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ORDER ISOMORPHISMS AND ORDER ANTI-ISOMORPHISMS ON SPACES OF CONVEX FUNCTIONS

D. H. LEUNG

Dedicated to Professor A. T.-M. Lau

ABSTRACT. For $i = 1, 2$, let C_i be a convex set in a locally convex Hausdorff topological vector space X_i. Denote by $\text{conv}(C_i)$ the space of all convex, proper, lower semicontinuous functions on C_i. A representation is given of any bijection $T : \text{conv}(C_1) \to \text{conv}(C_2)$ that preserves the pointwise order. For $X_i = \mathbb{R}^n$, this recovers a result of Artstein-Avidan and Milman and its generalization by Cheng and Luo. If X_1 is a Banach space and $X_2 = X_1^*$ with the weak*-topology, it gives a result due to Iusem, Reem and Svaiter. We also obtain representation of order reversing bijections and thus a characterization of the Legendre transform, generalizing the same result by Artstein-Avidan and Milman for the $\mathbb{R}^n$ case. The result on order isomorphisms actually holds for convex functions with values in ordered topological vector spaces.

1. Introduction

Let X be a Hausdorff topological vector space and let $\text{conv}(X)$ be the space of convex, proper, lower semicontinuous extended real-valued functions on X. In [2, 3], Artstein-Avidan and Milman characterized order preserving and order reversing maps acting on $\text{conv}(\mathbb{R}^n)$. As a result, they discovered a fundamental characterization of the Legendre transform from convex analysis as the essentially unique order reversing idempotent map on $\text{conv}(\mathbb{R}^n)$. Subsequently, for a convex subset C of $\mathbb{R}^n$, a characterization of order preserving maps on $\text{conv}(C)$ in terms of epigraphs was obtained by Artstein-Avidan, Florentin and Milman [1]. Recently, Cheng and Luo [4] obtained an explicit formula for such

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mappings. Moving to the infinite dimensional realm, Iusem, Reem and Svaiter [6] characterized order preserving as well as order reversing maps from $\mathrm{conv}(X)$ to $\mathrm{conv}(X^*, w^*)$, where X is a Banach space and w^* signifies the weak*-topology on X^*. Cheng and Luo [5] also showed that for a Banach space X, there is an order preserving bijection from $\mathrm{conv}(X)$ onto $\mathrm{conv}(X^*)$ if and only if X is reflexive and X and X^* are isomorphic as Banach spaces.

In this paper, we unify and generalize the aforementioned results. First of all, Theorem 2.13 gives a representation of a general order isomorphism $T : \mathrm{conv}(C_1, E_1) \to \mathrm{conv}(C_2, E_2)$, where, for $i = 1, 2$, C_i is a convex set in a Hausdorff topological vector space X_i and E_i is an ordered topological vector space with a generating positive cone. A point worth noting is that the proof uses only elementary calculations. Continuity of the constituents of the representation is shown under additional assumptions (Corollary 3.4). In particular, if C_i has nonempty interior in X_i, where X_i is locally convex Hausdorff, then X_1 and X_2 must be linearly homeomorphic *in their weak topologies*. In addition to $E_i = \mathbb{R}$, Corollary 3.4 also applies if, e.g., E_i is the space of self-adjoint elements in a C^*-algebra, or if E_i is the space of regular operators on an ordered Banach space. See the last remark in §2. In the final section, we consider order reversing bijections from $\mathrm{conv}(X_1, E_1)$ onto $\mathrm{conv}(X_2, E_2)$. Theorem 4.1 shows that such a map must be essentially unique if it exists. As a result, if X_i is locally convex Hausdorff and $E_i = \mathbb{R}$, then an order reversing bijection $T : \mathrm{conv}(X_1) \to \mathrm{conv}(X_2)$ exists if and only if (X_1^*, w^*) and (X_2, w) are linearly homeomorphic. In this case, T must be essentially the Legendre transform (Corollary 4.3). This allows us to obtain a characterization of the Legendre transform generalizing [3, Theorem 1] (Corollary 4.4).

2. Characterization of order isomorphisms

An ***ordered topological vector space*** E is a topological vector space with a partial order $\leq$ so that (a) $x + z \leq y + z$ and $\lambda x \leq \lambda y$ if $x, y, z \in E$, $x \leq y$ and $0 \leq \lambda \in \mathbb{R}$, (b) the ***positive cone*** $E_+ = \{x \in E : x \geq 0\}$ is closed. The positive cone E_+ is ***generating*** if $E = E_+ - E_+$. If E_+ is generating and $u_1, u_2 \in E$, let $v_1, v_2 \in E_+$ be such that $u_i \leq v_i$, $i = 1, 2$. Then $u_i \leq v_i \leq v_1 + v_2$. Let C be a nonempty convex set in a Hausdorff topological vector space and let E be an ordered Hausdorff topological vector space. A function $f : A \to E$ defined on a convex subset A of C is

(1) ***convex*** if

$$f((1-\alpha)x_1 + \alpha x_2) \leq (1-\alpha)f(x_1) + \alpha f(x_2)$$

for any $\alpha \in [0, 1]$ and $x_1, x_2 \in A$.

(2) ***lower semicontinuous (lsc)*** if the set $\{x \in A : f(x) \leq u\}$ is closed in C for any $u \in E$.

The set A is called the ***domain*** of f and is denoted by $\mathrm{dom}\, f$. Let $\mathrm{conv}(C, E)$ be the set of all convex lsc functions $f : \mathrm{dom}\, f \to E$, where $\mathrm{dom}\, f$ is a nonempty convex subset of C. For $f, g \in \mathrm{conv}(C, E)$, say that $f \leq g$ if $\mathrm{dom}\, g \subseteq \mathrm{dom}\, f$ and $f(x) \leq g(x)$ for all $x \in \mathrm{dom}\, g$. We begin by identifying some functions in $\mathrm{conv}(C, E)$. The first lemma is immediate.

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Lemma 2.1. *Let A be a nonempty closed convex subset of C and $u_0 \in E$. Define the function $\xi_{A,u_0} : A \to E$ by $\xi_{A,u_0}(x) = u_0$ for all $x \in A$. Then $\xi_{A,u_0} \in \operatorname{conv}(C, E)$.*

If $A = \{x_0\}$ for some $x_0 \in C$, then ξ_{A,u_0} is also written as ξ_{x_0,u_0}. If $x_1, x_2 \in C$, denote the line segment joining x_1, x_2 by $[x_1, x_2]$.

Lemma 2.2. *Let x_1, x_2 be distinct points in C and let $u_1, u_2 \in E$. The function $f : [x_1, x_2] \to E$ defined by*

$$f((1-\alpha)x_1 + \alpha x_2) = (1-\alpha)u_1 + \alpha u_2$$

belongs to $\operatorname{conv}(C, E)$.

Proof. The convexity of f is clear. Define $\tau : [0,1] \to C$ by $\tau(\alpha) = (1-\alpha)x_1 + \alpha x_2$. Clearly τ is a continuous function. Let $u \in E$. Then

$$\{x \in [x_1, x_2] : f(x) \le u\} = \tau\{\alpha \in [0,1] : f(\tau(\alpha)) \le u\}.$$

Since the positive cone E_+ is closed and $f \circ \tau$ is a continuous function, $\{\alpha \in [0,1] : f(\tau(\alpha)) \le u\}$ is closed in $[0,1]$. Thus

$$\{x \in [x_1, x_2] : f(x) \le u\} = \tau\{\alpha \in [0,1] : f(\tau(\alpha)) \le u\}$$

is compact and hence closed in C. This proves that f is lsc. □

From hereon, let C_1, C_2 be (nonempty) convex sets in Hausdorff topological vector spaces X_1, X_2 respectively and let E_1, E_2 be Hausdorff ordered topological vector spaces with generating positive cones. A bijection $T : \operatorname{conv}(C_1, E_1) \to \operatorname{conv}(C_2, E_2)$ such that $f_1 \le f_2 \iff Tf_1 \le Tf_2$ for any $f_1, f_2 \in \operatorname{conv}(C_1, E_1)$ is called an ***order isomorphism***. For the remainder of a section, fix an order isomorphism $T : \operatorname{conv}(C_1, E_1) \to \operatorname{conv}(C_2, E_2)$.

Lemma 2.3. *Let $f_1, f_2 \in \operatorname{conv}(X_1, E_1)$. Then $\operatorname{dom} f_1 \cap \operatorname{dom} f_2 = \emptyset$ if and only if $\operatorname{dom} Tf_1 \cap \operatorname{dom} Tf_2 = \emptyset$.*

Proof. Suppose that $x_0 \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2$. Since $(E_1)_+$ is generating, there exists $u_0 \in E_1$ such that $f_1(x_0), f_2(x_0) \le u_0$, By Lemma 2.1, $\xi_{x_0,u_0} \in \operatorname{conv}(X_1, E_1)$. Obviously, $f_i \le \xi_{x_0,u_0}$. Thus $Tf_i \le T\xi_{x_0,u_0}$. Hence $\emptyset \ne \operatorname{dom} T\xi_{x_0,u_0} \subseteq \operatorname{dom} Tf_1 \cap \operatorname{dom} Tf_2$. The reverse direction follows by symmetry. □

Lemma 2.4. *For any $x \in C_1$ and $u_1, u_2 \in E_1$, $\operatorname{dom} T\xi_{x,u_1} = \operatorname{dom} T\xi_{x,u_2}$ has exactly one point. Define $\varphi : C_1 \to C_2$ by $\{\varphi(x)\} = \operatorname{dom} T\xi_{x,u}$ for any $u \in E$. Then φ is a bijection so that $\varphi(\operatorname{dom} f) = \operatorname{dom} Tf$ for any $f \in \operatorname{conv}(C_1, E_1)$. In particular, $\varphi([x_1, x_2]) = [\varphi(x_1), \varphi(x_2)]$ for any $x_1, x_2 \in C_1$; hence φ maps convex subsets of C_1 onto convex subsets of C_2.*

Proof. Suppose that $y_i \in \operatorname{dom} T\xi_{x,u_i}$, $i = 1, 2$. Let $v \in E_2$. Then $\operatorname{dom} \xi_{y_i,v} \cap \operatorname{dom} T\xi_{x,u_i} \ne \emptyset$. Hence $\operatorname{dom} T^{-1}\xi_{y_i,v} \cap \operatorname{dom} \xi_{x,u_i} \ne \emptyset$ by Lemma 2.3. Thus $x \in \operatorname{dom} T^{-1}\xi_{y_i,v}$. Therefore, $\operatorname{dom} T^{-1}\xi_{y_1,v} \cap \operatorname{dom} T^{-1}\xi_{y_2,v} \ne \emptyset$. It follows from Lemma 2.3 again that $\operatorname{dom} \xi_{y_1,v} \cap \operatorname{dom} \xi_{y_2,v} \ne \emptyset$. So $y_1 = y_2$. This proves that $\operatorname{dom} T\xi_{x,u_1} = \operatorname{dom} T\xi_{x,u_2}$ has exactly one point.

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Define φ as above. By symmetry, there exists $\psi : C_2 \to C_1$ such that $\{\psi(y)\} = \operatorname{dom} T^{-1}\xi_{y,v}$ for any $(y, v) \in C_2 \times E_2$. In this case, $T^{-1}\xi_{y,v} = \xi_{\psi(y),u}$ for some $u \in E_1$. Then

$$\varphi(\psi(y)) = \operatorname{dom} T\xi_{\psi(y),u} = \operatorname{dom} \xi_{y,v} = y.$$

By symmetry, $\psi(\varphi(x)) = x$ for any $x \in C_1$. Hence φ is a bijection.

Let $f \in \operatorname{conv}(C_1, E_1)$. Then

$$\begin{aligned} x \in \operatorname{dom} f &\iff \operatorname{dom} \xi_{x,u} \cap \operatorname{dom} f \neq \emptyset \text{ for some } u \in E_1 \\ &\iff \operatorname{dom} T\xi_{x,u} \cap \operatorname{dom} Tf \neq \emptyset \text{ for some } u \in E_1 \\ &\iff \varphi(x) \in \operatorname{dom} Tf. \end{aligned}$$

Suppose that $x_1, x_2 \in C_1$. By Lemma 2.1, $\xi_{[x_1,x_2],u} \in \operatorname{conv}(C_1, E_1)$ for any $u \in E_1$. By the above, $\varphi([x_1, x_2]) = \operatorname{dom} T\xi_{[x_1,x_2],u}$ is a convex set in C_2. Thus $[\varphi(x_1), \varphi(x_2)] \subseteq \varphi([x_1, x_2])$. Similarly, $[x_1, x_2] \subseteq \varphi^{-1}([\varphi(x_1), \varphi(x_2)])$. Therefore, $\varphi([x_1, x_2]) = [\varphi(x_1), \varphi(x_2)]$. The final statement of the lemma follows readily. □

Lemma 2.5. *If $f \in \operatorname{conv}(C_1, E_1)$ and $x \in \operatorname{dom} f$, then $Tf(\varphi(x)) = T\xi_{x,f(x)}(\varphi(x))$.*

Proof. By Lemma 2.4, $\operatorname{dom} T\xi_{x,f(x)} = \varphi(\operatorname{dom} \xi_{x,f(x)}) = \{\varphi(x)\} \subseteq \operatorname{dom} Tf$. In particular, there exists $v \in E_2$ so that $T\xi_{x,f(x)} = \xi_{\varphi(x),v}$. Since $f \leq \xi_{x,f(x)}$, $Tf \leq \xi_{\varphi(x),v}$ and so $Tf(\varphi(x)) \leq v$. Let $w = \frac{1}{2}(Tf(\varphi(x)) + v)$. Then

$$Tf \leq \xi_{\varphi(x),w} \leq \xi_{\varphi(x),v} \implies f \leq T^{-1}\xi_{\varphi(x),w} \leq \xi_{x,f(x)}.$$

By Lemma 2.4, $\varphi(\operatorname{dom} T^{-1}\xi_{\varphi(x),w}) = \operatorname{dom} \xi_{\varphi(x),w} = \{\varphi(x)\}$. Thus there exists $u' \in E_1$ so that $T^{-1}\xi_{\varphi(x),w} = \xi_{x,u'}$. But then $f(x) \leq u' \leq f(x)$ and so $u' = f(x)$. Hence, $T^{-1}\xi_{\varphi(x),w} = \xi_{x,f(x)}$. Therefore, $\xi_{\varphi(x),v} = T\xi_{x,f(x)} = \xi_{\varphi(x),w}$, whence $v = w$. It follows that $Tf(\varphi(x)) = v = T\xi_{x,f(x)}(\varphi(x))$. □

Lemma 2.6. *There is a function $\Phi : C_1 \times E_1 \to E_2$ such that $\Phi(x, \cdot) : E_1 \to E_2$ is a bijection for all $x \in C_1$ and that*

$$Tf(y) = \Phi(\varphi^{-1}(y), f \circ \varphi^{-1}(y)) \text{ for all } f \in \operatorname{conv}(C_1, E_1) \text{ and } y \in \operatorname{dom} Tf.$$

Proof. By Lemma 2.4, $\varphi(x) \in \operatorname{dom} T\xi_{x,u}$ for any $(x, u) \in C_1 \times E_1$. Define $\Phi : C_1 \times E_1 \to E_2$ by $\Phi(x, u) = T\xi_{x,u}(\varphi(x))$. Let $f \in \operatorname{conv}(C_1, E_1)$ and let $y \in \operatorname{dom} Tf$. Then $x := \varphi^{-1}(y) \in \operatorname{dom} f$ by Lemma 2.4. By Lemma 2.5,

$$Tf(y) = T\xi_{x,f(x)}(y) = \Phi(x, f(x)) = \Phi(\varphi^{-1}(y), f \circ \varphi^{-1}(y)).$$

Note that from the proof of Lemma 2.4, the bijection $\psi : C_2 \to C_1$ associated with T^{-1} is φ^{-1}. Therefore, applying the above to T^{-1}, there exists $\Psi : C_2 \times E_2 \to E_1$ so that

$$T^{-1}g(x) = \Psi(\varphi(x), g \circ \varphi(x)) \text{ for all } g \in \operatorname{conv}(C_2, E_2) \text{ and } x \in \operatorname{dom} T^{-1}g.$$

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Take any $(x,u) \in C_1 \times E_1$. Then $T\xi_{x,u}(\varphi(x)) = \Phi(x,u)$. Hence

$$u = (T^{-1}T\xi_{x,u})(x) = \Psi(\varphi(x), T\xi_{\varphi(x),u} \circ \varphi(x)) = \Psi(\varphi(x), \Phi(x,u)).$$

Similarly, for any $v \in E_2$, $v = \Phi(x, \Psi(\varphi(x), v))$. This proves that $\Psi(\varphi(x), \cdot)$ is the inverse of $\Phi(x,\cdot)$. Therefore, $\Phi(x,\cdot) : E_1 \to E_2$ is a bijection. □

Lemma 2.7. *Let x_1, x_2 be distinct points in C_1 and let $u_1, u_2 \in E_1$. Define $f : [x_1, x_2] \to E_1$ by $f((1-\alpha)x_1 + \alpha x_2) = (1-\alpha)u_1 + \alpha u_2$. Let $g : [\varphi(x_1), \varphi(x_2)] \to E_2$ be given by*

$$g((1-\alpha)\varphi(x_1) + \alpha\varphi(x_2)) = (1-\alpha)v_1 + \alpha v_2, \ v_i = Tf(\varphi(x_i)), \ i = 1, 2.$$

Then $g = Tf$.

Proof. First of all, $f \in \operatorname{conv}(C_1, E_1)$ by Lemma 2.2. It follows from Lemma 2.4 that

$$\operatorname{dom} Tf = \varphi(\operatorname{dom} f) = \varphi([x_1, x_2]) = [\varphi(x_1), \varphi(x_2)] = \operatorname{dom} g.$$

Similarly, $\operatorname{dom} T^{-1}g = [x_1, x_2]$. Since Tf is convex, $Tf \le g$. Hence $f \le T^{-1}g$. Label $T^{-1}g(x_i)$ as u_i', $i = 1, 2$. Then

$$\Phi(x_i, u_i) = Tf(\varphi(x_i)) = g(\varphi(x_i)) = \Phi(x_i, u_i').$$

So $u_i' = u_i$. Therefore,

$$\begin{aligned} f((1-\alpha)x_1 + \alpha x_2) &\le T^{-1}g((1-\alpha)x_1 + \alpha x_2) \\ &\le (1-\alpha)T^{-1}g(x_1) + \alpha T^{-1}g(x_2) = f((1-\alpha)x_1 + \alpha x_2). \end{aligned}$$

Thus $T^{-1}g = f$ and hence $g = Tf$. □

In what follows, we seek to discover formulas for the mappings φ and Φ. Denote the zero element in the ambient space X_i of C_i, as well as the zero element in E_i, $i = 1, 2$, by the generic symbol 0. If u is a vector in E_i, the constant function with domain C_i and value u is also denoted by u; so that $u \in \operatorname{conv}(C_i, E_i)$. To simplify the notation, we make the following temporary assumption until further notice.

$$0 \in C_i, \ i = 1, 2, \text{ and } \varphi(0) = 0. \tag{2.1}$$

Also set $g_0 = T0 \in \operatorname{conv}(C_2, E_2)$. By Lemmas 2.4 and 2.6, for any $[x_1, x_2] \subseteq C_1$ and $u \in E_1$, $Tu = T(u|_{[x_1,x_2]})$ on the set $\varphi([x_1, x_2]) = [\varphi(x_1), \varphi(x_2)]$. Thus it follows from Lemma 2.7 that Tu, in particular g_0, is an affine function with domain C_2.

Lemma 2.8. *Let x_1, x_2 be distinct points in C_1. There exists a real number $c > 0$ (depending on x_1, x_2) so that*

$$Tu(\varphi(x_2)) - g_0(\varphi(x_2)) = c[Tu(\varphi(x_1)) - g_0(\varphi(x_1))] \text{ for any } u \in E_1. \tag{2.2}$$

Furthermore,

$$\varphi(\frac{x_1 + x_2}{2}) = \frac{1}{1+c}(c\varphi(x_1) + \varphi(x_2)).$$

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Proof. For simplicity, let $\tau_i : [0,1] \to E_i$ be given by

$$\tau_1(\alpha) = (1-\alpha)x_1 + \alpha x_2 \text{ and } \tau_2(\alpha) = (1-\alpha)\varphi(x_1) + \alpha\varphi(x_2).$$

Let $u \in E_1$. Define $f_1, f_2 : [x_1, x_2] \to E_1$ by

$$f_1(\tau_1(\alpha)) = \alpha u \text{ and } f_2(\tau_1(\alpha)) = (1-\alpha)u.$$

By Lemma 2.2, $f_i \in \operatorname{conv}(C_1, E_1)$. Since $f_1(x_1) = 0$, $f_1(x_2) = u$, $f_2(x_1) = u$ and $f_2(x_2) = 0$, Lemma 2.6 gives

$$Tf_1(\varphi(x_1)) = T0(\varphi(x_1)) = g_0(\varphi(x_1)),\ Tf_1(\varphi(x_2)) = Tu(\varphi(x_2)),$$
$$Tf_2(\varphi(x_1)) = Tu(\varphi(x_1)),\ Tf_2(\varphi(x_2)) = T0(\varphi(x_2)) = g_0(\varphi(x_2)).$$

Similarly, let $x_3 = \tau_1(\frac{1}{2})$. Then $f_1(x_3) = f_2(x_3)$ and thus $Tf_1(\varphi(x_3)) = Tf_2(\varphi(x_3))$ by Lemma 2.6. By Lemma 2.4, $\varphi(x_3) \in [\varphi(x_1), \varphi(x_2)]$. As φ is a bijection, there exists $\beta \in (0,1)$ such that $\varphi(x_3) = \tau_2(\beta)$. By Lemma 2.7, $Tf_i(\tau_2(\beta)) = (1-\beta)Tf_i(\varphi(x_1)) + \beta Tf_i(\varphi(x_2))$. Hence

$$Tf_1(\varphi(x_3)) = Tf_1(\tau_2(\beta)) = (1-\beta)g_0(\varphi(x_1)) + \beta\, Tu(\varphi(x_2)),$$
$$Tf_2(\varphi(x_3)) = Tf_2(\tau_2(\beta)) = (1-\beta)Tu(\varphi(x_1)) + \beta g_0(\varphi(x_2)).$$

Setting the two lines equal gives (2.2) with $c := \frac{1-\beta}{\beta} > 0$. Furthermore, c is independent of u. Finally,

$$\varphi(\frac{x_1+x_2}{2}) = \varphi(x_3) = \tau_2(\beta) = \tau_2(\frac{1}{1+c}) = \frac{1}{1+c}(c\varphi(x_1) + \varphi(x_2)).$$

This completes the proof of the lemma. □

Taking $x_1 = 0$ in Lemma 2.8 and recalling (2.1) gives a positive function $c : C_1 \to \mathbb{R}$ so that

$$Tu(\varphi(x)) - g_0(\varphi(x)) = c(x)[Tu(0) - g_0(0)] \text{ for any } u \in E_1. \tag{2.3}$$

Proposition 2.9. *Let x_1, x_2 be distinct points in C_1. For any $\alpha \in [0,1]$,*

$$\varphi((1-\alpha)x_1 + \alpha x_2) = \frac{(1-\alpha)c(x_2)\varphi(x_1) + \alpha c(x_1)\varphi(x_2)}{(1-\alpha)c(x_2) + \alpha c(x_1)}. \tag{2.4}$$

Proof. Determine $\gamma : [0,1] \to [0,1]$ by $\tau_2 \circ \gamma = \varphi \circ \tau_1$. Let $c = \frac{c(x_2)}{c(x_1)}$. An easy calculation shows that (2.4) is equivalent to $\gamma(t) = \frac{t}{(1-t)c+t}$, $t \in [0,1]$. Clearly, γ is a bijection so that $\gamma(0) = 0$ and $\gamma(1) = 1$. We claim that γ is increasing. Assume that $s, t \in [0,1]$ with $s \leq t$. Using Lemma 2.4 in the third step,

$$\tau_2(\gamma(s)) = \varphi(\tau_1(s)) \in \varphi([\tau_1(0), \tau_1(t)]) = [\varphi \circ \tau_1(0), \varphi \circ \tau_1(t)]$$
$$= [\tau_2(\gamma(0)), \tau_2(\gamma(t))] = \tau_2([\gamma(0), \gamma(t)]).$$

Thus $\gamma(s) \in [0, \gamma(t)]$. Hence $\gamma(s) \leq \gamma(t)$. This proves the claim.

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Fix a nonzero u in E_1 and let $v = Tu(\varphi(0)) - g_0(0) \neq 0$. Since both Tu and g_0 are affine on C_2, for $\alpha \in [0,1]$,

$$\begin{aligned} Tu(\tau_2(\alpha)) - g_0(\tau_2(\alpha)) &= (1-\alpha)[Tu(\varphi(x_1)) - g_0(\varphi(x_1))] \\ &\quad + \alpha[Tu(\varphi(x_2)) - g_0(\varphi(x_2))] \\ &= (1-\alpha)c(x_1)v + \alpha c(x_2)v \quad \text{by (2.3)} \\ &= c(x_1)(1-\alpha(1-c))v. \end{aligned} \tag{2.5}$$

Now we prove that $\gamma(\alpha) = \frac{\alpha}{(1-\alpha)c+\alpha}$ when $\alpha = \frac{i}{2^n}$ for some $n \in \mathbb{N} \cup \{0\}$ and $0 \le i \le 2^n$. The proof is by induction on n. The case $n = 0$ has already been observed. Assume that it holds for some n. In particular, it holds for $\alpha = \frac{i}{2^{n+1}}$ if $0 \le i \le 2^{n+1}$ and i is even. Consider $\alpha = \frac{2i-1}{2^{n+1}}$, where $1 \le i \le 2^n$. Let $x_1' = \tau_1(\frac{i-1}{2^n})$ and $x_2' = \tau_1(\frac{i}{2^n})$. By the inductive hypothesis,

$$\varphi(x_1') = \varphi(\tau_1(\frac{i-1}{2^n})) = \tau_2(\gamma(\frac{i-1}{2^n})) = \tau_2(\frac{i-1}{(2^n-i+1)c+i-1}) := \tau_2(\beta_1),$$
$$\varphi(x_2') = \tau_2(\gamma(\frac{i}{2^n})) = \tau_2(\frac{i}{(2^n-i)c+i}) := \tau_2(\beta_2).$$

Using (2.5), we see that

$$Tu(\varphi(x_i')) - g_0(\varphi(x_i')) = c(x_1)[(1-\beta_i(1-c)]v.$$

So $Tu(\varphi(x_2')) - g_0(\varphi(x_2')) = c'[Tu(\varphi(x_1')) - g_0(\varphi(x_1'))]$, where

$$c' = \frac{1-\beta_2(1-c)}{1-\beta_1(1-c)} = \frac{(2^n-i+1)c+i-1}{(2^n-i)c+i} = \frac{\frac{i-1}{\beta_1}}{\frac{i}{\beta_2}}.$$

Thus Lemma 2.8 holds for $[x_1', x_2']$ with the constant c' in place of c. Hence

$$\begin{aligned} \varphi(\tau_1(\frac{2i-1}{2^{n+1}})) &= \varphi(\frac{x_1'+x_2'}{2}) = \frac{1}{1+c'}(c'\varphi(x_1') + \varphi(x_2')) \\ &= \frac{1}{1+c'}(c'\tau_2(\beta_1) + \tau_2(\beta_2)) \\ &= \frac{1}{1+c'}[(1+c'-(c'\beta_1+\beta_2))\varphi(x_1) + (c'\beta_1+\beta_2)\varphi(x_2)] \\ &= \tau_2(\frac{c'\beta_1+\beta_2}{1+c'}) = \tau_2(\frac{2i-1}{\frac{i-1}{\beta_1}+\frac{i}{\beta_2}}) \\ &= \tau_2(\frac{\frac{2i-1}{2^{n+1}}}{(1-\frac{2i-1}{2^{n+1}})c + \frac{2i-1}{2^{n+1}}}). \end{aligned}$$

Thus $\gamma(\alpha) = \frac{\alpha}{(1-\alpha)c+\alpha}$ for $\alpha = \frac{2i-1}{2^{n+1}}$, completing the induction.

Since $\gamma : [0,1] \to [0,1]$ is an increasing bijection, it is continuous. Hence $\gamma(\alpha) = \frac{\alpha}{(1-\alpha)c+\alpha}$ for all $\alpha \in [0,1]$. □

Corollary 2.10. *If $x_1, x_2 \in C_1$ and $\alpha \in [0,1]$, then*

$$c((1-\alpha)x_1 + \alpha x_2) = \frac{c(x_1)c(x_2)}{(1-\alpha)c(x_2) + \alpha c(x_1)}.$$

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Thus $\frac{1}{c}$ is a positive affine function on C_1.

Proof. The formula is trivial if $x_1 = x_2$. Assume that x_1, x_2 are distinct. As before, let $\tau_1(\alpha) = (1-\alpha)x_1 + \alpha x_2$, $\alpha \in [0,1]$. Fix a nonzero u in E_1. By Proposition 2.9, (2.3) and the fact that Tu and g_0 are affine,

$$\begin{aligned} c(\tau_1(\alpha))[Tu(0) - g_0(0)] &= Tu(\varphi(\tau_1(\alpha))) - g_0(\varphi(\tau_1(\alpha))) \\ &= (Tu - g_0)\Big(\frac{(1-\alpha)c(x_2)\varphi(x_1) + \alpha c(x_1)\varphi(x_2)}{(1-\alpha)c(x_2) + \alpha c(x_1)}\Big) \\ &= \frac{(1-\alpha)c(x_2)}{(1-\alpha)c(x_2) + \alpha c(x_1)}\,(Tu - g_0)(\varphi(x_1)) \\ &\quad + \frac{\alpha c(x_1)}{(1-\alpha)c(x_2) + \alpha c(x_1)}\,(Tu - g_0)(\varphi(x_2)) \\ &= \frac{(1-\alpha)c(x_2)}{(1-\alpha)c(x_2) + \alpha c(x_1)}\,c(x_1)(Tu - g_0)(0) \\ &\quad + \frac{\alpha c(x_1)}{(1-\alpha)c(x_2) + \alpha c(x_1)}\,c(x_2)(Tu - g_0)(0) \\ &= \frac{c(x_1)c(x_2)}{(1-\alpha)c(x_2) + \alpha c(x_1)}\,[Tu(0) - g_0(0)]. \end{aligned}$$

Since $Tu(0) \neq g_0(0)$, the equation in the statement of the corollary is proved. The final assertion follows immediately. □

Keeping assumptions (2.1), we find a Hamel basis $(x_i)_{i\in I} \subseteq C_1$ of $\operatorname{span} C_1$. To rule out trivialities, suppose that C_1 contains more than one point, so that $\operatorname{span} C_1 \neq \{0\}$. In this case, C_2 also contains more than one point, since $\varphi : C_1 \to C_2$ is a bijection.

Proposition 2.11. *Let $c_i = c(x_i)$, $i \in I$. Define a linear transformation $L : \operatorname{span} C_1 \to \operatorname{span} C_2$ and a linear functional ℓ on $\operatorname{span} C_1$ by*

$$L(\sum a_i x_i) = \sum_i \frac{a_i}{c_i}\,\varphi(x_i) \quad \text{and} \quad \ell(\sum a_i x_i) = \sum_i a_i(\frac{1}{c_i} - 1)$$

for any real family $(a_i)_{i\in I}$ with finitely many nonzero terms. Then $\ell(x) \neq -1$ and $\varphi(x) = \frac{Lx}{1+\ell(x)}$ for any $x \in C_1$. Moreover, L is a vector space isomorphism from $\operatorname{span} C_1$ onto $\operatorname{span} C_2$.

Proof. First note that $1 + \ell$ and $\frac{1}{c}$ are two affine functions on C_1 satisfying $(1+\ell)(x_i) = \frac{1}{c(x_i)}$ for all $i \in I$ and $(1+\ell)(0) = c(0)$, since $c(0) = 1$ by (2.3). Hence $1 + \ell = \frac{1}{c}$ for all $x \in C_1$. In particular, $1 + \ell(x) \neq 0$ for $x \in C_1$.

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Suppose that $u, v, w \in C_1$ and $w = (1-\alpha)u + \alpha v$ for some $\alpha \in \mathbb{R}$. By Proposition 2.9 and the affineness of $\frac{1}{c}$,

$$\begin{aligned} \varphi(w) &= \frac{(1-\alpha)c(v)\varphi(u) + \alpha c(u)\varphi(v)}{(1-\alpha)c(v) + \alpha c(u)} \\ &= \frac{(1-\alpha)\frac{\varphi(u)}{c(u)} + \alpha\frac{\varphi(v)}{c(v)}}{\frac{1-\alpha}{c(u)} + \frac{\alpha}{c(v)}} \\ &= c(w) \cdot [(1-\alpha)\frac{\varphi(u)}{c(u)} + \alpha\frac{\varphi(v)}{c(v)}]. \end{aligned} \tag{2.6}$$

Let $C = \{x \in C_1 : \varphi(x) = c(x)L(x)\}$. By definition of L and (2.1), $x_i \in C$ for all $i \in I$ and $0 \in C$. If $u, v \in C$ and $w = (1-\alpha)u + \alpha v \in C_1$, where $\alpha \in \mathbb{R}$, then by (2.6),

$$\begin{aligned} \varphi(w) &= c(w) \cdot [(1-\alpha)\frac{\varphi(u)}{c(u)} + \alpha\frac{\varphi(v)}{c(v)}] \\ &= c(w) \cdot [(1-\alpha)L(u) + \alpha L(v)] = c(w)L(w). \end{aligned}$$

Thus $w \in C$. In particular, C is convex. If $x \in C_1 \subseteq \operatorname{span}\{x_i : i \in I\}$, there are $u, v \in \operatorname{co}\{x_i : i \in I\} \subseteq C$ and numbers $a, b \geq 0$ so that $x = au - bv$. If $a = 0$, then $w = -bv + (1+b)0 \in C$ by the above. If $a > 0$, choose k so that $0 < k < \min\{\frac{a}{1+b}, 1\}$. Then $ku \in [0, u] \subseteq C$ and $\frac{bkv}{a-k} \in [0, v] \subseteq C$. Let $\alpha = \frac{a}{k}$. We have

$$w = au - bv = (1-\alpha)\frac{bkv}{a-k} + \alpha \cdot ku \in C.$$

This proves that $C = C_1$ and hence $\varphi(x) = c(x)L(x)$ for all $x \in C_1$. Since $\frac{1}{c} = 1 + \ell$, it follows that $\varphi(x) = \frac{Lx}{1+\ell(x)}$ for any $x \in C_1$.

By symmetry, there are a linear transformation $M : \operatorname{span} C_2 \to \operatorname{span} C_1$ and a linear functional m on $\operatorname{span} C_2$ so that $\varphi^{-1}(y) = \frac{My}{1+m(y)}$ for any $y \in C_2$. (Note from the proof of Lemma 2.4 that the map $\psi : C_2 \to C_1$ associated with T^{-1} is precisely φ^{-1}.) If $x \in C_1$, then

$$x = \varphi^{-1}(\varphi(x)) = \varphi^{-1}(\frac{Lx}{1+\ell(x)}) = \frac{MLx}{1+\ell(x)+m(Lx)}. \tag{2.7}$$

Take any nonzero $x \in C_1$ and apply (2.7) to x and $\frac{x}{2}$. We find that

$$\frac{MLx}{1+\ell(x)+m(Lx)} = x = \frac{2MLx}{2+\ell(x)+m(Lx)}.$$

In particular, $MLx \neq 0$ and thus $\ell(x) + m(Lx) = 0$. Hence $MLx = x$ for any $x \in C_1$ and so the same holds for any $x \in \operatorname{span} C_1$. By symmetry, $LMy = y$ for any $y \in \operatorname{span} C_2$. Therefore, L is a vector space isomorphism from $\operatorname{span} C_1$ onto $\operatorname{span} C_2$. □

Recall the map $\Phi : C_1 \times E_1 \to E_2$ from Lemma 2.6.

Proposition 2.12. *Let $\Phi_0(u) = \Phi(0,u) - \Phi(0,0)$ for all $u \in E_1$. Then $\Phi_0 : E_1 \to E_2$ is an order preserving vector space isomorphism. For any $(x, u) \in C_1 \times E_1$,*

$$\Phi(x,u) = g_0(\varphi(x)) + \frac{\Phi_0(u)}{1+\ell(x)}. \tag{2.8}$$

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Proof. First note that if $f \in \operatorname{conv}(C_1, E_1)$ and $x \in \operatorname{dom} f$, then since $\varphi(0) = 0$ and $g_0 = T0$,

$$\Phi_0(f(x)) = \Phi(0, f(x)) - \Phi(0, 0) = Tw(0) - g_0(0),$$

where $w = f(x)$. By Lemma 2.6 and (2.3),

$$(Tf - g_0)(\varphi(x)) = \Phi(x, w) - \Phi(x, 0) = (Tw - g_0)(\varphi(x)) = c(x)\Phi_0(f(x)). \tag{2.9}$$

From the proof of Proposition 2.11, $c = \frac{1}{1+\ell}$. Hence, for any $(x, u) \in C_1 \times E_1$,

$$\Phi(x, u) = Tu(\varphi(x)) = g_0(\varphi(x)) + (Tu - g_0)(\varphi(x)) = g_0(\varphi(x)) + \frac{\Phi_0(u)}{1 + \ell(x)}.$$

Thus (2.8) holds. Let $u, v \in E_1$ be given. Choose a nonzero $x \in C_1$. By (2.8),

$$(Tu - g_0)(\varphi(x)) = \Phi(x, u) - \Phi(x, 0) = \frac{\Phi_0(u)}{1 + \ell(x)}.$$

Similarly, $(Tv - g_0)(\varphi(x)) = \frac{\Phi_0(v)}{1+\ell(x)}$. Define $f : [0, x] \to E_1$ by $f(\alpha x) = (1-\alpha)u + \alpha v$. Since $f(\frac{x}{2}) = \frac{u+v}{2}$, by (2.9),

$$(Tf - g_0)(\varphi(\frac{x}{2})) = c(\frac{x}{2})\Phi_0(\frac{u+v}{2}) = \frac{2\Phi_0(\frac{u+v}{2})}{2 + \ell(x)}.$$

By Proposition 2.11, $\varphi(x) = \frac{Lx}{1+\ell(x)}$ and

$$\varphi(\frac{x}{2}) = \frac{Lx}{2 + \ell(x)} = \frac{\varphi(0)}{2 + \ell(x)} + \frac{(1 + \ell(x))\varphi(x)}{2 + \ell(x)}.$$

Thus by Lemma 2.7, the affineness of g_0 and (2.9),

$$(Tf - g_0)(\varphi(\frac{x}{2})) = \frac{(Tf - g_0)(0) + (1 + \ell(x))(Tf - g_0)(\varphi(x))}{2 + \ell(x)} = \frac{\Phi_0(u) + \Phi_0(v)}{2 + \ell(x)}.$$

This proves that

$$\Phi_0(u) + \Phi_0(v) = 2\Phi_0(\frac{u+v}{2}).$$

Setting $v = 0$ and using $\Phi_0(0) = 0$, we find that $\Phi_0(\frac{u}{2}) = \frac{1}{2}\Phi_0(u)$. Thus, for any $u, v \in E_1$, $\Phi_0(u+v) = \Phi_0(u) + \Phi_0(v)$.

Since T is an order isomorphism, if $u \leq v$ in E_1, then $Tu \leq Tv$. In particular,

$$\Phi_0(u) = Tu(0) - \Phi(0, 0) \leq Tv(0) - \Phi(0, 0) = \Phi_0(v).$$

In the proof of Lemma 2.6, we find that if $\Psi : C_2 \times E_2 \to E_1$ is such that $\Psi(\varphi(x), \cdot)$ is the inverse of $\Phi(x, \cdot)$ for all $x \in C_1$, then $T^{-1}g(x) = \Psi(\varphi(x), g(\varphi(x)))$ for all $g \in \operatorname{conv}(C_2, E_2)$ and $x \in C_1$. Assume that $u, v \in E_1$ and $\Phi_0(u) \leq \Phi_0(v)$. Then $u' = Tu(0) \leq Tv(0) = v'$. Hence

$$u = \Psi(0, \Phi(0, u)) = \Psi(0, Tu(0)) = T^{-1}u'(0) \leq T^{-1}v'(0) = v.$$

This proves that Φ_0 preserves order (in both directions). Next, we show that Φ_0 is homogeneous. Since $E_1 = (E_1)_+ - (E_1)_+$ and Φ_0 is additive, it suffices to show that $\Phi_0(\alpha u) = \alpha\Phi_0(u)$ if $u \in (E_1)_+$ and $\alpha \geq 0$. From the additivity of Φ_0, it is easy to see that $\Phi_0(ru) = r\Phi_0(u)$ for any $r \in \mathbb{Q}$. Let

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$(r_n), (s_n)$ be nonnegative rational sequences that increase and decrease to α respectively. Since Φ_0 preserves order and is $\mathbb{Q}$-homogeneous,

$$r_n\Phi_0(u) = \Phi_0(r_n u) \leq \Phi_0(\alpha u) \leq \Phi_0(s_n u) = s_n\Phi_0(u) \text{ for all } n.$$

Thus $\alpha\Phi_0(u) \leq \Phi_0(\alpha u) \leq \alpha\Phi_0(u)$; hence $\Phi_0(\alpha u) = \alpha\Phi_0(u)$. This proves that Φ_0 is linear. As $\Phi(0,\cdot) : E_1 \to E_2$ is a bijection, we see that $\Phi_0 : E_1 \to E_2$ is a linear bijection and therefore a vector space isomorphism. □

We have reached the main result of the section. The temporary assumption (2.1) is removed from now on. Suppose that C is a convex set in a vector space X. For any $x_1, x_2 \in C$, $\text{span}(C - x_1) = \text{span}(C - x_2)$.

Theorem 2.13. *For $i = 1, 2$, let C_i be a convex set with more than one point in a Hausdorff topological vector space X_i and let E_i be a nonzero Hausdorff ordered topological vector space whose positive cone is generating. Assume that $T : \text{conv}(C_1, E_1) \to \text{conv}(C_2, E_2)$ is an order isomorphism. Let $\varphi : C_1 \to C_2$ be the bijection associated with T from Lemma 2.4. Take $x_0 \in C_1$ and set $D_1 = \text{span}(C_1 - x_0)$, $y_0 = \varphi(x_0)$ and $D_2 = \text{span}(C_2 - y_0)$. There are a linear functional $\ell : D_1 \to \mathbb{R}$ and a vector space isomorphism $L : D_1 \to D_2$ so that*

$$\ell(x - x_0) \neq -1 \text{ and } \varphi(x) = y_0 + \frac{L(x - x_0)}{1 + \ell(x - x_0)} \text{ for all } x, x_0 \in C_1.$$

Futhermore, there are a lsc affine function $g : C_2 \to E_2$ and an order preserving vector space isomorphism $\Phi_0 : E_1 \to E_2$ so that

$$Tf(y) = g(y) + \frac{\Phi_0(f \circ \varphi^{-1}(y))}{1 + \ell(\varphi^{-1}(y) - x_0)}$$

for all $f \in \text{conv}(C_1, E_1)$ and $y \in \text{dom}\, Tf = \varphi(\text{dom}\, f)$.

Proof. Let $x_0 \in C_1$ and $y_0 = \varphi(x_0)$. Obtain a function $\Phi : C_1 \times E_1 \to E_2$ from Lemma 2.6. Set $C_1' = C_1 - x_0$ and $C_2' = C_2 - y_0$. For any $f \in \text{conv}(C_1, E_1)$ and $g \in \text{conv}(C_2, E_2)$, define $j_1 f : C_1' \to E_1$ and $j_2 g : C_2' \to E_2$ by

$$j_1 f(x - x_0) = f(x) \text{ and } j_2 g(y - y_0) = g(y) \text{ for all } x \in C_1, y \in C_2.$$

Clearly $j_i : \text{conv}(C_i, E_i) \to \text{conv}(C_i', E_i)$ is an order isomorphism. Thus $T' := j_2 T j_1^{-1} : \text{conv}(C_1', E_1) \to \text{conv}(C_2', E_2)$ is an order isomorphism. Using Lemmas 2.4 and 2.6, obtain φ' and Φ' with respect to T'. For any $x \in C_1$ and $f \in \text{conv}(C_1, E_1)$, let $z = \varphi^{-1}(\varphi'(x - x_0) + y_0)$. Then

$$\begin{aligned} \Phi'(x - x_0, f(x)) &= \Phi'(x - x_0, (j_1 f)(x - x_0)) \\ &= (T' j_1 f)(\varphi'(x - x_0)) \\ &= (j_2 T f)(\varphi'(x - x_0)) \\ &= Tf(\varphi'(x - x_0) + y_0) = \Phi(z, f(z)). \end{aligned} \tag{2.10}$$

If $z \neq x$, for any $u, v \in E_1$, there exists $f \in \text{conv}(C_1, E_1)$ so that $f(x) = u$ and $f(z) = v$. Thus $\Phi'(x - x_0, u) = \Phi(z, v)$ for all $u, v \in E_1$. This is absurd since $\Phi'(x - x_0, \cdot)$ and $\Phi(x_0, \cdot)$ both map

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onto $E_2 \neq \{0\}$. Thus $\varphi'(x - x_0) = \varphi(x) - y_0$. In particular, $\varphi'(0) = 0$. So assumptions (2.1) hold for $\varphi' : C_1' \to C_2'$. Set $g_0 = T'0 \in \text{conv}(C_2', E_2)$. Note that $\text{dom}\, g_0 = C_2'$ and that $D_i = \text{span}\, C_i'$. By Propositions 2.11 and 2.12, there are a vector space isomorphism $L : D_1 \to D_2$, a linear functional ℓ on D_1 and an order preserving vector space isomorphism $\Phi_0 : E_1 \to E_2$ so that

$$\varphi'(x - x_0) = \frac{L(x - x_0)}{1 + \ell(x - x_0)} \text{ and } \Phi'(x - x_0, \cdot) = g_0(\varphi'(x - x_0)) + \frac{\Phi_0(\cdot)}{1 + \ell(x - x_0)}$$

for all $x \in C_1$. Furthermore, $\ell(x - x_0) \neq -1$ for all $x \in C_1$. By (2.10) and since $\varphi'(x - x_0) = \varphi(x) - y_0$, for any $f \in \text{conv}(C_1, E_1)$ and $y \in \text{dom}\, Tf$,

$$\varphi(x) = y_0 + \frac{L(x - x_0)}{1 + \ell(x - x_0)} \text{ and} \tag{2.11}$$

$$\begin{aligned} Tf(y) &= \Phi(\varphi^{-1}(y), f \circ \varphi^{-1}(y)) = \Phi'(\varphi^{-1}(y) - x_0, f \circ \varphi^{-1}(y)) \\ &= g_0(y - y_0) + \frac{\Phi_0(f \circ \varphi^{-1}(y))}{1 + \ell(\varphi^{-1}(y) - x_0)}. \end{aligned} \tag{2.12}$$

Define g by $g(y) = g_0(y - y_0)$. Then g is lsc, affine and $\text{dom}\, g = C_2$. This completes the proof of the theorem. □

The corollary below concerns the special case when C_i is the entire topological vector space X_i.

Corollary 2.14. *For $i = 1, 2$, let $X_i \neq \{0\}$ be a Hausdorff topological vector space and let $E_i \neq \{0\}$ be a Hausdorff ordered topological vector space whose positive cone is generating. Assume that $T : \text{conv}(X_1, E_1) \to \text{conv}(X_2, E_2)$ is an order isomorphism. Let $y_0 = \varphi(0) \in C_2$. There are an lsc affine function $g_0 : X_2 \to E_2$, a vector space isomorphism $L : X_1 \to X_2$ and an order preserving vector space isomorphism $\Phi_0 : E_1 \to E_2$ such that for all $f \in \text{conv}(X_1, E_1)$, $\text{dom}\, Tf = y_0 + L(\text{dom}\, f)$ and that*

$$Tf(y) = g_0(y) + \Phi_0(f(L^{-1}(y - y_0))) \textit{ for all } f \in \text{conv}(X_1, E_1) \textit{ and } y \in \text{dom}\, Tf.$$

Proof. Take $x_0 = 0$ in Theorem 2.13 to obtain maps ℓ, L, g and Φ_0. Since $\ell(x) \neq -1$ for all $x \in X_1$, $\ell = 0$. By Theorem 2.13, $\varphi(x) = \varphi(0) + L(x - x_0) = y_0 + Lx$. By Lemma 2.4, $\text{dom}\, Tf = \varphi(\text{dom}\, f) = y_0 + L(\text{dom}\, f)$. Clearly, $\varphi^{-1}(y) = L^{-1}(y - y_0)$. Thus

$$Tf(y) = g(y) + \Phi_0(f \circ \varphi^{-1}(y)) = g(y) + \Phi_0(f(L^{-1}(y - y_0))).$$

□

Remark. Assume that $E_i = \mathbb{R}$ for $i = 1, 2$. The order preserving linear isomorphism $\Phi_0 : \mathbb{R} \to \mathbb{R}$ is given by multiplication by some $a > 0$. Thus Corollary 2.14 gives [3, Corollary 8]. When C_i is a convex set in $\mathbb{R}^n$ and $E_i = \mathbb{R}$, Theorem 2.13 is obtained by Cheng and Luo [4].

In Theorem 2.13, the maps ℓ, L, g and Φ_0 may depend on the "base point" x_0. To anticipate the next section, we will work out the the form of the corresponding maps ℓ_1, L_1, g_1 and Φ_1 when the base point changes to some $x_1 \in C_1$. For any $u \in E_1$, take u to be the constant function on C_1 with value u. Using (2.12) at both base points and taking $y_1 = \varphi(x_1)$,

$$g(y) + \frac{\Phi_0(u)}{1 + \ell(\varphi^{-1}(y) - x_0)} = Tu(y) = g_1(y) + \frac{\Phi_1(u)}{1 + \ell_1(\varphi^{-1}(y) - x_1)}$$

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for all $y \in C_2$. Since Φ_0 and Φ_1 are linear, taking $u = 0$ shows that $g_1 = g$. Set $a_0 = \ell(x_1 - x_0)$ and $a_1 = \ell_1(x_0 - x_1)$, Taking $y = y_1$ gives $\Phi_1(u) = \frac{\Phi_0(u)}{1+a_0}$. Put this back into the equation above and substitute $z = \varphi^{-1}(y) - x_0$. As Φ_0 is not the zero map,

$$1 + \ell(z) = (1 + a_0)\,[1 + a_1 + \ell_1(z)] \text{ for all } z \in C_1 - x_0.$$

By the linearity of ℓ and ℓ_1, the foregoing equation holds for all $z \in D_1$. Thus

$$(1 + a_0)(1 + a_1) = 1 \text{ and } \ell_1(z) = \frac{\ell(z)}{1 + a_0} = (1 + a_1)\ell(z).$$

Simiarly, for any $x \in C_1$, let $z = x - x_0 \in C_1 - x_0$. From (2.11),

$$\begin{aligned} y_0 + \frac{L(z)}{1 + \ell(z)} = \varphi(x) &= y_1 + \frac{L_1(z + x_0 - x_1)}{1 + \ell_1(z) + \ell_1(x_0 - x_1)} \\ &= y_1 + \frac{L_1(z + x_0 - x_1)}{1 + (1 + a_1)\ell(z) + a_1} \\ &= y_1 + \frac{L_1(z + x_0 - x_1)}{(1 + a_1)(1 + \ell(z))}. \end{aligned}$$

In particular, at $z = 0$, we find that $L_1(x_0 - x_1) = (1 + a_1)(y_0 - y_1)$. Hence

$$L_1(z) = (1 + a_1)[L(z) - (y_1 - y_0)\ell(z)] = \frac{L(z) - (y_1 - y_0)\ell(z)}{1 + a_0}.$$

3. **Continuity**

In this section, we investigate the continuity of the the maps L, ℓ and Φ_0 arising from Theorem 2.13, under appropriate settings.

Lemma 3.1. *In the situation of Theorem 2.13, the map φ maps closed convex subsets of C_1 onto closed convex subsets of C_2.*

Proof. Let W be a closed convex set in C_1. Let $h = T^{-1}0 \in \operatorname{conv}(C_1, E_1)$. Then $\operatorname{dom} h = C_1$. Now $h_0 = h|_W : W \to E_1$ is convex and lsc since

$$\{x : h_0(x) \le u\} = \{x : h(x) \le u\} \cap W$$

is closed in C for any $u \in E$. Thus $h_0 \in \operatorname{conv}(C_1, E_1)$. By Lemma 2.4, $\operatorname{dom} Th_0 = \varphi(\operatorname{dom} h_0) = \varphi(W)$. By Theorem 2.13, for any $y \in \operatorname{dom} Th_0$, $Th_0(y) = Th(y) = 0$. Thus $0|_{\varphi(W)} = Th_0 \in \operatorname{conv}(C_2, E_2)$. Therefore,

$$\varphi(W) = \{y \in \operatorname{dom} Th_0 : Th_0(y) \le 0\}$$

is closed in C_2. □

The next theorem is the main result on continuity. Denote the weak topology $\sigma(X_i, X_i^*)$ by σ_i, $i = 1, 2$.

Theorem 3.2. *In the notation of Theorem 2.13, assume that X_i is locally convex Hausdorff and that C_i has nonempty interior in X_i. Then ℓ is a continuous linear functional on X_1 and $L : X_1 \to X_2$ is an isomorphism of the topological vector space (X_1, σ_1) onto (X_2, σ_2). Thus $\varphi : (C_1, \sigma_1) \to (C_2, \sigma_2)$ is a homeomorphism.*

Proof. Since C_i has nonempty interior, $D_i = X_1$, $i = 1, 2$, in the notation of Theorem 2.13. Thus ℓ is a linear functional on X_1 and $L : X_1 \to X_2$ is a vector space isomorphism. (These maps are obtained at the base point x_0.) Let x_1 be an interior point of C_1, with corresponding maps ℓ_1 and L_1. Let U be a circled open neighborhood of 0 in X_1 so that $x_1 + U \subseteq C_1$. By Theorem 2.13, $\ell_1(x) \neq -1$ for all $x \in U$. Thus $|\ell_1(x)| < 1$ for all $x \in U$. It follows easily that ℓ_1 is continuous at 0 and hence continuous on X_1.

Next, we show that φ is σ_1-to-σ_2 continuous at x_0. Otherwise, there are a net (x_α) in C_1 that σ_1-converges to x_0, $y^* \in X_2^*$ and $r > 0$ so that $y^*(\varphi(x_\alpha)) > y^*(\varphi(x_0)) + r$ for all α. Let

$$W = \{y \in C_2 : y^*(y) \geq y^*(\varphi(x_0)) + r\}.$$

Then W is a closed convex set in C_2. Apply Lemma 2.4 to φ^{-1} to see that $\varphi^{-1}(W)$ is a closed convex set in C_1. By choice, $x_\alpha \in \varphi^{-1}(W)$ for all α and hence $x_0 \in \varphi^{-1}(W)$, i.e., $\varphi(x_0) \in W$, which is obviously absurd. This completes the proof of the claim.

Since ℓ_1 is continuous and x_1 is an interior point of C_1, it follows from the expression for φ in Theorem 2.13 (at x_1) that L_1 is σ_1-to-σ_2 continuous at x_1. Hence L_1 is σ_1-to-σ_2 continuous on X_1. Let $y_i = \varphi(x_i), i = 0, 1$. By the final paragraph in §2, there is a real constant a_0 so that for all $z \in X_1$,

$$\ell_1(z) = \frac{\ell(z)}{1 + a_0} \text{ and } L_1(z) = \frac{L(z) - (y_1 - y_0)\ell(z)}{1 + a_0}.$$

Clearly the continuity of ℓ and the σ_1-to-σ_2 continuity of L follow from that of ℓ_1 and L_1.

Applying the above to T^{-1} at y_0 gives a continuous linear functional m and a σ_2-to-σ_1 continuous linear map $M : X_2 \to X_1$ so that

$$\varphi^{-1}(y) = x_0 + \frac{M(y - y_0)}{1 + m(y - y_0)},\ y \in C_2.$$

Since $y = \varphi(\varphi^{-1}(y))$ for all $y \in C_2$, one easily deduces that $M = L^{-1}$. This proves that L is a an isomorphism of the topological vector space (X_1, σ_1) onto (X_2, σ_2). Therefore, $\varphi : (C_1, \sigma_1) \to (C_2, \sigma_2)$ is a homeomorphism by the formula for φ in Theorem 2.13 and the formula for φ^{-1} above. □

Remark. It follows from the σ_1-σ_2 continuity of L that the graph of L is closed in $X_1 \times X_2$ (in their original topologies). Therefore, if X_i's are in addition completely metrizable, then it follows from the Closed Graph Theorem that $L : X_1 \to X_2$ is a topological vector space isomorphism with respect to the original topologies on X_i, $i = 1, 2$.

Let E be an ordered vector space. A subset A of E is ***solid*** if $x, y \in A$ and $x \leq z \leq y$ imply that $z \in A$. The topology on E is ***locally solid*** if there exists a local basis at 0 consisting of solid sets. If E is locally solid and $(a_n), (b_n)$ are sequences in E so that $0 \leq a_n \leq b_n$ and (b_n) converges to 0, then (a_n) converges to 0 as well.

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Theorem 3.3. *Let the notation and assumptions be as in Theorems 2.13 and 3.2. Assume additionally that for $i = 1, 2$, E_i is locally solid, completely metrizable and that for every null sequence (u_n) in E_i, there is a positive null sequence (v_n) so that $u_n \leq v_n$ for all n. Then $\Phi_0 : E_1 \to E_2$ is an isomorphism of ordered topological vector spaces.*

Proof. By Theorem 2.13, Φ_0 is an order preserving vector space isomorphism. It suffices to show that Φ_0 is continuous (at 0); continuity of Φ_0^{-1} follows by symmetry. Let (u_n) be a null sequence in E_1, we show that there is a subsequence (u_{n_k}) so that $(\Phi_0(u_{n_k}))_k$ converges to 0 in E_2. Let (v_n) be as given in the statement of the theorem and set $w_n = v_n - u_n$. Then $(v_n), (w_n)$ are positive null sequences. Since E_1 is completely metrizable, by a result of Klee [7], we may assume that the metric d on E_1 is complete and translation invariant. Let $n_1 < n_2 < \cdots$ be chosen so that $d(v_{n_k} + w_{n_k}, 0) \leq \frac{1}{k2^k}$ for all k. By translation invariance $d(k(v_{n_k} + w_{n_k}), 0) \leq \frac{1}{2^k}$ and hence $\sum_{k=1}^{\infty} k(v_{n_k} + w_{n_k})$ converges to an element a in E_1. Then $0 \leq v_{n_k}, w_{n_k} \leq \frac{a}{k}$ for all k. Since Φ_0 is order preserving and linear,

$$0 \leq \Phi_0(v_{n_k}), \Phi_0(w_{n_k}) \leq \frac{1}{k}\Phi_0(a) \text{ for all } k.$$

Use the local solidity of E_2 to conclude that $(\Phi_0(v_{n_k})), (\Phi_0(w_{n_k}))$ both converge to 0. Thus $(\Phi_0(u_{n_k}))$ converges to 0, as claimed. □

We collect together the foregoing results in the following corollary.

Corollary 3.4. *For $i = 1, 2$, let C_i be a convex set with nonempty interior in a locally convex Hausdorff topological vector space $X_i \neq \{0\}$ and let E_i be a nonzero locally solid completely metrizable ordered topological vector space whose positive cone is generating. Moreover, assume that if (u_n) is a null sequence in E_i, then there is a positive null sequence (v_n) in E_i so that $u_n \leq v_n$ for all n. Suppose that $T : \mathrm{conv}(C_1, E_1) \to \mathrm{conv}(C_2, E_2)$ is an order isomorphism. If $x_0 \in C_1$ and $y_0 = \varphi(x_0) \in C_2$, then there are*

(1) a lsc affine function $g_0 : C_2 \to E_2$,
(2) an isomorphism of topological vector spaces $L : (X_1, \sigma_1) \to (X_2, \sigma_2)$,
(3) a continuous linear functional $\ell : X_1 \to \mathbb{R}$ and
(4) an isomorphism of ordered topological vector spaces $\Phi_0 : E_1 \to E_2$

so that $\ell(x - x_0) \neq -1$, $\varphi(x) = y_0 + \frac{L(x-x_0)}{1+\ell(x-x_0)}$ for all $x \in C_1$ and

$$Tf(y) = g_0(y) + \frac{\Phi_0(f \circ \varphi^{-1}(y))}{1 + \ell(\varphi^{-1}(y) - x_0)},\ y \in \mathrm{dom}\, Tf = \varphi(\mathrm{dom}\, f),\ f \in \mathrm{conv}(C_1, E_1).$$

Remark. Clearly, if E_i is a completely metrizable locally solid topological vector lattice, then it satisfies the assumptions of Corollary 3.4. In particular, this occurs if E_i is a Banach lattice. We give two other examples which are not necessarily lattices.

(1) Let E_i be the space of self-adjoint elements in a C^*-algebra A, equipped with the norm topology and the usual order ($0 \leq a$ if and only if $a = b^*b$ for some $b \in A$).

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(2) Let E_i be the space of regular operators on a Banach lattice F; i..e., the space of operators $T : F \to F$ that can be written as $T = S - R$, where $S, R : F \to F$ are positive (linear) operators, with the order $T_1 \le T_2$ if $T_2 - T_1$ is positive, and the norm

$$|||T||| = \inf\{\|S\| + \|R\| : T = S - R, S, R \text{ positive}\}.$$

Here $\|\cdot\|$ is the operator norm.

4. Order anti-isomorphisms

As before, for $i = 1, 2$, let C_i be a convex set in a Hausdorff topological vector space X_i and let E_i be an ordered vector space. A bijection $T : \mathrm{conv}(C_1, E_1) \to \mathrm{conv}(C_2, E_2)$ is an ***order anti-isomorphism*** if $f \le g$ if and only if $Tg \le Tf$ for all $f, g \in \mathrm{conv}(C_1, E_1)$. If $E_i = \mathbb{R}$, abbreviate $\mathrm{conv}(C_i, \mathbb{R})$ to $\mathrm{conv}(C_i)$. Order anti-isomorphisms $T : \mathrm{conv}(\mathbb{R}^n) \to \mathrm{conv}(\mathbb{R}^n)$ are characterized in [3, Theorem 7], which then leads to a characterization of the Legendre transform [3, Theorem 1]. A generalization [6, Theorem 2] characterizes order anti-isomorphisms $T : \mathrm{conv}(X) \to \mathrm{conv}(X^*, \sigma^*)$, where X is a Banach space and (X^*, σ^*) means X^* with the weak*-topology. It is shown in [4] that if C is a nonempty convex set in $\mathbb{R}^n$ and there is an order anti-isomorphism from $T : \mathrm{conv}(C) \to \mathrm{conv}(C)$, then C is either $\mathbb{R}^n$ or a singleton. Another result by the same authors [5] shows that for a Banach space X, there is an order anti-isomorphism $T : \mathrm{conv}(X) \to \mathrm{conv}(X)$ if and only if X is reflexive and X is isomorphic to X^*.

Corollary 2.14 allows us to prove the essential uniqueness of order anti-isomorphisms *if such a mapping exists*. Let X_3 be a Hausdorff topological vector space and let E_3 be a Hausdorff ordered topological vector space. Denote the weak topology $\sigma(X_i, X_i^*)$ by σ_i, $i = 1, 2, 3$.

Theorem 4.1. *Let $T : \mathrm{conv}(X_1, E_1) \to \mathrm{conv}(X_2, E_2)$ and $S : \mathrm{conv}(X_1, E_1) \to \mathrm{conv}(X_3, E_3)$ be order anti-isomorphisms. Then there are $y_0 \in X_2$, an lsc affine function $g_0 : X_2 \to E_2$, a vector space isomorphism $L : X_3 \to X_2$ and an order preserving vector space isomorphism $\Phi_0 : E_3 \to E_2$ such that for all $f \in \mathrm{conv}(X_1, E_1)$, $\mathrm{dom}\, Tf = y_0 + L(\mathrm{dom}\, Sf)$ and*

$$Tf(y) = g_0(y) + \Phi_0((Sf)(L^{-1}(y - y_0))) \text{ for all } y \in \mathrm{dom}\, Tf. \tag{4.1}$$

Furthermore, if X_i, $i = 1, 2, 3$, are locally convex Hausdorff, then (X_3, σ_3) and (X_2, σ_2) are linearly homeomorphic via L.

Proof. The map $TS^{-1} : \mathrm{conv}(X_3, E_3) \to \mathrm{conv}(X_2, E_2)$ is an order isomorphism. Obtain $y_0 \in X_2$, an lsc affine function $g_0 : X_2 \to E_2$, a vector space isomorphism $L : X_3 \to X_2$ and an order preserving vector space isomorphism $\Phi_0 : E_3 \to E_2$ by Corollary 2.14 with respect to TS^{-1}. For all $f \in \mathrm{conv}(X_1, E_1)$,

$$\mathrm{dom}\, Tf = \mathrm{dom}\, TS^{-1}(Sf) = y_0 + L(\mathrm{dom}\, Sf) \text{ and}$$
$$Tf(y) = (TS^{-1})(Sf)(y) = g_0(y) + \Phi_0((Sf)(L^{-1}(y - y_0)))$$

for all $y \in \mathrm{dom}(TS^{-1})(Sf) = \mathrm{dom}\, Tf$. The final asssertion follows from Theorem 3.2. □

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Lemma 4.2. *If $g : X_2 \to \mathbb{R}$ is a lsc affine function, then there exist a continuous linear functional $y^* \in X_2^*$ and $a \in \mathbb{R}$ so that $g(y) = a + y^*(y)$ for all $y \in X_2$.*

Proof. The functional $h : X_2 \to \mathbb{R}$ defined by $h(y) = g(y) - g(0)$ is linear since g is affine. By the lower semicontinuity of g, there is a balanced open neighborhood V of 0 in X_2 so that

$$V \subseteq \{y : g(y + y_0) > g(y_0) - 1\} = \{y : h(y) > -1\}.$$

Then $|h(y)| < 1$ for all $y \in V$. Thus $y^* := h \in X_2^*$. Finally, set $a = g(0)$ to complete the proof of the lemma. □

If X_1 is locally convex Hausdorff and σ_1^* is the topology $\sigma(X_1^*, X_1)$ on X_1^*, then the ***Legendre transform*** $\mathcal{L} : \text{conv}(X_1) \to \text{conv}(X_1^*, \sigma_1^*)$ is known to be an order anti-isomorphism, where

$$(\mathcal{L}f)(x^*) = \sup\{x^*(x) - f(x) : x \in \text{dom}\, f\}$$

and dom $\mathcal{L}f$ is the set where the sup is finite. Thus we have the following corollary of Theorem 4.1.

Corollary 4.3. *Let X_1, X_2 be locally convex Hausdorff spaces and let $\mathcal{L} : \text{conv}(X_1) \to \text{conv}(X_1^*, \sigma_1^*)$ be the Legendre transform. If $T : \text{conv}(X_1) \to \text{conv}(X_2)$ is an order anti-isomorphism, then there are $y_0 \in X_2$, $y_0^* \in X_2^*$, $a, b \in \mathbb{R}$ with $b > 0$, an isomorphism of topological vector spaces $L : (X_1^*, \sigma_1^*) \to (X_2, \sigma_2)$ so that for all $f \in \text{conv}(X_1)$, $\text{dom}\, Tf = y_0 + L(\text{dom}\, \mathcal{L}f)$ and*

$$Tf(y) = a + y_0^*(y) + b \cdot (\mathcal{L}f)(L^{-1}(y - y_0)) \text{ for all } y \in \text{dom}\, Tf.$$

Proof. Take $X_3 = (X_1^*, \sigma_1^*)$ and $S = \mathcal{L}$ in Theorem 4.1 to obtain y_0, g_0, L and Φ_0. By Lemma 4.2, there are $a \in \mathbb{R}$ and $y_0^* \in X_2^*$ so that $g_0(y) = a + y^*(y)$ for all $y \in X_2$. Also, $\Phi_0 : \mathbb{R} \to \mathbb{R}$ is an order preserving linear map and hence is given by multiplication by some $b > 0$. Note that $X_3^* = X_1$ and thus $\sigma(X_3, X_3^*) = \sigma_1^*$. So L is a topological vector space isomorphism from (X_1^*, σ_1^*) onto (X_2, σ_2). The corollary now follows from (4.1). □

If $X_1 = X_2 = \mathbb{R}^n$, then Corollary 4.3 yields [3, Theorem 7]. Suppose that $X_1 = X$ is a Banach space and $(X_2, \sigma_2) = (X^*, \sigma^*)$. Then $L : (X^*, \sigma^*) \to (X^*, \sigma^*)$ is a linear homeomorphism. Hence $L = M^*$, where $M : X \to X$ is a Banach space isomorphism. So in this case we obtain [6, Theorem 2]. Finally, if $X_1 = X_2 = X$ is a Banach space, then L is a linear homeomorphism from (X^*, σ^*) onto (X, σ), where σ is the weak topology on X. Hence the ball of X is the image under L of a relatively compact set in (X^*, σ^*). In particular, X is reflexive. Hence $L : X^* \to X$ is weak-to-weak continuous and thus it is a Banach space isomorphism. This gives the result in [5] mentioned above. It is also possible to obtain a generalization of [3, Theorem 1].

Corollary 4.4. *Let X be a locally convex Hausdorff space. Suppose that $T : \text{conv}(X) \to \text{conv}(X)$ is an order anti-isomorphism such that $T(Tf) = f$ for all $f \in \text{conv}(X)$. Then there are $x_0 \in X$, $x_0^* \in X^*$, $a, b \in \mathbb{R}$ with $b > 0$ and a linear homeomorphism $L : (X^*, \sigma^*) \to (X, \sigma)$ such that*

$$Tf(x) = a + x_0^*(x) + b \cdot (\mathcal{L}f)(L^{-1}(x - x_0)) \text{ for all } f \in \text{conv}(X) \text{ and } x \in \text{dom}\, Tf.$$

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Proof. Use Corollary 4.3 with $X_1 = X_2 = X$ to obtain x_0, x_0^*, a, b and $L : (X^*, \sigma^*) \to (X, \sigma)$ corresponding to T^{-1}. For any $f \in \mathrm{conv}(X)$ and $x \in \mathrm{dom}\, Tf$,

$$Tf(x) = T^{-1}f(x) = a + x_0^*(x) + b \cdot (\mathcal{L}f)(L^{-1}(x - x_0)).$$

□

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Denny H. Leung

Department of Mathematics, National University of Singapore Singapore, Republic of Singapore.

Email: dennyhl@u.nus.edu