Journal of the Iranian Mathematical Society
ISSN (on-line): 2717-1612
J. Iran. Math. Soc. 4 (2023), no. 1, 5-25

ARENS REGULARITY OF IDEALS IN $A(G)$, $A_{cb}(G)$ AND $A_M(G)$

B. FORREST*, J. SAWATZKY AND A. THAMIZHAZHAGAN

Dedicated to professor A. T.-M. Lau

Abstract. n this paper, we look at the question of when various ideals in the Fourier algebra $A(G)$ or its closures $A_M(G)$ and $A_{cb}(G)$ in, respectively, its multiplier and cb-multiplier algebra are Arens regular. We show that in each case, if a non-zero ideal is Arens regular, then the underlying group G must be discrete. In addition, we show that if an ideal I in $A(G)$ has a bounded approximate identity, then it is Arens regular if and only if it is finite-dimensional.

1. Introduction

Let G be a locally compact group with a fixed left Haar measure μ. Let Σ_G denote the collection of equivalence classes of weakly continuous unitary representations of G. The Fourier-Stieltjes algebra, $B(G)$ is the space of all coefficient functions of weakly continuous unitary representations on G. That is

$$B(G) = \{u(x) = \langle \pi(x)\xi, \eta\rangle | \pi \in \Sigma_G, \xi, \eta \in \mathcal{H}_\pi\}.$$

$B(G)$ is a commutative Banach algebra under pointwise operations when given the norm it inherits as the dual of the group C^*-algebra $C^*(G)$.

The left regular representation λ on $L^2(G)$ is defined

$$\lambda(x)(f)(y) = f(x^{-1}y)$$

Communicated by Massoud Amini
MSC(2020): Primary: 43A07; Secondary: 43A22, 46J10, 47L25.
Keywords: Fourier algebra, multipliers, Arens regularity, uniformly continuous functionals, topologically invariant means.
This research is partially supported by the Natural Sciences and Engineering Research Council of Canada.
Received: 13 February 2023, Accepted: 19 May 2023.
∗Corresponding author
DOI: https://dx.doi.org/10.30504/JIMS.2023.385500.1091

.

for each $x, y \in G$. The Fourier algebra $A(G)$ consists of all coefficient functions of λ. That is all functions of the form

$$u(x) = \langle \lambda(x)f, g\rangle$$

where $f, g \in L^2(G)$. The Fourier algebra is a closed ideal of $B(G)$ with $\Delta(A(G)) = G$. It is also the predual of the von Neumann subalgebra $VN(G)$ of $B(L^2(G))$ generated by $\{\lambda(x) \mid x \in G\}$. It can also be realized as the closure of the space of elements in $B(G)$ with compact support. (See [8] and [18] for more on the nature of $B(G)$ and $A(G)$.)

It is well known that the multiplication on any Banach algebra $\mathcal{A}$ can be extended to its second dual in two natural ways as demonstrated by Arens in [1]. Generally, these two Arens products are different. If they agree, we say that the Banach algebra $\mathcal{A}$ is *Arens regular*. For a commutative Banach algebra $\mathcal{A}$, Arens regularity occurs precisely when the second dual $\mathcal{A}^{**}$ is commutative.

The most well-known examples of Arens regular algebras are the closed subalgebras of $B(\mathcal{H})$, the bounded operators on a Hilbert space $\mathcal{H}$. Typically, many of the algebras arising from locally compact groups fail to be Arens regular unless the group structure is significantly restricted. For example, the first author showed that if $A(G)$ is Arens regular, then G must be discrete and every amenable subgroup of G must be finite [10, Theorem 3.2 and Proposition 3.7] extending an earlier result of Lau's for amenable groups [19, Proposition 3.3]. In fact, it is conjectured, and strongly believed to be true, that whenever $A(G)$ is Arens regular, G must be finite.

The literature concerning Arens regularity of algebras arising from locally compact groups is extensive. For the Fourier algebra in particular see for example [10, 11, 15, 19, 20] and [23]

In this paper, we will consider the possible Arens regularity of ideals in $A(G)$ and in two related Banach algebras $A_M(G)$ and $A_{cb}(G)$, that arise from $A(G)$ as its closure in its space of multipliers and completely bounded multipliers respectively. We will show that if I is a non-zero closed ideal in any of these three algebras that is Arens regular, then G must be discrete.

2. **Preliminaries and notation**

Throughout this paper, $\mathcal{A}$ will denote a Banach algebra. In this case, the dual $\mathcal{A}^*$ becomes a Banach $\mathcal{A}$-bimodule with respect to the module actions

$$\langle u \cdot T, v\rangle = \langle T, vu\rangle$$

and

$$\langle T \square u, v\rangle = \langle T, uv\rangle$$

for every $u, v \in \mathcal{A}$ and $T \in \mathcal{A}^*$.

It is well known that there are two natural products that can be used to extend the multiplication of $\mathcal{A}$ to its second dual $\mathcal{A}^{**}$. These two Arens products are defined as follows:

1a) $\langle u \cdot T, v\rangle = \langle T, vu\rangle$ for every $u, v \in \mathcal{A}$ and $T \in \mathcal{A}^*$.

1b) $\langle T \odot m, u\rangle = \langle m, u \cdot T\rangle$ for every $u \in \mathcal{A}$ and $T \in \mathcal{A}^*$ and $m \in \mathcal{A}^{**}$.

DOI: https://dx.doi.org/10.30504/JIMS.2023.385500.1091

1c) $\langle m \odot n, T\rangle = \langle n, T \odot m\rangle$ for every $T \in \mathcal{A}^*$ and $m, n \in \mathcal{A}^{**}$.

2a) $\langle T \Box u, v\rangle = \langle T, uv\rangle$ for every $u, v \in \mathcal{A}$ and $T \in \mathcal{A}^*$.

2b) $\langle m \Box T, u\rangle = \langle m, T \Box u\rangle$ for every $u \in \mathcal{A}$ and $T \in \mathcal{A}^*$ and $m \in \mathcal{A}^{**}$.

2c) $\langle m \Box n, T\rangle = \langle m, n \Box T\rangle$ for every $T \in \mathcal{A}^*$ and $m, n \in \mathcal{A}^{**}$.

Remark 2.1. *If $\mathcal{A}$ is a commutative Banach algebra, then it is easy to see that $u \cdot T = T \Box u$ for every $u \in \mathcal{A}$ and $T \in \mathcal{A}^*$. In particular, if $m \in \mathcal{A}^{**}$ and $u \in \mathcal{A}$, then*

$$m \odot u = u \odot m = m \Box u = u \Box m.$$

However, as is well known, even if $\mathcal{A}$ is commutative this does not mean that the two multiplications agree on $\mathcal{A}^{**}$. Moreover, even though $\mathcal{A}$ is assumed to be commutative, it is generally not the case that $\mathcal{A}^{**}$ would be as well.

Definition 2.2. *If $\mathcal{A}$ is a Banach algebra for which $m \odot n = m \Box n$ for every $m, n \in \mathcal{A}^{**}$, we say that $\mathcal{A}$ is Arens regular.*

From here on we will assume that $\mathcal{A}$ is a commutative Banach Algebra with maximal ideal space $\Delta(\mathcal{A})$. Moreover, if we are speaking of $\mathcal{A}^{**}$, we will assume that the product we are using is $\odot$ unless otherwise specified.

We will proceed with the following definitions and notational conventions.

Definition 2.3. *We call the space*

$$UCB(\mathcal{A}) = \overline{\text{span}\{v \cdot T \mid v \in \mathcal{A}, T \in \mathcal{A}^*\}}^{-\|\cdot\|_{\mathcal{A}^*}}$$

the uniformly continuous functionals on $\mathcal{A}$.

We call $T \in \mathcal{A}^*$ a (weakly) almost periodic functional on $\mathcal{A}$ if

$$\{u \cdot T \mid u \in \mathcal{A}, \|u\|_{\mathcal{A}} \leq 1\}$$

is relatively (weakly) compact in $\mathcal{A}^*$ and we denote the space of all (weakly) almost periodic functionals on $\mathcal{A}$ by $AP(\mathcal{A})$ $(WAP(\mathcal{A}))$.

Remark 2.4. *It is a well-known criterion of Grothendieck that $T \in \mathcal{A}^*$ is weakly almost periodic if and only if given two nets $\{u_\alpha\}_{\alpha \in \Omega_1}$ and $\{v_\beta\}_{\beta \in \Omega_2}$ in $\mathcal{A}$ we have that*

$$\lim_\alpha \lim_\beta \langle T, u_\alpha v_\beta\rangle = \lim_\beta \lim_\alpha \langle T, u_\alpha v_\beta\rangle$$

whenever both limits exist. From here it is easy to see that $\mathcal{A}$ is Arens regular if and only if $WAP(\mathcal{A}) = \mathcal{A}^*$. Moreover, as a consequence, this shows that if $\mathcal{A}$ is commutative, then $\mathcal{A}$ is Arens regular if and only if $\mathcal{A}^{**}$ is also commutative in either, and hence both Arens products.(See [6] and [21].)

Definition 2.5. *We say that a closed subspace $X \subseteq \mathcal{A}^*$ is invariant if $u \cdot T \in X$ for every $u \in \mathcal{A}$ and $T \in X$.*

DOI: https://dx.doi.org/10.30504/JIMS.2023.385500.1091

Given a closed invariant subspace X of $\mathcal{A}^*$ and an $m \in X^*$, we define the linear operator $m_L : X \to \mathcal{A}^*$ by

$$\langle m_L(T), u\rangle := \langle m, u \cdot T\rangle$$

for every $T \in X$ and $u \in \mathcal{A}$. We say that X is topologically introverted if $m_L(T) \in X$ for every $m \in X^*$ and $T \in X$.

It is also well known that if X is topologically introverted then X^* can be made into a Banach algebra by mimicking what we did for $\mathcal{A}^{**}$ as follows.

1) *For each $T \in X$ and $m \in X^*$, we define $T \odot m = m_L(T)$.*
2) *For each $T \in X$ and $n, m \in X^*$, we define $\langle m \odot n, T\rangle = \langle n, T \odot m\rangle$*

It is well-known and straightforward to show that all three of $AP(\mathcal{A}), WAP(\mathcal{A})$, and $UCB(\mathcal{A})$ are closed introverted subspaces of $\mathcal{A}^*$. Moreover, if $\mathcal{A}$ is commutative and X is topologically introverted, then X^* is commutative if and only if $X \subseteq WAP(\mathcal{A})$.

3. Multipliers of the Fourier algebra

Let G be a locally compact group. We let $A(G)$ and $B(G)$ denote the Fourier and Fourier-Stieltjes algebras of G, which are Banach algebras of continuous functions on G and were introduced in [5]. A *multiplier* of $A(G)$ is a bounded continuous function $v\colon G \to \mathbb{C}$ such that $vA(G) \subseteq A(G)$. Each multiplier v of $A(G)$ determines a linear operator M_v on $A(G)$ defined by $M_v(u) = vu$ for each $u \in A(G)$. It is a routine consequence of the Closed Graph Theorem that each M_v is also bounded. The *multiplier algebra* of $A(G)$ is the closed subalgebra

$$MA(G) := \{M_v : v \text{ is a multiplier of } A(G)\}$$

of $B(A(G))$, where $B(A(G))$ denotes the algebra of all bounded linear operators from $A(G)$ to $A(G)$. Throughout this paper, we will typically use v in place of the operator M_v for notational convenience and we will write $\| v \|_{MA(G)}$ to represent the norm of M_v in $B(A(G))$.

Let G be a locally compact group and let $VN(G)$ denote its group von Neumann algebra. The duality

$$A(G) = VN(G)_*$$

equips $A(G)$ with a natural operator space structure. Given this operator space structures, we can define the *cb-multiplier algebra* of $A(G)$ to be

$$M_{cb}A(G) := CB(A(G)) \cap MA(G),$$

where $CB(A(G))$ denotes the algebra of all completely bounded linear maps from $A(G)$ into itself. We let $\| v \|_{cb}$ denote the *cb*-norm of the operator M_v. It is well known that $M_{cb}A(G)$ is a closed subalgebra of $CB(A(G))$ and is thus a (completely contractive) Banach algebra with respect to the norm $\| \cdot \|_{cb}$.

It is clear that,

$$A(G) \subseteq B(G) \subseteq M_{cb}A(G) \subseteq MA(G).$$

DOI: https://dx.doi.org/10.30504/JIMS.2023.385500.1091

Moreover, for $v \in A(G)$ we have that

$$\| v \|_{A(G)} = \| v \|_{B(G)} \geq \| v \|_{M_{cb}A(G)} \geq \| v \|_{MA(G)} .$$

In case G is an amenable group, we have

$$B(G) = M_{cb}A(G) = MA(G)$$

and that

$$\| v \|_{B(G)} = \| v \|_{M_{cb}A(G)} = \| v \|_{MA(G)}$$

for any $v \in B(G)$.

Definition 3.1. *Given a locally compact group G let*

$$A_M(G) \overset{def}{=} A(G)^{-\|\cdot\|_{MA(G)}} \subseteq MA(G).$$

and

$$A_{cb}(G) \overset{def}{=} A(G)^{-\|\cdot\|_{M_{cb}A(G)}} \subseteq M_{cb}A(G).$$

We refer the reader to [5] for the basic properties of $M_{cb}A(G)$.

Remark 3.2. *The algebra $A_{cb}(G)$ was introduced by the first author in [9] where it was denoted by $A_0(G)$. In that paper, we show that in the case of $\mathbb{F}_2$, the free group on two generators, $A_{cb}(G)$ shares many of the properties characteristic of the Fourier-algebra of an amenable group. In particular, the algebra $A_{cb}(\mathbb{F}_2)$ is known to have a bounded approximate identity. The locally compact groups G for which $A_{cb}(G)$ has a bounded approximate identity are called weakly amenable groups. All amenable groups are weakly amenable, but many classical non-amenable groups such as $\mathbb{F}_2$ and $SL(2,\mathbb{R})$ are weakly amenable.*

Remark 3.3. *Let $\mathcal{A}(G)$ denote either $A_{cb}(G)$ or $A_M(G)$. Consider the following map and its adjoints:*

$i_{\mathcal{A}} : A(G) \to \mathcal{A}(G)$

$i_{\mathcal{A}}^* : \mathcal{A}(G)^* \to VN(G)$

$i_{\mathcal{A}}^{**} : VN(G)^* \to \mathcal{A}(G)^{**}$,

where $i_{\mathcal{A}}$ denotes the inclusion map. Since $i_{\mathcal{A}}$ has dense range, $i_{\mathcal{A}}^*$ is injective and as such is invertible with inverse ${i_{\mathcal{A}}^*}^{-1}$ on Range$(i_{\mathcal{A}}^*)$. It is easy to see that $i_{\mathcal{A}}^*$ is simply the restriction map. That is

$$i_{\mathcal{A}}^*(T) = T_{|_{A(G)}}.$$

It will also be useful to view all of the above maps as embeddings. That is, when G is non-amenable $\mathcal{A}(G)^*$ can be viewed as a proper subset of $VN(G)$ and $VN(G)^*$ as a proper subset of $\mathcal{A}(G)^{**}$.

Remark 3.4. *Let $\mathcal{A}(G)$ denote either $A_{cb}(G)$ or $A_M(G)$. Let I be an ideal in $\mathcal{A}(G)$. We let*

$$Z(I) = \{x \in G \mid u(x) = 0 \text{ for every } u \in I\},$$

denote the hull of I. It is easy to see that $Z(I)$ is closed in G. Conversely, if $E \subset G$ is closed, we let

$$I_{\mathcal{A}(G)}(E) = \{u \in \mathcal{A}(G) \mid u(x) = 0 \text{ for every } x \in E\},$$

DOI: https://dx.doi.org/10.30504/JIMS.2023.385500.1091

$$j_{\mathcal{A}(G)}(E) = \{u \in \mathcal{A}(G) \mid supp(u) \text{ is compact, and disjoint from } E\}$$

and

$$J_{\mathcal{A}(G)}(E) = \overline{j_{\mathcal{A}(G)}(E)}^{-\|\cdot\|_{\mathcal{A}(G)}}.$$

It is well known that $I_{\mathcal{A}(G)}(E)$ and $J_{\mathcal{A}(G)}(E)$ are respectively the largest and smallest closed ideals of $\mathcal{A}(G)$ with hull equal to E.

We say that a closed set $E \subset G$ is a set of spectral synthesis for $\mathcal{A}(G)$ if $I_{\mathcal{A}(G)}(E) = J_{\mathcal{A}(G)}(E)$. That is if $I_{\mathcal{A}(G)}(E)$ is the unique closed ideal in $\mathcal{A}(G)$ with hull E.

4. Topological invariant means

Definition 4.1. *Let $\mathcal{A}$ be a commutative Banach algebra with maximal ideal space $\Delta(\mathcal{A})$. Let X be a closed submodule of $\mathcal{A}^*$ containing $\phi \in \Delta(\mathcal{A})$. Then $m \in X^*$ is called a topologically invariant mean (TIM) on X at ϕ if*

i) $\|m\|_{X^*} = \langle m, \phi \rangle = 1$,
ii) $\langle m, v \cdot T \rangle = \phi(v)\langle m, T \rangle$ *for every* $v \in \mathcal{A}$ *and* $T \in X$.

We denote the set of topologically invariant means on X at ϕ by $TIM_{\mathcal{A}}(X, \phi)$.

Note: For the rest of this section we will focus our attention on the algebras $A(G)$, $A_{cb}(G)$, $A_M(G)$, and their closed ideals.

Definition 4.2. *Let $\mathcal{A}$ be one of the algebras $A(G)$, $A_{cb}(G)$ or $A_M(G)$. Let $x \in G$ define the isometry $L_x : \mathcal{A} \to \mathcal{A}$ by*

$$L_x(u)(y) = u(xy),$$

for each $y \in G$.

The following proposition will prove useful.

Proposition 4.3. *Let $\mathcal{A}(G)$ be one of the algebras $A(G)$, $A_{cb}(G)$ or $A_M(G)$. Let $X \subseteq \mathcal{A}(G)^*$ be a closed submodule. Let $x \in G$.*

i) *If $Y = L_x^*(X)$, then Y is a closed submodule of $\mathcal{A}(G)^*$ and*

$$u \cdot L_x^*(T) = L_x^*(L_x(u) \cdot T)$$

for every $u \in \mathcal{A}$ and $T \in X$.

ii) *Let $x \in G$. Then $L_x^*(\phi_e) = \phi_x$ where e denotes the identity of G.*
iii) *Let $m \in TIM_{\mathcal{A}(G)}(X, \phi_e)$. If $x \in G$, Then $\phi_x \in L_x^*(X)$ and $L_{x^{-1}}^{**}(m) \in TIM_{\mathcal{A}(G)}(L_x^*(X), \phi_x)$.*

Proof. Let $\mathcal{A}(G)$ be one of the algebras $A(G)$, $A_{cb}(G)$ or $A_M(G)$. Let $X \subset \mathcal{A}(G)^*$ be a closed submodule. Let $x \in G$.

i) Since L_x^* is an isometry, it is clear that Y is a closed subspace.
Let $T \in X$ and $u, v \in \mathcal{A}(G)$. Observe that

DOI: https://dx.doi.org/10.30504/JIMS.2023.385500.1091

$$\begin{aligned}\langle u\cdot L_x^*(T),v\rangle &= \langle L_x^*(T),uv\rangle\\ &= \langle T,L_x(uv)\rangle\\ &= \langle T,L_x(u)L_x(v)\rangle\\ &= \langle L_x(u)\cdot T,L_x(v)\rangle\\ &= \langle L_x^*(L_x(u)\cdot T),v\rangle\end{aligned}$$

Hence $u\cdot L_x^*(T)=L_x^*(L_x(u)\cdot T)\in L_x^*(X)=Y$.

ii) Let $u\in\mathcal{A}(G)$. Then

$$\langle L_x^*(\phi_e),u\rangle=\langle\phi_e,L_x(u)\rangle=L_x(u)(e)=u(x)=\langle\phi_x,u\rangle.$$

This shows that $L_x^*(\phi_e)=\phi_x$.

iii) Let $T=L_x^*(T_1)$ with $T_1\in X$. If $u\in\mathcal{A}(G)$, we have

$$\begin{aligned}\langle L_{x^{-1}}^{**}(m),u\cdot T\rangle &= \langle m,L_{x^{-1}}^*(u\cdot T)\rangle\\ &= \langle m,L_{x^{-1}}^*(u\cdot L_x^*(T_1))\rangle\\ &= \langle m,,L_{x^{-1}}^*(L_x^*((L_x(u)\cdot T_1)\rangle\\ &= \langle m,L_x(u)\cdot T_1\rangle\\ &= L_x(u)(e)\langle m,T_1\rangle\\ &= u(e)\langle m,T_1\rangle\\ &= u(e)\langle m,L_{x^{-1}}^*(L_x^*(T_1))\rangle\\ &= u(e)\langle L_{x^{-1}}^{**}(m),T\rangle\end{aligned}$$

Since $L_{x^{-1}}^{**}$ is an isometry we have that

$$\|L_{x^{-1}}^{**}(m)\|_{\mathcal{A}^*}=\|m\|_{\mathcal{A}^*}=1.$$

Finally, we have that

$$\langle L_{x^{-1}}^{**}(m),\phi_x\rangle=\langle m,L_{x^{-1}}^*(\phi_x)\rangle=\langle m,\phi_e\rangle=1.$$

□

The next result follows immediately from Proposition 4.3 iii).

Corollary 4.4. *Let $\mathcal{A}(G)$ be any of $A(G)$, $A_{cb}(G)$ or $A_M(G)$. Let $x\in G$. Then*

$$|TIM_{\mathcal{A}(G)}(\mathcal{A}(G)^*,\phi_x)|=|TIM_{\mathcal{A}(G)}(\mathcal{A}(G)^*,\phi_e)|,$$

where $|\cdot|$ represents the cardinality of the underlying set.

DOI: https://dx.doi.org/10.30504/JIMS.2023.385500.1091

The first author together with T. Miao established the next result for $TIM_{A_M(G)}(A_M(G)^*, \phi_e)$ in [13].

Proposition 4.5. *Let $\mathcal{A}$ be either $A_{cb}(G)$ or $A_M(G)$. Let $\mathcal{A}(G) \cdot VN(G) = \{u \cdot T : u \in \mathcal{A}(G), T \in VN(G)\}$. Then*

i) $\mathcal{A}(G) \cdot VN(G) \subseteq UCB(A(G))$.

ii $i^*(v \cdot T) = v \cdot i^*(T)$ *for each* $v \in \mathcal{A}(G), T \in \mathcal{A}(G)^*$.

iii) $i^*(UCB(\mathcal{A}(G))) \subseteq UCB(A(G))$.

iv) If $\mathcal{A}$ has a bounded approximate identity, then $\mathcal{A}(G) \cdot VN(G) \cdot VN(G) = UCB(\mathcal{A}(G))$.

v) $u \cdot T \in i^*(\mathcal{A}(G)^*)$ *for each* $u \in A(G), T \in VN(G)$.

Proof. i) Since $UCB(A(G))$ is a closed subspace of $VN(G)$ and since $A(G)$ is dense in $\mathcal{A}(G)$ with respect to the norm $\| \cdot \|_{\mathcal{A}(G)}$, to establish $i)$ we need only show that for any sequence $\{v_n\} \subset A(G)$ and $v \in \mathcal{A}(G)$ with $\| v_n - v \|_{\mathcal{A}(G)} \to 0$ and any $T \in VN(G)$, we have $\|v_n \cdot T - v \cdot T\|_{VN(G)} \to 0$. However, this follows immediately since for any $u \in A(G)$

$$\begin{aligned} | \langle v_n \cdot T - v \cdot T, u \rangle | &= | \langle (v_n - v) \cdot T, u \rangle | \\ &= | \langle T, (v_n - v)u \rangle | \\ &\leq \| T \|_{VN(G)} \| v_n - v \|_{\mathcal{A}(G)} \| u \|_{A(G)} . \end{aligned}$$

ii) Let $v \in \mathcal{A}(G)$, $T \in \mathcal{A}(G)^*$ and $u \in A(G)$. We have

$$\begin{aligned} \langle i^*(v \cdot T), u \rangle &= \langle v \cdot T, i(u) \rangle \\ &= \langle T, vi(u) \rangle \\ &= \langle T, i(vu) \rangle \\ &= \langle i^*(T), vu \rangle \\ &= \langle v \cdot i^*(T), u \rangle \end{aligned}$$

Hence $i^*(v \cdot T) = v \cdot i^*(T)$.

iii) Because of i) above we need only show that $i^*(v \cdot T) \in \mathcal{A}(G) \cdot VN(G)$ for any $v \in \mathcal{A}(G)$ and $T \in \mathcal{A}(G)^*$. However, this follows immediately from ii).

iv) Since $\mathcal{A}(G)$ has a bounded approximate identity it follows from Cohen's Factorization Theorem and from $i)$ above that $\mathcal{A}(G) \cdot VN(G)$ is a closed subspace of $UCB(A(G))$. However, since $A(G) \cdot VN(G) \subseteq \mathcal{A}(G) \cdot VN(G)$, it is also clear that $\mathcal{A}(G) \cdot VN(G)$ is dense in $UCB(\hat{G})$. It follows that $\mathcal{A}(G) \cdot VN(G) = UCB(A(G))$

v) Let $u \in A(G), T \in VN(G)$. Then we can define a linear functional on $\mathcal{A}(G)$ by

$$\varphi_{u,T}(v) = \langle T, uv \rangle$$

DOI: https://dx.doi.org/10.30504/JIMS.2023.385500.1091

for each $v \in \mathcal{A}(G)$. It is also clear that $\varphi_{u,T}$ has norm at most $\|u\|_{A(G)}\|T\|_{VN(G)}$. Moreover, this linear functional agrees with $u \cdot T$ on $A(G)$ and as such $u \cdot T = i^*(\varphi_{u,T})$.

□

Theorem 4.6. *Let $\mathcal{A}(G)$ denote either $A_{cb}(G)$ or $A_M(G)$. For any locally compact group, $i^{**}(TIM_{A(G)}$ $(A(G)^*, \phi_x) \subseteq TIM_{\mathcal{A}(G)}(\mathcal{A}(G)^*, \phi_x)$. Moreover, $i^{**} : TIM_{A(G)}(A(G)^*, \phi_x) \to TIM_{\mathcal{A}(G)}(\mathcal{A}(G)^*, \phi_x)$ is a bijection.*

Proof. We will first show that $i^{**}(TIM_{A(G)}(A(G)^*, \phi_x)) \subseteq TIM_{\mathcal{A}(G)}(\mathcal{A}(G)^*, \phi_x)$.

Let $m \in TIM_{A(G)}(A(G)^*, \phi_x))$. Let $v \in \mathcal{A}(G)$ and $T \in \mathcal{A}(G)^*$. Then there exists $\{u_n\} \subset A(G)$ such that $\| u_n - v \|_{\mathcal{A}(G)} \to 0$. Since $\| u_n - v \|_\infty \leq \| u_n - v \|_{\mathcal{A}(G)}$ it follows that $u_n(x) \to v(x)$.

Next, we note that in a similar manner to the proof of Proposition 4i), we can show that $u_n \cdot T \to v \cdot T$ in the norm $\| \cdot \|_{\mathcal{A}(G)^*}$ for each $T \in \mathcal{A}(G)^*$. Appealing this time to Proposition 4ii), it follows that

$$\begin{aligned}
\langle i^{**}(m), v \cdot T \rangle &= \lim_{n\to\infty} \langle i^{**}(m), u_n \cdot T \rangle \\
&= \lim_{n\to\infty} \langle m, i^*(u_n \cdot T) \rangle \\
&= \lim_{n\to\infty} \langle m, u_n \cdot i^*(T) \rangle \\
&= \lim_{n\to\infty} u_n(x) \langle m, i^*(T) \rangle \\
&= \lim_{n\to\infty} v(x) \langle m, i^*(T) \rangle \\
&= v(x) \langle i^{**}(m), T \rangle.
\end{aligned}$$

This shows that $i^{**}(TIM_{A(G)}(A(G)^*, \phi_x)) \subseteq TIM_{\mathcal{A}(G)}(\mathcal{A}(G)^*, \phi_x)$.

We next show that $i^{**} : TIM_{A(G)}(A(G)^*, \phi_x) \to TIM_{\mathcal{A}(G)}(\mathcal{A}(G)^*, \phi_x)$ is injective. To see this, we first note that if $m_1, m_2 \in TIM_{A(G)}(A(G)^*, \phi_x)$ with $m_1 \neq m_2$, then there exists an $T \in VN(G)$ for which

$$\langle m_1, T \rangle \neq \langle m_2, T \rangle.$$

Next choose $u_0 \in A(G)$ with $u_0(x) = 1$. Then

$$\langle m_1, u_0 \cdot T \rangle = \langle m_1, T \rangle \neq \langle m_2, T \rangle = \langle m_2, u_0 \cdot T \rangle.$$

Since $u_0 \cdot T \in \mathcal{A}(G)^*$, we have

$$\begin{aligned}
\langle i^{**}(m_1), u_0 \cdot T \rangle &= \langle m_1, i^*(u_0 \cdot T) \rangle \\
&= \langle m_1, u_0 \cdot T \rangle \\
&\neq \langle m_2, u_0 \cdot T \rangle \\
&= \langle m_2, i^*(u_0 \cdot T) \rangle \\
&= \langle i^{**}(m_2), u_0 \cdot T \rangle
\end{aligned}$$

DOI: https://dx.doi.org/10.30504/JIMS.2023.385500.1091

so that $i^{**}(m_1) \neq i^{**}(m_2)$.

Finally, we show that $i^{**} : TIM_{A(G)}(A(G)^*, \phi_x) \to TIM_{\mathcal{A}(G)}(\mathcal{A}(G)^*, \phi_x)$ is surjective.

Let $M \in TIM_{\mathcal{A}(G)}(\mathcal{A}(G)^*, \phi_x)$. First note that if $u, v \in A(G)$, with $u(x) = 1 = v(x)$ and if $T \in VN(G)$, then $u \cdot T$ and $v \cdot T$ are in $\mathcal{A}(G)^*$ and

$$\langle M, u \cdot T\rangle = \langle M, v \cdot (u \cdot T)\rangle = \langle M, u \cdot (v \cdot T)\rangle = \langle M, v \cdot T\rangle.$$

Pick a $u_0 \in A(G)$ with $\| u_0 \|_{A(G)} = 1$ and $u_0(x) = 1$. We can define $m_M \in A(G)^{**}$ by

$$\langle m_M, T\rangle = \langle M, u_0 \cdot T\rangle$$

for $T \in VN(G)$.

Note that $\|m_M\|_{A(G)^{**}} \leq 1$. It is clear from the observation above that if $v \in A(G)$ is such that $v(x) = 1$, then $\langle m_M, v \cdot T\rangle = \langle m_M, T\rangle$. We also have that

$$\langle m_M, \phi_x\rangle = \langle M, u_0 \cdot \phi_x\rangle = \langle M, u_0(x)\phi_x\rangle = \langle M, \phi_x\rangle = 1.$$

That is, $m_M \in TIM_{A(G)}(A(G)^*, \phi_x)$.

Finally, if $T \in \mathcal{A}(G)^*$, then

$$\langle i^{**}(m_M), T\rangle = \langle m_M, i^*(T)\rangle = \langle M, u_0 \cdot i^*(T)\rangle = \langle M, u_0.T\rangle = \langle M, T\rangle.$$

Therefore, $i^{**}(m_M) = M$.

□

Definition 4.7. *Given a locally compact group G we let $b(G)$ denote the smallest cardinality of a neighbourhood basis at the identity e for G.*

The next corollary follows immediately from the previous theorem and from Hu [15].

Corollary 4.8. *Let G be a non-discrete locally compact group. Let $\mathcal{A}(G)$ be $A(G)$, $A_{cb}(G)$ or $A_M(G)$. Then*

$$\mid TIM_{\mathcal{A}(G)}(\mathcal{A}(G)^*, \phi_x) \mid = 2^{2^{b(G)}}.$$

In particular, $\mathcal{A}(G)^*$ admits a unique topological invariant mean if and only if G is discrete.

We now turn our attention to ideals in the algebra $\mathcal{A}(G)$ where $\mathcal{A}(G)$ is any of $A(G)$, $A_{cb}(G)$ or $A_M(G)$.

Lemma 4.9. *Let $\mathcal{A}(G)$ be $A(G)$, $A_{cb}(G)$ or $A_M(G)$. Let I be a closed ideal in $\mathcal{A}(G)$. Assume that $x \notin Z(I)$. Let $M \in TIM_{\mathcal{A}(G)}(\mathcal{A}(G)^*, \phi_x)$. Then $M \in (I^\perp)^\perp$.*

Proof. Let $T \in I^\perp$. Since $Z(I)$ is closed and since $x \notin Z(I)$, we can find an open neighborhood U of x such $U \cap Z(I) = \emptyset$. We can then find a $u_0 \in A(G) \cap C_{00}(G)$ so that $supp(u_0) \subseteq U$, and $u_0(x) = 1$. It follows that $u_0 \in I$.

Since $T \in I^\perp$, we have that for any $u \in \mathcal{A}(G)$ that

$$\langle u_0 \cdot T, u\rangle = \langle T, u_0 u\rangle = 0$$

DOI: https://dx.doi.org/10.30504/JIMS.2023.385500.1091

so $u_0 \cdot T = 0$. However, since $M \in TIM_{\mathcal{A}(G)}(\mathcal{A}(G)^*, \phi_x)$, and since $u_0(x) = 1$,

$$\langle M, T\rangle = \langle M, u_0 \cdot T\rangle = 0.$$

□

Remark 4.10. *The previous lemma shows that if I is a closed ideal in $\mathcal{A}(G)$ with $x \notin Z(I)$, and if $M \in TIM_{\mathcal{A}(G)}(\mathcal{A}(G)^*, \phi_x)$, then we can view M as an element $\hat{M}$ of I^{**} in a canonical way. Specifically, if $T \in I^{**}$ and T_1 is any extension of T we can define*

$$\hat{M}(T) = M(T_1)$$

and $\hat{M}$ is well defined since $M \in (I^{\perp})^{\perp}$. We claim that $\hat{M} \in TIM_I(I^*, \phi_{x_{|_I}})$. To see that this is the case we note that

$$\|\hat{M}\|_{I^*} = \|M\|_{\mathcal{A}(G)^{**}} = M(\phi_x) = \hat{M}(\phi_{x_{|_I}}).$$

If $u \in I$, then if $T_{1_{|_I}} = T$, then $u \cdot T_{1_{|_I}} = u \cdot T$ and as such

$$\hat{M}(u \cdot T) = M(u \cdot T_1) = u(x)M(T_1) = \phi_{x_{|_I}}(u)\hat{M}(T).$$

This gives us a map $\Gamma : TIM_{\mathcal{A}(G)}(\mathcal{A}(G)^*, \phi_x) \to TIM_I(I^*, \phi_{x_{|_I}})$, given by

$$\Gamma M = \hat{M}.$$

The map $\Gamma : TIM_{\mathcal{A}(G)}(\mathcal{A}(G)^*, \phi_x) \to TIM_I(I^*, \phi_{x_{|_I}})$, given by

$$\Gamma(M) = \hat{M}.$$

Theorem 4.11. *Let $\mathcal{A}(G)$ be $A(G)$, $A_{cb}(G)$ or $A_M(G)$. Let I be a closed ideal in $\mathcal{A}(G)$. Assume that $x \notin Z(I)$. The map $\Gamma : TIM_{\mathcal{A}(G)}(\mathcal{A}(G)^*, \phi_x) \to TIM_I(I^*, \phi_{x_{|_I}})$, given by*

$$\Gamma(M) = \hat{M}$$

is a bijection.

Proof. Assume that $M_1, M_2 \in TIM_{\mathcal{A}(G)}(\mathcal{A}(G)^*, \phi_x)$ and that $T_0 \in \mathcal{A}(G)^*$ is such that $M_1(T_0) \neq M_2(T_0)$. Let $T \in I^* = T_{0_{|_I}}$. Then

$$\hat{M}_1(T) = M_1(T_0) \neq M_2(T_0) = \hat{M}_2(T)$$

so

$$\Gamma(M_1) = \hat{M}_1 \neq \hat{M}_2 = \Gamma(M_2)$$

and hence Γ is injective.

Next we let $m \in TIM_I(I^*, \phi_{x_{|_I}})$. We let $u_0 \in I$ be such that $u_0(x) = 1$ with $\|u_0\|_{\mathcal{A}(G)} = 1$. First, observe that $(u_0 \cdot T)_{|_I} = u_0 \cdot (T_{|_I})$ for each $T \in \mathcal{A}(G)^*$. Then we define $M \in \mathcal{A}(G)^{**}$ by

$$\langle M, T\rangle = \langle m, u_0 \cdot (T_{|_I})\rangle = \langle m, T_{|_I}\rangle$$

for each $T \in \mathcal{A}(G)^*$. We have that

$$\|M\|_{\mathcal{A}(G)^{**}} \leq \|m\|_{I^{**}}\|u_0\|_{\mathcal{A}(G)} = 1.$$

DOI: https://dx.doi.org/10.30504/JIMS.2023.385500.1091

Moreover

$$\langle M, \phi_x \rangle = \langle m, u_0 \cdot \phi_{x_{|_I}} \rangle = u_0(x)\langle m, \phi_{x_{|_I}} \rangle = 1.$$

Next let $T \in \mathcal{A}(G)^*$ and let $u \in \mathcal{A}(G)$. Then

$$\begin{aligned}
\langle M, u \cdot T \rangle &= \langle m, u_0 \cdot (u \cdot T)_{|_I} \rangle \\
&= \langle m, (u_0 u) \cdot T_{|_I} \rangle \\
&= (u_0 u)(x) \langle m, T_{|_I} \rangle \\
&= u(x)(u_0(x)\langle m, T_{|_I} \rangle) \\
&= u(x)\langle m, u_0 \cdot (T_{|_I}) \rangle \\
&= u(x)\langle M, T \rangle
\end{aligned}$$

It follows that $M \in TIM_{\mathcal{A}(G)}(\mathcal{A}(G)^*, \phi_x)$. Finally, if $T \in I^*$ and if $T_1 \in \mathcal{A}(G)$ with $T_{1_{|_I}} = T$, then

$$\begin{aligned}
\langle \hat{M}, T \rangle &= \langle M, T_1 \rangle \\
&= \langle m, T_{1_{|_I}} \rangle \\
&= \langle m, T \rangle
\end{aligned}$$

Hence $\Gamma(M) = \hat{M} = m$ and Γ is surjective.

□

The following result follows immediately from Corollary 4.8.

Corollary 4.12. *Let $\mathcal{A}(G)$ be $A(G)$, $A_{cb}(G)$ or $A_M(G)$. Let I be a closed ideal in $\mathcal{A}(G)$. Assume that $x \notin Z(I)$. If G is a non-discrete group, then*

$$|TIM_I(I^*, \phi_{x_{|_I}})| = 2^{2^{b(G)}}.$$

In particular, I^* admits a unique topological invariant mean if and only if G is discrete.

Lemma 4.13. *Let $\mathcal{A}$ be a commutative Banach algebra with maximal ideal space $\Delta(\mathcal{A})$. Let X be a closed submodule of $\mathcal{A}^*$ containing $\phi \in \Delta(\mathcal{A})$. Let $M \in TIM_{\mathcal{A}}(\mathcal{A}, \phi)$. Let $m = M_{|_X}$ be the restriction of M to X. Then $m \in TIM_{\mathcal{A}}(X, \phi)$.*

In particular, if we let $\mathcal{A}(G)$ be one of the algebras $A(G)$, $A_{cb}(G)$ or $A_M(G)$, I be a closed ideal in $\mathcal{A}(G)$ with $x \notin Z(I)$ and X is any of $UCB(I)$, $WAP(I)$ or $AP(I)$, we have that $\phi_{x_{|_I}} \in X$ and hence that for any $M \in TIM_I(I^*, \phi_{x_{|_I}})$, if $m = M_{|_X}$ we get that $m \in TIM_I(X, \phi_{x_{|_I}})$.

Proof. Let $M \in TIM_{\mathcal{A}}(\mathcal{A}, \phi)$. Let $m = M_{|_X}$ be the restriction of M to X. We have that

$$1 = \langle M, \phi \rangle = \langle m, \phi \rangle = \|m\|_{X^*}$$

and that if $u \in \mathcal{A}$ and $T \in X$, then

$$\langle m, u \cdot T \rangle = \langle M, u \cdot T \rangle = \langle \phi, u \rangle \langle M, T \rangle = \langle \phi, u \rangle \langle m, T \rangle.$$

DOI: https://dx.doi.org/10.30504/JIMS.2023.385500.1091

Hence $m \in TIM_{\mathcal{A}}(X, \phi)$.

Let $\mathcal{A}(G)$ be one of the algebras $A(G)$, $A_{cb}(G)$ or $A_M(G)$ and I a closed ideal in $\mathcal{A}(G)$ with $x \notin Z(I)$. Let $X = UCB(I)$. We can choose an open neighborhood U of X with $U \cap Z(I) = \emptyset$ and a $u_0 \in \mathcal{A}(G) \cap C_{00}(G)$ such that $supp(u_0) \subseteq U$ and $u_0(x) = 1$. Then $u_0 \in I$. Moreover, if $v \in I$,

$$\langle u_0 \cdot \phi_{x_{|_I}}, v\rangle = \langle \phi_{x_{|_I}}, u_0 v\rangle = u_0(x)v(x) = \langle \phi_{x_{|_I}}, u_0 v\rangle.$$

Hence $\phi_{x_{|_I}} = u_0 \cdot \phi_{x_{|_I}} \in UCB(I)$.

To see that $\phi_{x_{|_I}} \in AP(I)$ note that $\{u(x) | \|u\|_I \leq 1\} = \{\lambda \in \mathbb{C} | \mid \lambda \mid \leq 1\}$ is compact and hence

$$\{u \cdot \phi_{x_{|_I}} | \|u\|_I \leq 1\} = \{\lambda \phi_{x_{|_I}} \in \mathbb{C} | \mid \lambda \mid \leq 1\}$$

is compact in I^* so $\phi_{x_{|_I}} \in AP(I)$. As $AP(I) \subseteq WAP(I)$ we also have that $\phi_{x_{|_I}} \in WAP(I)$.

□

Theorem 4.14. *Let $\mathcal{A}(G)$ be one of the algebras $A(G)$, $A_{cb}(G)$ or $A_M(G)$. Let I be a closed ideal in $\mathcal{A}(G)$. Assume that $x \notin Z(I)$. The restriction map $R : TIM_I(I^*, \phi_{x_{|_I}})) \to TIM_I(UCB(I), \phi_{x_{|_I}}))$ is a bijection. In particular, if G is non-discrete, then*

$$\mid TIM_{\mathcal{A}(G)}(\mathcal{A}(G)^*, \phi_x) \mid = \mid TIM_{\mathcal{A}(G)}(UCB(\mathcal{A}(G), \phi_x) \mid = 2^{2^{b(G)}}.$$

Proof. Choose an open neighbourhood U of x with $U \cap Z(I) = \emptyset$ and a $u_0 \in \mathcal{A}(G) \cap C_{00}(G)$ with $supp(u_0) \subseteq U$, and $u_0(x) = 1$. Then $u_0 \in I$ and

$$\phi_{x_{|_I}} = u_0 \cdot \phi_{x_{|_I}} \in UCB(I).$$

Next we let $M \in TIM_I(I^*, \phi_{x_{|_i}}))$. Let $m = R(M)$. It follows from Lemma 4.13 that $m \in TIM_I(UCB(I), \phi_{x_{|_I}}))$.

If $M_1, M_2 \in TIM_I(I^*, \phi_{x_{|_I}})$ with $M_1 \neq M_2$, then there exists a $T \in I^*$ for which

$$\langle M_1, T\rangle \neq \langle M_2, T\rangle.$$

Choose an open neighbourhood U of x with $U \cap Z(I) = \emptyset$ and a $u_0 \in \mathcal{A}(G)$ with $supp(u_0) \subseteq U$, and $u_0(x) = 1$, then $u_0 \cdot T \in UCB(I)$ with

$$\langle M_1, u_0 \cdot T\rangle = \langle M_1, T\rangle \neq \langle M_2, T\rangle = \langle M_2, u_0 \cdot T\rangle.$$

This shows that $R(M_1) \neq R(M_2)$ and hence R is injective.

Next, let $m \in TIM_I(UCB(I), \phi_{x_{|_I}}))$. Pick a $u_0 \in I$ with $\| u_0 \|_{A(G)} = 1 = u_0(x)$. Define $M \in I^{**}$ by

$$\langle M, T\rangle = \langle m, u_0 \cdot T\rangle, \quad T \in I^*.$$

Since $u_0(x) = 1$, it follows that

$$\langle M, \phi_{x_{|_I}}\rangle = \langle m, u_0 \cdot \phi_{x_{|_I}}\rangle = \langle m, \phi_{x_{|_I}}\rangle = 1.$$

From this and the fact that $\| u_0 \|_{A(G)} = 1$, we get that $\| M \| = 1$.

DOI: https://dx.doi.org/10.30504/JIMS.2023.385500.1091

Next, if $v \in I, T \in I^*$, then

$$\langle M, v \cdot T\rangle = \langle m, u_0 \cdot (v \cdot T)\rangle = \langle m, v \cdot (u_0 \cdot T)\rangle = v(x)\langle m, u_0 \cdot T\rangle = v(x)\langle M, T\rangle.$$

This shows that $M \in TIM_I(I^*, \phi_{x_{|_I}})$.

Finally, if $T \in UCB(I))$, then

$$\langle M, T\rangle = \langle m, u_0 \cdot T\rangle = \langle m, T\rangle$$

since $m \in TIM_I(UCB(I), \phi_{x_{|_I}})$. Therefore, $R(M) = m$ and R is surjective.

□

Remark 4.15. *In the proof of the previous theorem we were able to explicitly show how each* $m \in TIM_I(UCB(I), \phi_{x_{|_I}})$ *extends to an element* $M \in TIM_I(I^*, \phi_{x_{|_I}})$. *The next proposition shows that such extensions hold in greater generality.*

We need the following lemma.

Lemma 4.16. *Let* $\mathcal{A}(G)$ *be one of the algebras* $A(G)$, $A_{cb}(G)$ *or* $A_M(G)$. *Let* I *be a closed ideal in* $\mathcal{A}(G)$ *with* $Z(I)$ *being a set of spectral synthesis for* $\mathcal{A}(G)$. *Assume that* $x \notin Z(I)$. *Then* $\{x\}$ *is a set of spectral synthesis for* I.

Proof. Let $u \in I$ be such that $u(x) = 0$. Let $E = Z(I)$. Let $\epsilon > 0$. Since E is a set of spectral synthesis for $\mathcal{A}(G)$, we can find $w \in \mathcal{A}(G) \cap C_{00}(G)$ which is such $K = supp(w) \cap E = \emptyset$ and

$$\|u - w\|_{\mathcal{A}(G)} < \frac{\epsilon}{2}.$$

Since $u(x) = 0$, this means that $|w(x)| < \frac{\epsilon}{2}$.

Next we choose neighbourhoods V_1 and V_2 of x with compact closure disjoint from E with $V_1 \subseteq V_2$. Then choose a $v \in \mathcal{A}(G)$ such that $v(y) = 0$ if $y \notin V_2$, $v(y) = w(x)$ on V_1 and $\|v\|_{\mathcal{A}(G)} =| w(x) |$. Then $w - v \in I = I(E)$ has compact support K_1 with $K_1 \cap (E \cup \{x\}) = \emptyset$ and

$$\|u - (w - v)\|_{\mathcal{A}(G)} \le \|u - w\|_{\mathcal{A}(G)} + \|v\|_{\mathcal{A}(G)} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

□

Proposition 4.17. *Let* $\mathcal{A}(G)$ *be one of the algebras* $A(G)$, $A_{cb}(G)$ *or* $A_M(G)$. *Let* I *be a closed ideal in* $\mathcal{A}(G)$ *with* $Z(I)$ *being a set of spectral synthesis for* $\mathcal{A}(G)$. *Assume that* $x \notin Z(I)$. *Assume also that* $Y \subset X$ *are two closed submodules of* I^* *each containing* $\phi_{x_{|_I}}$. *Let* $m \in TIM_I(Y, \phi_{x_{|_I}})$. *Then there exists some* $M \in TIM_I(X, \phi_{x_{|_I}})$ *such that* $M_{|_Y} = m$.

Proof. Let $m \in TIM_I(Y, \phi_{x_{|_I}})$. By the Hahn-Banach Theorem we can find a $\Psi \in X^*$ so that

$$\|\Psi\|_{X^*} = \|m\|_{Y^*} = 1.$$

Next we let

$$S = \{u \in I \mid \|u\|_{\mathcal{A}(G)} = u(x) = 1\}.$$

Then S is a commutative semigroup under pointwise multiplication and is hence amenable. Let $\Phi \in \ell_\infty(S)^*$ be an invariant mean. Then for each $T \in X$ we define $f_T : S \to \mathbb{C}$ by

$$f_T(u) = \langle \Psi, u \cdot T \rangle.$$

It follows that $f_T \in \ell_\infty(S)$ as $|f_T(u)| \leq \|T\|_X$ for each $u \in S$. Moreover, if $v \in S$, then

$$f_{v \cdot T}(u) = \langle \Psi, u \cdot (v \cdot T) \rangle = \langle \Psi, vu \cdot T \rangle = L_v(f_T)(u)$$

where L_v is the left translation operator on $\ell_\infty(S)$.

Next, we let

$$\langle M, T \rangle = \langle \Phi, f_T \rangle$$

for each $T \in X$. Note that $f_{\alpha T_1 + \beta T_2} = \alpha f_{T_1} + \beta f_{T_2}$ and $| \langle M, T \rangle | \leq \|T\|_X$ so that in fact $M \in X^*$ with $\|M\|_{X^*} \leq 1$.

Now if $T = \phi_{x_{|_I}}$, then

$$f_T(u) = \langle \Psi, u \cdot \phi_{x_{|_I}} \rangle = 1$$

for every $u \in S$ and hence $\langle M, \phi_{x_{|_I}} \rangle = 1$ and $\|M\|_{X*} = 1$.

Finally, since Φ is a left invariant mean on $\ell_\infty(S)^*$ we have that if $u \in S$, then

$$\langle M, u \cdot T \rangle = \langle \Phi, f_{u \cdot T} \rangle = \langle \Phi, L_u(f_T) \rangle = \langle M, T \rangle.$$

Finally, we must show that $\langle M, u \cdot T \rangle = u(x)\langle M, T \rangle$ for all $u \in I$, and $T \in X$.

First we will show that if $v \in I$ and if there exists a neighborhood U of x such that $v(y) = 1$ on U, then $\langle M, v \cdot T \rangle = \langle M, T \rangle$ for all $T \in X$. To see why this is the case, we choose $u \in I$ such that $u(x) = 1 = \|u\|_{\mathcal{A}(G)}$ with $u(y) = 0$ if $y \notin U$. Then $uv = u$, hence

$$\langle M, v \cdot T \rangle = \langle M, u \cdot (v \cdot T) \rangle = \langle M, uv \cdot T \rangle = \langle M, u \cdot T \rangle = \langle M, T \rangle.$$

Next assume that $v \in I$ satisfies $v(y) = 0$ on some neighborhood U of x. Since $1 \in B(G)$, the function $1 - u$ is a multiplier of I, that is $(1 - v)w \in I$ for every $w \in I$ and hence $(1 - v) \cdot T \in X$. We note that $1 - v(y) = 1$ on U. Again choosing $u \in I$ such that $u(x) = 1 = \|u\|_{\mathcal{A}(G)}$ with $u(y) = 0$ if $y \notin U$. Once more we get that

$$(\langle M, T \rangle - \langle M, v \cdot T \rangle) = \langle M, (1 - v) \cdot T \rangle = \langle M, u \cdot ((1 - v) \cdot T) \rangle = \langle M, u \cdot T \rangle = \langle M, T \rangle.$$

Hence $\langle M, v \cdot T \rangle = 0$.

Now let $u \in I$ be such that $u(x) = 0$. By Lemma 4.16, $\{x\}$ is a set of spectral synthesis for I. It follows that we can find a sequence of functions $\{w_n\} \subset I$ and as sequence $\{U_n\}$ of open neighbourhoods of x such that each w_n has compact support, $w_n(y) = 0$ for all $y \in U_n$ and $\lim_{n \to \infty} \|u - w_n\|_{\mathcal{A}(G)} = 0$. In particular, from what we have seen above, $\langle M, w_n \cdot T \rangle = 0$ and hence

$$\langle M, u \cdot T \rangle = \langle M, (u - w_n) \cdot T \rangle$$

Moreover,

$$\lim_{n \to \infty} \|u \cdot T - w_n \cdot T\|_{I^*} = 0,$$

DOI: https://dx.doi.org/10.30504/JIMS.2023.385500.1091

as hence we have that $\langle M, u\rangle = 0$. Finally, choose $u \in I$ be such that $u(x) = 1$. Once more choose $v \in I$ so that $v = 1$ on a neighbourhood U of x. Then $u - v(x) = 0$ and

$$\langle M, (u-v)\cdot T\rangle = 0$$

which means that

$$\langle M, u\cdot T\rangle = \langle M, v\cdot T\rangle = v(x)\langle M, T\rangle = u(x)\langle M, T\rangle.$$

For here, if $u(x) \neq 0$, let $w = \frac{1}{u(x)}u$. Then

$$\langle M, u\cdot T\rangle = \langle M, u(x)(\frac{1}{u(x)}u\cdot T)\rangle = u(x)\langle M, (\frac{1}{u(x)}u\cdot T)\rangle = u(x)\langle M, T\rangle.$$

This shows that $M \in TIM_I(X, \phi_{x_{|_I}})$.

Finally, if $T \in Y$, then

$$f_T(u) = \langle \Psi, u\cdot T\rangle = \langle m, u\cdot T\rangle = \langle m, T\rangle$$

for all $u \in S$. In particular, $\langle M, T\rangle = \langle \Phi, f_T\rangle = \langle m, T\rangle$ so $M_{|_Y} = m$ as desired.

□

Theorem 4.18. *Let $\mathcal{A}(G)$ be one of the algebras $A(G)$, $A_{cb}(G)$ or $A_M(G)$. Let I be a closed ideal in $\mathcal{A}(G)$. Assume that $x \notin Z(I)$. Then $WAP(I)$ has a unique topologically invariant mean at $\phi_{x_{|_I}}$.*

Proof. The fact that $TIM_I(WAP(I), \phi_{x_{|_i}})) \neq \emptyset$ follows immediately from the observation that every TIM on I^* restricts to a TIM on $WAP(I)$ and we know that $TIM_I(I^*, \phi_{x_{|_i}})) \neq \emptyset$.

Next, we let $n, m \in TIM_I(WAP(I), \phi_{x_{|_i}}))$. Then there exits a net $\{u_\alpha\}_{\alpha\in\Omega} \subseteq I$ so that

$$m = \lim_{\alpha\in\Omega}\{u_\alpha\}$$

in the weak* topology on I^{**}. Hence, for each $T \in I^*$,

$$\langle n \odot m, T\rangle = \lim_\alpha \langle n \odot u_\alpha, T\rangle = \lim_\alpha \langle n, u_\alpha \cdot T\rangle = \lim_\alpha u_\alpha(x)\langle n, T\rangle.$$

But we also know that

$$1 = \langle n, \phi_{x_{|_i}}\rangle = \lim_\alpha \langle u_\alpha, \phi_{x_{|_i}}\rangle = \lim_\alpha u_\alpha(x).$$

Hence $n \odot m = n$. But we also know that $n \odot m = m \odot n = m$. Hence, $n = m$ and the TIM is unique. □

Corollary 4.19. *Let $\mathcal{A}(G)$ be one of the algebras $A(G)$, $A_{cb}(G)$ or $A_M(G)$. Let I be a closed ideal in $\mathcal{A}(G)$. If $UCB(I) \subseteq WAP(I)$, then G is discrete.*

Proof. Assume that G is non discrete and that $x \notin Z(I)$. Let $M_1, M_2 \in TIM_I(I^*, \phi_{x_{|_i}}))$ with $M_1 \neq M_2$, Then $m_1 = M_{1_{|_{WAP(I)}}}, m_2 = M_{2_{|_{WAP(I)}}} \in TIM_I(WAP(I), \phi_{x_{|_i}}))$. Let $T \in I^*$ be such that $\langle M_1, T\rangle \neq \langle M_2, T\rangle$ and choose $u \in I$ so that $u(x) = 1$. Then $u \cdot T \in UCB(I) \subseteq WAP(I)$, and

$$\langle m_1, u\cdot T\rangle = \langle M_1, u\cdot T\rangle = \langle M_1, T\rangle \neq \langle M_2, T\rangle = \langle M_2, u\cdot T\rangle = \langle m_2, u\cdot T\rangle$$

which contradicts the uniqueness of the TIM on $WAP(I)$.

□

DOI: https://dx.doi.org/10.30504/JIMS.2023.385500.1091

5. Arens regularity of ideals in $A(G)$, $A_{cb}(G)$ and $A_M(G)$

In this section, we will apply what we know about topologically invariant means to questions concerning the possible Arens regularity of ideals in $A(G)$, $A_{cb}(G)$, and $A_M(G)$. The key observation is the following which improves on [10, Corollary 3.13]:

Theorem 5.1. *Let $\mathcal{A}(G)$ be any of the algebras $A(G)$, $A_{cb}(G)$ or $A_M(G)$. Let I be a non-zero closed ideal in $\mathcal{A}(G)$. If I is Arens regular, then G is discrete.*

Proof. If I is Arens regular, then $I^* = WAP(I)$. Hence I^* has a unique topologically invariant mean. However, by Corollary 4.12, this implies that G must be discrete.

□

The following corollary is immediate. See also [10, Theorem 3.2] and [13, Corollary 3.9].

Corollary 5.2. *Let $\mathcal{A}(G)$ be any of the algebras $A(G)$, $A_{cb}(G)$ or $A_M(G)$. If $\mathcal{A}(G)$ is Arens regular, then G is discrete.*

Corollary 5.3. *Let G be non-discrete. If $\mathcal{A}(G)$ is one of $A(G)$, $A_{cb}(G)$ or$A_M(G)$, then $\mathcal{A}(G)$ has no non-zero reflexive closed ideal.*

Remark 5.4. *Granirer* [14, Theorem 5] *has shown that every infinite discrete group contains an infinite set $E \subset G$ such that the ideal $I_{A(G)}(G \setminus E)$ is isomorphic to ℓ_2. In particular, this ideal is reflexive and hence Arens regular. However, if we ask that I also has a bounded approximate identity, then at least for the Fourier algebra such an ideal can only be Arens regular if it is finite-dimensional.*

Lemma 5.5. *Let $\mathcal{A}(G)$ be any of the algebras $A(G)$, $A_{cb}(G)$ or $A_M(G)$. Let H be a subgroup of G. If $\mathcal{A}(G)$ is Arens regular, then so is $\mathcal{A}(H)$. In particular, if H is amenable, then H is finite.*

Proof. As G is discrete, H is open in G. In this case, the restriction map $R : \mathcal{A}(G) \to \mathcal{A}(H)$ is a contractive homomorphism that is also surjective. As such $\mathcal{A}(H)$ is Arens regular.

The last statement is simply [19, Proposition 3.3].

□

Definition 5.6. *Let $\mathcal{R}(G)$, the coset ring of G, denote the Boolean ring of sets generated by cosets of subgroups of G. A subset E of G is in $\mathcal{R}(G)$ if and only if*

$$E = \bigcup_{i=1}^{n} (x_i H_i \setminus \bigcup_{j=1}^{m_i} b_{i,j} K_{i,j}),$$

where H_i is a subgroup of G, $x_i \in G$, $K_{i,j}$ is a subgroup of H_i, and $b_{i,j} \in K_{i,j}$.

By $\mathcal{R}_a(G)$, the amenable coset ring of G, we will mean all sets of the form

$$E = \bigcup_{i=1}^{n} (x_i H_i \setminus \bigcup_{j=1}^{m_i} b_{i,j} K_{i,j}),$$

where H_i is an amenable subgroup of G, $x_i \in G$, $K_{i,j}$ is a subgroup of H_i, and $b_{i,j} \in K_{i,j}$.

DOI: https://dx.doi.org/10.30504/JIMS.2023.385500.1091

Theorem 5.7. *Let I be a closed ideal of $A(G)$ with a bounded approximate identity that is Arens regular. Then I is finite-dimensional.*

Proof. Assume that $I \subseteq A(G)$ is Arens regular. By Theorem 5.1 G must be discrete. If I has a bounded approximate identity, then since $A(G)$ is weakly sequentially complete (ref), I must be unital. It follows that $1_{G\setminus Z(I)} \in I$. In particular, $G \setminus Z(I)$ must be compact and hence finite. This shows that I is finite-dimensional.

□

Remark 5.8. *The fact that $A(G)$ is weakly sequentially complete was crucial in establishing the previous theorem. Unfortunately, we do not know whether or not either or both of $A_{cb}(G)$ or $A_M(G)$ would be weakly sequentially complete.*

For the remainder of this section, we will assume that G is a discrete group.

Lemma 5.9. *Let $\mathcal{A}(G)$ be any of the algebras $A(G)$, $A_{cb}(G)$ or $A_M(G)$. Let H be a proper amenable subgroup of G. If $I_{\mathcal{A}(G)}(H)$ is Arens regular, then H is finite and $\mathcal{A}(G)$ is also Arens regular.*

Proof. Since H is proper, there exists an $x \in G \setminus H$. The ideal $I_{\mathcal{A}(G)}(xH)$ is isometrically isomorphic to $I_{\mathcal{A}(G)}(H)$ and hence is also Arens regular.

If $u \in \mathcal{A}(H)$, then the function u° defined by $u^\circ(y) = u(y)$ if $y \in H$ and $u^\circ(y) = 0$ if $y \in G \setminus H$ is in $\mathcal{A}(G)$. Now if $R : \mathcal{A}(G) \to \mathcal{A}(H)$ is the restriction map, then R is contractive homomorphism that maps $I_{\mathcal{A}(G)}(xH)$ onto $\mathcal{A}(H)$. In particular, $\mathcal{A}(H) = A(G)$ is also Arens regular. It follows that H is finite.

Let $\mathcal{B}$ be the algebra $1_H\mathcal{A}(G) \oplus_1 1_{G\setminus H}\mathcal{A}(G)$. Then B is a commutative Banach algebra and the mapping $\Gamma : \mathcal{A}(G) \to \mathcal{B}$ given by $\Gamma(u) = (1_H u, 1_{G\setminus H} u)$ is a continuous isomorphism that maps $I_{\mathcal{A}(G)}(H)$ isometrically onto the ideal $(I_{\mathcal{A}(G)}(H), 0)$ in $\mathcal{B}$. Since $1_{G\setminus H})\mathcal{A}(G)$ is finite-dimensional, it is Arens regular. We get that

$$(1_H\mathcal{A}(G) \oplus_1 1_{G\setminus H}\mathcal{A}(G))^{**} = (1_H\mathcal{A}(G))^{**} \oplus_1 (1_{G\setminus H}\mathcal{A}(G))^{**}$$

which is commutative since each of its components is commutative. Hence $1_H\mathcal{A}(G) \oplus_1 1_{G\setminus H}\mathcal{A}(G)$ is Arens regular, and so is $\mathcal{A}(G)$ □

Theorem 5.10. *Let $\mathcal{A}(G)$ be any of the algebras $A(G)$, $A_{cb}(G)$ or $A_M(G)$. Let*

$$E = \bigcup_{i=1}^{n} (x_i H_i \setminus \bigcup_{j=1}^{m_i} b_{i,j} K_{i,j}),$$

where H_i is an amenable subgroup of G, $x_i \in G$, $K_{i,j}$ is a subgroup of H_i, and $b_{i,j} \in K_{i,j}$. If $I_{\mathcal{A}(G)}(E)$ is non-zero and Arens regular, then either E is finite and $\mathcal{A}(G)$ is also Arens regular, or G is amenable and $I(E)$ is finite-dimensional.

DOI: https://dx.doi.org/10.30504/JIMS.2023.385500.1091

Proof. We begin by first assuming that $E = \bigcup_{i=1}^{n} x_i H_i$. In this case, we will prove the conclusion by induction on n. That is we let $P(n)$ be the statement that if $E = \bigcup_{i=1}^{n} x_i H_i$ is a proper subset of G and if $I_{\mathcal{A}(G)}(E)$ is Arens regular, then E is finite and $\mathcal{A}(G)$ is Arens regular.

If $n = 1$, $E = xH$ where H is a proper amenable subgroup. Since $I(H)$ is isometrically isomorphic to $I(xH)$, Lemma 5.9 shows that E is finite and $\mathcal{A}(G)$ is Arens regular.

Assume that $P(n)$ is true for all $n \leq k$. Let $E = \bigcup_{i=1}^{k+1} x_i H_i$ where each H_i is an amenable subgroup of G. By translating if necessary we can assume that $x_{k+1} = e$. If $H_{k+1} \subseteq \bigcup_{i=1}^{k} x_i H_i$, then we have $E = \bigcup_{i=1}^{k} x_i H_i$ and we are done. So we may assume that

$$F = H_{k+1} \setminus (\bigcup_{i=1}^{k} x_i H_i) \neq \emptyset.$$

Note that $H_{k+1} \setminus F \in \mathcal{R}(H_{k+1})$ and

$$I_{\mathcal{A}(H)}(H_{k+1} \setminus F) = I_{\mathcal{A}(G)}(E)_{|_{H_{k+1}}}.$$

In particular, since the restriction map is a homomorphism, $I_{\mathcal{A}(H)}(H_{k+1} \setminus F)$ is Arens regular. But as $H_{k+1} \setminus F \in \mathcal{R}(H_{k+1})$ and H_{k+1} is amenable, we have that $\mathcal{A}(H) = A(H)$ and $I_{\mathcal{A}(H)}(H_{k+1} \setminus F)$ has a bounded approximate identity. It then follows from Theorem 5.7 that F is finite.

Next we observe that E is the disjoint union of $\bigcup_{i=1}^{k} x_i H_i$ and the finite set F. But as F is finite we can proceed in a manner similar to that of the proof of Lemma 5.9 to conclude that $I_{\mathcal{A}(G)}(\bigcup_{i=1}^{k} x_i H_i)$ is also Arens regular. From here the induction hypothesis tells us that $\bigcup_{i=1}^{k} x_i H_i$ is finite. And as $F = H_{k+1} \setminus (\bigcup_{i=1}^{k} x_i H_i)$ is also finite, H_{k+1} is finite. Hence E is finite as well.

If we assume that

$$E = \bigcup_{i=1}^{n} (x_i H_i \setminus \bigcup_{j=1}^{m_i} b_{i,j} K_{i,j}),$$

where H_i is an amenable subgroup of G, $x_i \in G$, $K_{i,j}$ is a subgroup of H_i. We have two cases. The first is that $\bigcup_{i=1}^{n} x_i H_i \neq G$. If this is the case, then if $E_1 = \bigcup_{i=1}^{n} x_i H_i$ then $E \subseteq E_1$ and hence the non-zero closed ideal $I_{\mathcal{A}(G)}(E_1)$ is contained in the Arens regular ideal $I_{\mathcal{A}(G)}(E)$ and is therefore also Arens regular. But we have seen above that this means that E_1 is finite. It follows that E is also finite. As before, this would imply that $\mathcal{A}(G)$ would also be Arens regular.

Finally, if we assume that $\bigcup_{i=1}^{n} x_i H_i = G$. Then by [11, Corollary 3.3] one of the H_i's has finite index in G. Since each H_i is amenable, so is G. This means that we can express $G \backslash E$ as a disjoint union $\bigcup_{l=1}^{m} F_l$ where each F_l is a translate of an element of the coset ring of one of the open amenable subgroups $K_{i,j}$.

DOI: https://dx.doi.org/10.30504/JIMS.2023.385500.1091

Moreover, this means that $I_{\mathcal{A}(G)}(E) = I_{A(G)}(E)$ has a bounded approximate identity [11, Theorem 3.20]. It now follows from Theorem 5.7 that this ideal is finite-dimensional.

□

Acknowledgments

The authors wish to congratulate Professor Anthony To-Ming Lau on his many accomplishments over his distinguished career. In addition, the first author wishes to express his deep gratitude to Professor Lau for his exceptional mentorship, his unwavering support, and for the exemplary role model he has provided for all of us who have had the great pleasure to work with him.

References

[1] R. Arens, The adjoint of a bilinear operation, *Proc. Amer. Math. Soc. (2)* (1951) 839–848.

[2] F. F. Bonsall and J. Duncan, Complete Normed Algebras, Springer-Verlag, New York-Heidelberg Berlin, 1973.

[3] P. Civin and B. Yood, The second conjugate space of a Banach algebra as an algebra, *Pacific J. Math.* **11** (1961), no. 3, 847–870.

[4] M. Cowling and U. Haagerup, Completely bounded multipliers of the Fourier algebra of a simple Lie group of real rank one, *Invent. Math.* **96** (1989), no. 3, 507–549.

[5] J. De Canniere and U. Haagerup, Multipliers of the Fourier algebras of some simple Lie groups and their subgroups, *Amer. J. Math.* **107** (1985), no. 2, 455–500.

[6] J. Duncan and S. A. R. Hosseinium, The second dual of a Banach algebra, *Proc. Roy. Soc. Edinburgh Sect. A* **84** (1979), no. 3-4, 309–325.

[7] C. Dunkl and D. Ramirez, Weakly almost periodic functionals on the Fourier algebra, *Trans. Amer. Math. Soc.* **185** (1973) 501–514.

[8] P. Eymard, L'algèbre de Fourier d'un groupe localement compact, *Bull. Soc. Math. France* **92** (1964) 181–236.

[9] B. E. Forrest, Completely bounded multipliers and ideals in $A(G)$ vanishing on closed subgroups. In: A. T.-M. Lau and V. Runde, Banach Algebras and Their Applications, Contemp. Math., 363, pp. 89–94. Amer. Math. Soc., Providence, 2004.

[10] B. Forrest, Arens regularity and discrete groups, *Pacific J. Math.* **151** (1991), no. 2, 217–227.

[11] B. Forrest, Amenability and ideals in $A(G)$. *J. Austral. Math. Soc. Ser. A* **53** (1992), no. 2, 143–155.

[12] B. Forrest, Arens regularity and the $A_p(G)$ algebras, *Proc. Amer. Math. Soc.* **119** (1993), no. 2, 595–598.

[13] B. Forrest and T. Miao, Uniformly continuous functionals and M-weakly amenable groups, *Canad. J. Math.* **65** (2013), no. 5, 1005–1019.

[14] E. E. Granirer, On some properties of the Banach algebras $A_p(G)$ for locally compact groups, *Proc. Amer. Math. Soc.* **95** (1985), no. 3, 375–381.

[15] Z. Hu, On the set of topologically invariant means on the von Neumann algebra VN(G), *Illinois J. Math.* **39** (1995), no. 3, 463–490.

[16] Z. Hu, Inductive extreme non-Arens regularity of the Fourier algebra $A(G)$, *Studia Math.* **151** (2002), no. 3, 247–264.

[17] S. Kakutani, Two fixed point theorems concerning bicompact convex sets, *Proc. Imp. Akad. Tokyo* **14** (1938), no. 7, 242–245.

[18] E. Kaniuth and A. T. Lau, Fourier and Fourier-Stieltjes Algebras on Locally Compact Groups, Mathematical Surveys and Monographs, 231, Amer. Math. Soc., Providence, 2018.

DOI: https://dx.doi.org/10.30504/JIMS.2023.385500.1091

[19] A. T. Lau, The second conjugate of the Fourier algebra of a locally compact group, *Trans. Amer. Math. Soc.* **267** (1981), no. 1, 53–63

[20] V. Losert, The centre of the bidual of Fourier algebras (discrete groups), *Bull. Lond. Math. Soc.* **48** (2016), no. 6, 968–976.

[21] J. S. Pym, The convolution of functionals on spaces of bounded functions, *Proc. London Math. Soc. (3)* **15** (1965) 84–104.

[22] A. Ülger, Arens regularity sometimes implies the RNP, *Pacific J. Math.* **143** (1990), no. 2, 377–399.

[23] A. Ülger, Arens regularity of weakly sequentially complete Banach algebras, *Proc. Amer. Math. Soc.* **127** (1999), no. 11, 3221–3227.

Brian Forrest

Department of Pure Mathematics, University of Waterloo Waterloo, Ontario, Canada, N2L 3G1.

Email: beforres@uwaterloo.ca

John Sawatzky

Department of Pure Mathematics, University of Waterloo, Waterloo, Ontario, Canada, N2L 3G1.

Email: jmsawatzky@uwaterloo.ca

Aasaimani Thamizhazhagan

Department of Pure Mathematics, University of Waterloo, Waterloo, Ontario, Canada, N2L 3G1.

Email: athamizhazhagan@uwaterloo.ca

DOI: https://dx.doi.org/10.30504/JIMS.2023.385500.1091