq n \|x\| \max_{1 \leq j \leq n} \|Ae_j\|$$

that a sequence  $\{A_n\}$  converges to  $A$  in the norm topology if and only if  $\{A_n x\}$  converges to  $Ax$  for all  $x \in \mathbb{C}^n$ . In infinite dimensional Hilbert spaces, the pointwise convergence does not imply the norm convergence, in general. For example, let  $A_n \in \mathbb{B}(\ell_2)$  be defined by the infinite diagonal matrix  $\text{diag}(1, 1, \dots, 1, 0, 0, \dots)$ , whose first  $n$  diagonal entries are equal to 1, and all other entries are 0. Then clearly  $A_n x \rightarrow Ix$ , for all  $x \in \ell_2$ , but the sequence  $(A_n)$  is not a Cauchy sequence in  $\mathbb{B}(\ell_2)$  since  $\|A_n - A_m\| = 1$ ,  $n \neq m$ , and so cannot be convergent in the norm topology.

- Every linear mapping  $A : \mathbb{C}^n \rightarrow \mathbb{C}^n$  is automatically continuous while a linear mapping on an infinite dimensional inner product space may be discontinuous (unbounded). Suppose that  $\mathcal{K}$  is the dense subspace of  $\ell_2$  consisting of all sequences  $(x_n)$  with  $x_n = 0$  for sufficiently large  $n$ . Let  $A : \mathcal{K} \rightarrow \mathcal{K}$  denote the linear mapping  $(x_n) \mapsto (nx_n)$ . Then  $A$  is unbounded since if  $(e_n)$  is the orthonormal basis for  $\ell_2$ , then  $\|e_n\| = 1$  and  $\|Ae_n\| = n$  for all  $n$ .

In this example,  $A$  is defined on a dense subset of  $\ell_2$  but not on the whole space. Discontinuous linear operators defined on the whole space also exist and can be constructed with the use of Hamel bases. For example, following Halmos [11]: Extend the standard orthonormal basis  $(e_n)$  of  $\ell_2$  to a Hamel (linear algebra) basis  $\beta$  for  $\ell_2$ . Choose  $f \in \beta$  different from all  $e_n$ s, and define the linear operator  $A : \ell_2 \rightarrow \ell_2$  by

$$A(g) = \begin{cases} 1 & g = f \\ 0 & g \in \beta \setminus \{f\} \end{cases}$$

Then  $A(e_n) = 0$  and  $A$  is unbounded (otherwise,  $1 = A(f) = \sum_{n=1}^{\infty} \langle f, e_n \rangle A e_n = 0$ ).

- Given an operator  $A$ , the unique operator  $A^\dagger$  (if exists) satisfying (i)  $AA^\dagger A = A$ , (ii)  $A^\dagger AA^\dagger = A^\dagger$ , (iii)  $A^\dagger A$  is self-adjoint, and (iv)  $AA^\dagger$  is self-adjoint, is called the Moore–Penrose inverse of  $A$ . Every matrix has the Moore–Penrose inverse. However, there are operators having no Moore–Penrose inverses (precisely, those operators with non-closed ranges; see [16]). For example, the range of the operator  $A$  on  $\ell_2$  defined by  $A(x_1, x_2, x_3, \dots) = (x_1, \frac{1}{2}x_2, \frac{1}{3}x_3, \dots)$  contains all finitely nonzero sequences, and so is dense in  $\ell_2$ . Since this range does not contain the sequence  $(1/n)$ , it is not closed. This  $A$  has no Moore–Penrose inverse.
- It is known that every normal matrix can be written of the form  $A = UDU^*$  with the unitary matrix  $U$  and diagonal matrix  $D = \text{diag}(\lambda_1, \dots, \lambda_n)$ , where  $\lambda_1, \dots, \lambda_n$  are the eigenvalues of  $A$ ; see [21, Theorem 9.1]. Such a result does not hold in the infinite dimensional case. In other words, there exist normal operators  $A$  on an infinite dimensional Hilbert space  $\mathcal{H}$  for which there are no orthonormal bases of  $\mathcal{H}$  consisting of the eigenvectors of  $A$ . As an extreme manifestation of this phenomenon, the bilateral shift operator  $A(f_n) = f_{n+1}$  ( $n = 0, \pm 1, \pm 2, \dots$ ) on  $\ell_2(\mathbb{Z})$  is normal but has no eigenvalues [17, p. 56].
- An operator  $A$  is called hypercyclic if there exists a vector  $x_0 \in \mathcal{H}$  such that the set  $\{A^n x_0 : n = 0, 1, 2, \dots\}$  is dense in  $\mathcal{H}$ . The space  $\mathcal{H}$  must be separable in order to support a hypercyclic operator. By using the Jordan decomposition for a matrix, it is not difficult to prove that if  $\mathcal{H}$  is finite dimensional, then it has no hypercyclic operator (see [8, pp. 54–55]). The situation for Hilbert spaces of infinite dimension is quite different. For example, every scalar multiple  $\alpha A$  ( $|\alpha| > 1$ ) of the left shift operator  $A$  on  $\ell_2$  is a hypercyclic operator; see [18]. A close, weaker notion is the one of supercyclicity. An operator  $A$  is called hypercyclic if there exists a vector  $x_0 \in \mathcal{H}$  such that the projective orbit  $\{\lambda A^n x_0 : n = 0, 1, 2, \dots; \lambda \in \mathbb{K}\}$  ( $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ ) is dense in  $\mathcal{H}$ . Since any hypercyclic operator is supercyclic, we have that every infinite dimensional separable Hilbert space supports supercyclic operators (in fact,  $A$  itself is supercyclic on  $\ell_2$ ). In the finite dimensional case, Herzog [13] proved in 1992 that, if  $\mathbb{K} = \mathbb{R}$  ( $\mathbb{K} = \mathbb{C}$ , resp.), then  $\mathcal{H}$  supports a supercyclic operator if and only if  $\dim(\mathcal{H}) \in \{1, 2\}$

( $\dim(\mathcal{H}) = 1$ , resp.). In fact, every rotation on  $\mathbb{R}^2$  given by a matrix  $\begin{pmatrix} \cos(2\pi\alpha) & \sin(2\pi\alpha) \\ -\sin(2\pi\alpha) & \cos(2\pi\alpha) \end{pmatrix}$  with  $\alpha$  irrational is supercyclic.

- Factorization of matrices and operators acting on Hilbert spaces is a lively area of research in matrix analysis and operator theory. Problems of factorization ask whether a given operator in  $\mathbb{B}(\mathcal{H})$  can be factored into (real or complex) linear combination or product of finitely many operators in a class of operators and seek for the minimal number of factors in a factorization. Matrix versions of these problems have a long history and many of them have appropriate analogues (probably under some additional conditions) for operators acting on Hilbert spaces of arbitrary dimension. However, some of these problems having solutions for matrices cannot have any solution for operators acting on infinite dimensional Hilbert space. A nice survey on such problems is [20]. By the polar decomposition, every matrix  $A = U|A|$  is the product of two normal matrices, say  $U$  and  $|A|$ , whilst the right shift operator cannot be factored as the product of finitely many normal operators; cf. [11, Problem 144(a)].
- Bart et al. [1] showed that if  $P_1, \dots, P_k$  are idempotent matrices such that  $P_1 + \dots + P_k = 0$ , then  $P_j = 0$  for all  $j = 1, \dots, k$ . The situation changes in the infinite dimensional setting. As shown in [1], for  $k \geq 5$  there exist  $k$  different nonzero projections  $P_1, \dots, P_k$  on  $\mathcal{H}$  such that  $P_1 + \dots + P_k = 0$ . By the way, the number 5 is sharp in the sense that there is no nontrivial zero sum of four idempotents.
- For a long time, there has been considerable interest in the famous invariant subspace problem. This problem asks whether every operator  $\mathfrak{T}$  on a Banach space  $\mathcal{X}$  has a nontrivial (neither  $\{0\}$  nor  $\mathcal{X}$ ) invariant closed subspace. By an invariant subspace, we mean a subspace  $\mathcal{M}$  such that  $\mathfrak{T}\mathcal{M} \subseteq \mathcal{M}$ . Enflo [3] in 1975 proved that there exists a separable Banach space  $\mathcal{X}$  and a continuous linear operator on  $\mathcal{X}$  with dense range having no nontrivial closed invariant subspace. If  $\mathcal{H}$  is a nonseparable Hilbert space,  $x_0 \neq 0$ , and  $A \in \mathbb{B}(\mathcal{H})$ , then the closed linear span  $\{A^n x_0 : n = 0, 1, 2, \dots\}$  is a nontrivial invariant subspace for  $A$ . By the spectral theorem, all normal operators on an infinite dimensional Hilbert space admit nontrivial invariant subspaces. The problem, in its generality, remains still open for (separable) Hilbert spaces. However, if  $A \in \mathbb{M}_n$   $n \geq 2$  is a matrix and  $\lambda$  is an eigenvalue of  $A$ , then its eigenspace  $\{x \in \mathbb{C}^n : Ax = \lambda x\}$  is a nontrivial invariant subspace for  $A$ . Just around the time when this paper was accepted (namely on May, 25th, 2023), Per H. Enflo announced that he had obtained a potential definitive answer to the invariant subspace problem in Hilbert spaces (see <https://www.ucm.es/imi/boletin00091#noticia> and [4]), in which he claimed that that every bounded linear operator on a Hilbert space has a closed non-trivial invariant subspace.

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