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HOCHSCHILD COHOMOLOGY OF SULLIVAN ALGEBRAS AND MAPPING SPACES BETWEEN MANIFOLDS

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Abstract. Let $e : N^n \to M^m$ be an embedding of closed, oriented manifolds of dimension n and m respectively. We study the relationship between the homology of the free loop space LM on M and of the space $L_N M$ of loops of M based in N and define a shriek map $H_*(e)_! : H_*(LM, \mathbb{Q}) \to H_*(L_N M, \mathbb{Q})$ using Hochschild cohomology and study its properties. In particular we extend a result of Félix on the injectivity of the map induced by $\mathrm{aut}_1 M \to \mathrm{map}(N, M; f)$ on rational homotopy groups when M and N have the same dimension and $f : N \to M$ is a map of non zero degree.

1. **Introduction**

All spaces are assumed to be simply connected and (co)homology coefficients are taken in the field $\mathbb{Q}$ of rationals. If M is a compact oriented manifold of dimension m and $LM = \mathrm{map}(S^1, M)$ the space of free loops in M, then there is an intersection product

$$\mu : H_{p+m}(LM) \otimes H_{q+m}(LM) \to H_{p+q+m}(LM)$$

which induces a graded multiplication on $\mathbb{H}_*(LM) = H_{*+m}(LM)$, turning it into a commutative graded algebra [3]. Consider the embedding $e : N \to M$ of a closed, oriented submanifold of degree n.

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.

Construct the pullback

(1.1)
$$\begin{array}{ccc} L_N M & \xrightarrow{\tilde{e}} & LM \\ {\scriptstyle \tilde{p}}\downarrow & & \downarrow{\scriptstyle p} \\ N & \xrightarrow[e]{} & M, \end{array}$$

where p is the evaluation of a loop at $1 \in S^1$. There is also an intersection product on $\mathbb{H}_*(L_N M) = H_{*+n}(L_N M)$, turning it into a commutative graded algebra [16].

We consider a morphism $f : (A, d) \to (B, d)$ which models the embedding e, where (A, d) and (B, d) are Poincaré duality commutative differential graded algebras (cdga for short) [5]. We show that there is an A-linear shriek map $f_! : (B, d) \to (A, d)$ of degree $m - n$. We consider induced maps $HH^*(f) : HH^*(A, A) \to HH^*(A, B)$ and $HH^*(f_!) : HH^*(A, B) \to HH^*(A, A)$ in Hochschild cohomology. Moreover we obtain the following.

Theorem 1.1. *The composition map*

$$HH^*(f_!) \circ HH^*(f) : HH^*(A, A) \to HH^*(A, A)$$

is the multiplication by the Poincaré dual of the fundamental class of N in M.

Theorem 1.2. *Let $g : N^m \to M^m$ be a map between 1-connected compact smooth manifolds of same dimension m such $\deg f \neq 0$ and $f : (A, d) \to (B, d)$ a cdga model of g. Then*

$$HH^*(A, A) \to HH^*(A, B)$$

is injective.

The above result suggests that $\mathbb{H}(\tilde{g}) : \mathbb{H}_*(L_N M) \to \mathbb{H}_*(LM)$ is an injective algebra morphism, where $\tilde{g} : L_N M \to LM$ is the pullback of $g : N \to M$ along the fibration $p : LM \to M$ defined by $p(\gamma) = \gamma(0)$.

The paper is organized as follows: In Section 2 we define a shriek map $f_! : (B, d) \to (A, d)$ and prove Theorem 1.1. In Section 3, we recall a resolution to compute $HH^*(C^*(M), C^*(N))$ and in Section 4 we prove Theorem 1.2.

2. A shriek map

We first recall some facts in Rational Homotopy Theory. We make use of Sullivan models for which the standard reference is [6]. All vector spaces are over the ground field $\mathbb{Q}$. A differential graded algebra (A, d) (dga for short) is a direct sum of vector spaces A^p, that is, $A = \oplus_{p \geq 0} A^p$ together with a graded multiplication $\mu : A^p \otimes A^q \to A^{p+q}$ which is associative and has a unity $1 \in A^0$. An element $a \in A^p$ is called homogeneous of degree $|a| = p$. Moreover there is a differential $d : A^p \to A^{p+1}$ which an algebra derivation, that is, $d(ab) = (da)b + (-1)^{|a|} a(db)$, and satisfies $d^2 = 0$. A dga (A, d) is called connected if $A^0 = \mathbb{Q}$, and 1-connected if $A^0 = \mathbb{Q}$ and $H^1(A, d) = 0$.

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A graded algebra A is called commutative if $ab = (-1)^{|a||b|}ba$ for $a, b \in A$. If (A, d) is a cdga then $H^*(A, d)$ is graded commutative. A morphism $f : (A, d) \to (B, d)$ of cdga's is called a quasi-isomorphism if $H^*(f)$ is an isomorphism.

A commutative graded algebra A is free if it is of the form $\wedge V = S(V^{even}) \otimes E(V^{odd})$, where $V^{even} = \oplus_{i\geq 1} V^{2i}$ and $V^{odd} = \oplus_{i\geq 0} V^{2i+1}$. A Sullivan algebra is a cdga $(\wedge V, d)$, where $V = \oplus_{i\geq 1} V^i$ admits a homogeneous basis $\{x_i\}_{i\in I}$ indexed by a well ordered set I such $dx_i \in \wedge(\{x_i\})_{i<j}$. A Sullivan algebra is called minimal if $dV \subset \wedge^{\geq 2} V$ [6]. If there is a quasi-isomorphism $f : (\wedge V, d) \to (A, d)$, where $(\wedge V, d)$ is a (minimal) Sullivan algebra, then we say that $(\wedge V, d)$ is a (minimal) Sullivan model of (A, d). We state here the fundamental result of Sullivan algebras.

Proposition 2.1. *If (A, d) is a cdga such that $H^0(A, d) = \mathbb{Q}$ then (A, d) has a minimal Sullivan model $(\wedge V, d)$ which is unique up to isomorphism* [6, § 14].

To a simply connected topological space X of finite type, Sullivan associates in a functorial way a cdga $A_{PL}(X)$ of piecewise linear forms on X such $H^*(A_{PL}(X)) \cong H^*(X, \mathbb{Q})$ [17]. A Sullivan model of X is a Sullivan model of $A_{PL}(X)$. Moreover any cdga (A, d) is called a model of X if there is a sequence of quasi-isomorphisms

$$(A, d) \to (A_1, d) \leftarrow \ldots \to (A_{n-1}, d) \leftarrow A_{PL}(X).$$

Definition 2.2. A simply connected space X is called formal if there is a quasi-isomorphism $(\wedge V, d) \to H^*(\wedge V, d)$, where $(\wedge V, d)$ is a Sullivan model of X. Formal spaces include spheres, compact Lie groups and complex projective spaces.

Definition 2.3.
(1) A connected graded commutative algebra A over $\mathbb{Q}$ is called an oriented Poincaré duality algebra of formal dimension n if $A^i = 0$ for $i > n$ and there is a linear map $\epsilon : A^n \to \mathbb{Q}$ such that the induced bilinear forms, $\beta : A^k \otimes A^{n-k}$ defined by $\beta(x \otimes y) = \epsilon(ab)$, are non degenerate.
(2) A commutative differential graded algebra (A, d) is a Poincaré algebra of formal dimension n if A is an oriented Poincaré duality algebra such that $\epsilon(dA^{n-1}) = 0$.

Remark 2.4. If A is a connected Poincaré duality algebra, then $A^i \cong A^{n-i}$. In particular $\epsilon : A^n \to \mathbb{Q}$ is an isomorphism. Moreover if (A, d) is a connected oriented Poincaré duality cdga, then $H^*(A, d)$ is a Poincaré duality algebra of dimension n [13, Proposition 4.8]. Therefore there is a cocycle $\omega_A \in A^n$ such that $\epsilon(\omega_A) = 1$, which is unique up to a factor $q \in \mathbb{Q} \setminus \{0\}$. We call $[\omega_A]$ the fundamental class of (A, d).

If A is of finite type, then A is finite dimensional. Moreover if $\{a_1, \ldots, a_k\}$ is a homogeneous basis of A, then there is a dual homogeneous basis $\{a_j^*\}$ such that $\epsilon(a_i a_j^*) = \delta_{ij}$. We denote by $a^\#$ the dual of a in $A^\# = \mathrm{Hom}(A, \mathbb{Q})$. In particular $\omega_A = \epsilon^\# \in (A^\#)^\# \cong A$ is the fundamental class of A. Moreover the linear map $\pi_A : A^k \to (A^{n-k})^\#$ defined by $\pi_A(a)(x) = \epsilon(ax)$ is an isomorphism of A-modules of lower degree n.

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If $(\wedge V, d)$ is the minimal Sullivan model of a simply connected space X, where $H^*(X, \mathbb{Q})$ satisfies Poincaré duality, then $(\wedge V, d)$ is quasi-isomorphic to a Poincaré duality cdga (A, d) [14]. In particular, a simply connected smooth manifold M of dimension m has a cdga model (A, d) which satisfies Poincaré duality in dimension m.

Let $f : (A, d) \to (B, d)$ be a map between cdga's with Poincaré duality in dimensions m and n respectively, where $m \geq n$. We can now relate isomorphisms $\pi_A : A \overset{\cong}{\to} A^{\#}$ and $\pi_B : B \overset{\cong}{\to} B^{\#}$.

Proposition 2.5. *If f is surjective, then there exists a morphism of A-modules $f_! : B \to A$ making the following diagram commutative.*

$$\begin{array}{ccc} B & \xrightarrow{f_!} & A \\ \simeq \big\downarrow \pi_B & & \pi_A \big\downarrow \cong \\ B^{\#} & \xrightarrow{f^{\#}} & A^{\#} \end{array}$$

Proof. Let $1 \in B$, then $\pi_B(1) = \omega_B^{\#}$, where ω_B is a cocycle which represents the fundamental class $[\omega_B] \in H^n(B)$. As π_A is bijective, there exists a unique $\alpha \in A^{m-n}$ such that $\pi_A(\alpha) = f^{\#}(\omega_B^{\#})$. As f is surjective, then given $b \in B$, there exists $a \in A$ such that $b = f(a)$. Recall that B is an A-module through the action induced by f, hence $b = f(a)1 = a * 1$. Therefore we define $f_!(b) = a\alpha$.

Let $a \in \ker f$ and $x \in A$,

$$\begin{aligned} \pi_A(a\alpha)(x) &= a\pi_A(\alpha)(x) && (\pi_A \text{ is a morphism of } A\text{-modules}) \\ &= af^{\#}(\omega_B^{\#}(x)) && (\pi_A = f^{\#}(\omega_B^{\#})) \\ &= \omega_B(f(ax)) = 0. \end{aligned}$$

Therefore $a\alpha = 0$, hence $f_!$ is well defined. In particular $f_! f(a) = a\alpha$.

We show that the above diagram commutes. Let $b \in B$ and $a \in A$ such that $b = f(a)$. On one hand

$$f^{\#}(\pi_B(b)) = f^{\#}(\pi_B(b \times 1)) = f^{\#}(b\omega_B^{\#}), \tag{2.1}$$

whereas

$$\pi_A(f_!(b)) = \pi_A(a\alpha) = a\pi_A(\alpha) = af^{\#}(\omega_B^{\#}). \tag{2.2}$$

Let $x \in A$. Then

$$f^{\#}(b\omega_B^{\#})(x) = (b\omega_B^{\#})(f(x)) = \omega_B^{\#}(bf(x)), \tag{2.3}$$

and

$$\begin{aligned} (af^{\#}(\omega_B^{\#}))(x) &= (f^{\#}(\omega_B^{\#}))(ax) = \omega_B^{\#}(f(ax)) \\ &= \omega_B^{\#}(f(a)f(x)) = \omega_B^{\#}(bf(x)). \end{aligned} \tag{2.4}$$

Hence $f^{\#}(b\omega_B^{\#}) = af^{\#}(\omega_B)$ and the diagram commutes.

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Finally we show that $f_!$ is a morphism of A-modules. If $x \in A$ and $b \in B$, then

$$f_!(x * b) = f_!(f(x)b) = f_!(f(xa)) = (xa)\alpha = xf_!(b).$$

In particular $f_!(b) = f_!(b \times 1) = a * f_!(1)$. □

Remark 2.6. If ω_B is a cocycle representing the fundamental class of (B, d) and f is surjective, then there exists $x \in A$ such that $f(x) = \omega_B$. Then $f^{\#}(\omega_B^{\#}) = x^{\#} = \pi_A(x^*)$, where x^* is the dual of x under a choice of a basis $\{a_i\}$ of A and its dual $\{a_j^*\}$ (see Remark 2.4). If $dx = 0$, then $[x] \in H^*(A) \neq 0$ and $[x^*] \in H^{m-n}(A)$ is non zero.

Example 2.7. Consider the inclusion $i : \mathbb{C}P^n \to \mathbb{C}P^{n+k}$. As complex projective spaces are formal, a cdga model of the inclusion is

$$f : \wedge x_2/(x_2^{n+k+1}) \to \wedge y_2/(y_2^{n+1}),$$

where $f(x) = y$. Then $f_!$ is defined by $f_!(1) = x^k$. Hence $f_!(y^i) = x^{k+i}$, for $0 \leq i \leq n$.

3. Hochschild cohomology

If (A, d) is a dga, then A is an $A^e = A \otimes A^{op}$ differential graded module by the action $(a_1 \otimes a_2)a = (-1)^{|a_2||a|}a_1 a a_2$, where a, a_1 and a_2 are in A. For an A^e differential graded module (Q, d), the Hochschild cohomology of A with coefficients in Q is defined by $HH^*(A, Q) = \mathrm{Ext}_{A^e}(A, Q)$.

Let $A = (\wedge V, d)$ be the minimal Sullivan model of a simply connected space X and $\mu : (\wedge V \otimes \wedge V, d \otimes 1 + 1 \otimes d) \to (\wedge V, d)$ a model of the diagonal map $\Delta : X \to X \times X$. It converts into a quasi-isomorphism

$$m : (\wedge V \otimes \wedge V \otimes \wedge \bar{V}, \tilde{D}) \to (\wedge V, d),$$

where $\bar{V}^n = (sV)^n = V^{n+1}$ and the differential $\tilde{D}$ is defined by

$$\tilde{D}(v \otimes 1 \otimes 1) = dv \otimes 1 \otimes 1, \quad \tilde{D}(1 \otimes v \otimes 1) = 1 \otimes dv \otimes 1,$$
$$\tilde{D}(1 \otimes 1 \otimes \bar{v}) = (v \otimes 1 - 1 \otimes v) \otimes 1 - \sum_{i=1}^{\infty} \frac{(sd)^i}{i!}(v \otimes 1 \otimes 1),$$

for $v \in V$ and $\bar{v} \in \bar{V}$, and s is the unique derivation on $\wedge V \otimes \wedge V \otimes \wedge \bar{V}$ such that $s(v \otimes 1 \otimes 1) = s(1 \otimes v \otimes 1) = 1 \otimes 1 \otimes \bar{v}$ and $s(1 \otimes 1 \otimes \bar{v}) = 0$ (see [5, §15] or [7]). The map m is defined by $m(a_1 \otimes a_2 \otimes 1) = a_1 a_2$ and by zero on $\wedge V \otimes \wedge V \otimes \wedge^+ \bar{V}$.

Therefore

$$m : (\wedge V \otimes \wedge V \otimes \wedge \bar{V}, \tilde{D}) \to (\wedge V, d) \quad (3.1)$$

is a semi-free resolution of $\wedge V$ as a $\wedge V \otimes \wedge V$-module [7].

Moreover, the pushout

$$\begin{array}{ccc} (\wedge V \otimes \wedge V, d \otimes 1 + d \otimes 1) & \rightarrowtail & (\wedge V \otimes \wedge V \otimes \wedge \bar{V}, \tilde{D}) \\ \downarrow \mu & & \downarrow m \\ (\wedge V, d) & \rightarrowtail & (\wedge V \otimes \wedge \bar{V}, D) \end{array}$$

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yields a Sullivan model $(\wedge V \otimes \wedge \bar{V}, D)$ of the free loop space on X [18]. The differential is given by $Dv = dv$ for $v \in V$ and $D\bar{v} = -Sdv$, where S is the unique derivation on $\wedge V \otimes \wedge \bar{V}$ defined by $Sv = \bar{v}$ and $S\bar{v} = 0$.

Hence if (Q, d) is a $\wedge V$-differential module, then the Hochschild cochains $CH(A, Q)$ are given by

$$CH^*(A,Q) \begin{aligned} &= (\mathrm{Hom}_{\wedge V \otimes \wedge V}(\wedge V \otimes \wedge V \otimes \wedge \bar{V}, Q), D) \\ &\cong (\mathrm{Hom}_{\wedge V}(\wedge V \otimes \wedge \bar{V}, Q), D). \end{aligned} \tag{3.2}$$

As the differential of D on $\wedge V \otimes \wedge \bar{V}$ satisfies

$$D(\wedge V \otimes \wedge^n \bar{V}) \subset \wedge V \otimes \wedge^n \bar{V},$$

one gets a Hodge type decomposition

$$HH^*(A,Q) = \oplus_{i \geq 0} HH^*_{(i)}(A,Q),$$

where $HH^*_{(i)}(A,Q) = H^*(\mathrm{Hom}_{\wedge V}(\wedge V \otimes \wedge^i \bar{V}, \wedge V), D)$. Moreover, if $L = s^{-1}\,\mathrm{Der}\,\wedge V$, then the symmetric algebra $(\wedge_A L, d)$ is quasi-isomorphic to the Hochschild cochain complex $(\mathrm{Hom}_{\wedge V}(\wedge V \otimes \wedge \bar{V}, \wedge V), D)$ [9].
Furthermore if V is finite dimensional then $HH^*(\wedge V, \wedge V)$ is the homology of the complex $(\wedge V \otimes \wedge Z, D)$ where $Z \simeq s^{-1}V^{\#}$ [10].

Assume that M is a simply connected smooth orientable compact manifold of dimension m and $(\wedge V, d)$ its minimal Sullivan model. Then there is an isomorphism of BV-algebras $\mathbb{H}_*(LM) \cong HH^*(\wedge V, \wedge V)$ [4, 7, 8]. For closed oriented submanifolds N and N' of M, we denote by $P_N^{N'}M$ the space of paths in M starting in N and ending in N'. Let N_1, N_2 and N_3 be submanifolds of M. When coefficients are rationals (or in $\mathbb{Z}/n\mathbb{Z}$) Sullivan showed that there is an intersection product

$$\mu : H_{p+d}(P_{N_1}^{N_2}M) \otimes H_{q+d}(P_{N_2}^{N_3}M) \to H_{p+q+d}(P_{N_1}^{N_3}M)$$

where $d = \dim N_2$ [16]. In particular if $N_1 = N_2 = N_3 = N$, one gets a graded commutative algebra structure on $\mathbb{H}_*(P_N^N M, \mathbb{Q}) = H_{*+d}(P_N^N M, \mathbb{Q})$. We consider the subset of $P_N^N M$ consisting of loops that originate in N. This is exactly $L_N M$ defined by the pullback of the diagram (1.1). The restriction yields a product on $\mathbb{H}_*(L_N M) = H_{*+d}(L_N M)$.

Definition 3.1. Let $f : M \to N$ be a map between simply connected compact smooth manifolds of dimensions m and n respectively. A model $\phi : (A, d) \to (B, d)$ of f, where (A, d) and (B, d) are Poincaré duality cdga's, is called a Poincaré duality model of f.

Such a model exists if one assumes that $H^2(f)$ is injective and $m, n \geq 7$ [5, Proposition 1]. However $H^*(f)$ is a Poincaré duality model of f if M and N are formal spaces.

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Let $e : N^n \hookrightarrow M^m$ be an embedding where N is simply connected and $f : (A, d) \to (B, d)$ a Poincaré duality cdga model of e. Assume that f is surjective and let $[y] \in H^n(B)$ be the fundamental class. Let $x \in A$ such that $f(x) = y$. We will assume that x is a cocycle and consider $[x] \in H^n(A, d)$.

Theorem 3.2. *Under the above hypotheses, the composition*

$$HH^*(A, A) \xrightarrow{HH^*(f)} HH^*(A, B) \xrightarrow{HH^*(f_!)} HH^*(A, A)$$

is the multiplication with the Poincaré dual $[x^*] \in H^{m-n}(A, d)$ *of* $[x]$.

Proof. We consider the minimal Sullivan model $\phi : (\wedge V, d) \to (A, d)$. By Eq. (3.2), $HH^*(A, A)$ is obtained as the cohomology of the complex

$$\begin{aligned} \mathrm{Hom}_{\wedge V \otimes \wedge V}(\wedge V \otimes \wedge V \otimes \wedge \bar{V}, \wedge V) &\cong \mathrm{Hom}_{\wedge V}(\wedge V \otimes \wedge \bar{V}, \wedge V) \\ &\simeq \mathrm{Hom}_{\wedge V}(\wedge V \otimes \wedge \bar{V}, A). \end{aligned}$$

If $\gamma \in \mathrm{Hom}_{\wedge V}(\wedge V \otimes \wedge \bar{V}, A)$, then

$$(CH(f_!) \circ CH(f))(\gamma)(x) = (f_! \circ f)(\gamma)(x) = \alpha\gamma(x),$$

where $\alpha = x^*$, by Remark 2.6. Therefore, if γ is a cocycle, then

$$HH^*(f_!) \circ HH^*(f)([\gamma]) = [x^*][\gamma].$$

□

Example 3.3. We consider the embedding $e : \mathbb{C}P^n \hookrightarrow \mathbb{C}P^{n+k}$ for which a Poincaré duality model is given by

$$f : A = \wedge x_2/(x_2^{n+k+1}) \to \wedge y_2/(y_2^{n+1}) = B, \text{ where } f(x_2) = y_2.$$

As f is surjective, the hypotheses of Theorem 3.2 are satisfied. The complex to compute $HH^*(A, A)$ is given by $(A \otimes \wedge(z_1, z_{2(n+k)}), D)$ where subscripts indicate the lower degree, and $Dz_{2(n+k)} = 0$, $Dz_1 = (n + k + 1)x_2^{n+k} z_{2(n+k)}$ [10]. Here an element $x \in A^n = A_{-n}$ is assumed to be of lower degree $-n$. At chain's level, the composition

$$CH^*(f_!) \circ CH(f) : (A \otimes \wedge(z_1, z_{2(n+k)}), D) \to (A \otimes \wedge(z_1, z_{2(n+k)}), D)$$

is the multiplication by x_2^k.

Proposition 3.4. *Let* $e : N \to M$ *be an embedding between closed, oriented manifolds,* $(\wedge V, d)$ *the minimal Sullivan model of* M *and* $Z = s^{-1}V^\#$ *and* $L_N M$ *the pullback of Eq.* (1.1)*. If* $f : (A, d) \to (B, d)$ *is a model of* $e : N \to M$*, then* $HH^*(C^*(M), C^*(N))$ *is computed by the complex* $(B \otimes \wedge Z, D)$ *which is the pushout of the following diagram.*

(3.3)
$$\begin{array}{ccc} (A, d) & \longrightarrow & (A \otimes \wedge Z, D) \\ \downarrow & & \downarrow \\ (B, d) & \longrightarrow & (B \otimes \wedge Z, D) \end{array}$$

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Proof. Let $(\wedge V, d)$ be the minimal Sullivan model of M, where V is finite dimensional. Then $\mathbb{H}_*(LM)$ is the homology of the complex $(\wedge V \otimes \wedge Z, D)$ where $Z = s^{-1}V^{\#}$ and the differential D is induced by δ on $(\mathrm{Der} \wedge V, \delta)$, where $V^{\#} \subset \mathrm{Der} \wedge V$. As $(\wedge V, D) \to (A, d)$ is a quasi-isomorphism, then the pushout is a model of the pullback in Eq. 1.1. □

However, it is not known whether $\mathbb{H}_*(L_N M)$ and $H_*(B \otimes \wedge Z, D)$ are isomorphic as algebras.

4. Maps between manifolds of same dimension

Let $f : (A, d) \to (B, d)$ be a morphism of graded cochain algebras. An f-derivation of degree k is a linear map $\theta : A^* \to B^{*-k}$ such that $\theta(xy) = \theta(x)f(y) + (-1)^{k|x|} f(x)\theta(y)$. We denote by $\mathrm{Der}_k(A, B; f)$ the vector space of all f-derivations of degree k and $\mathrm{Der}(A, B; f) = \oplus_k \mathrm{Der}_k(A, B; f)$. Define a differential δ on $\mathrm{Der}(A, B; f)$ by $\delta\theta = d_B\theta - (-1)^{|\theta|}\theta d_A$. If $A = B$ and $f = 1_A$, we get the usual Lie algebra of derivations, $\mathrm{Der}\, A$, where the Lie bracket is the the commutator of two derivations. There is an action of A on $\mathrm{Der}\, A$, defined by $(a\theta)(x) = a\theta(x)$, making $(\mathrm{Der}\, A, \delta)$ a differential graded module over A.

Let M and N be compact, oriented manifolds of dimension n and $g : N \to M$ a smooth map such that $\deg g \neq 0$. Consider a Poincaré duality model $f : (A, d) \to (B, d)$ of g. Then f is injective and $B = f(A) \oplus Z$, where $dZ \subseteq Z$ and $f(A).Z$ [5]. Therefore Z is an A-submodule. Moreover the projection $p : B = f(A) \oplus Z \to A$ is a morphism of A-modules.

Theorem 4.1 ([5], Theorem 2)**.** *Consider a surjective Sullivan model $\phi : (\wedge V, D) \to (A, d)$. Then*

$$f_* : (\mathrm{Der}(\wedge V, A; \phi), \delta) \to (\mathrm{Der}(\wedge V, B; f \circ \phi), \delta) \tag{4.1}$$

induces an injective map in homology.

This can be interpreted in terms of rational homotopy groups of function spaces. Let $g : X \to Y$ be a continuous map between CW complexes where Y is finite and X of finite type and $\phi : (\wedge Z, d) \to (B, d)$ a Sullivan model of g. Consider $\mathrm{map}(X, Y; g)$ be the space of continuous mappings from X to Y which are homotopic to g. There is a natural isomorphism [1, 2, 15]

$$\pi_n(\mathrm{map}(X, Y; g)) \otimes \mathbb{Q} \cong H_n(\mathrm{Der}(\wedge V, B; \phi), \delta),\ \ n \geq 2.$$

Hence if $g : N \to M$ is a map between simply connected smooth manifolds such that $\deg g \neq 0$, then the map

$$j_M : \mathrm{aut}_1 M = \mathrm{map}(M, M; 1_M) \to \mathrm{map}(N, M; g)$$

induces an injective map

$$\pi_*(j_M) \otimes \mathbb{Q} : \pi_*(\mathrm{aut}_1 M) \otimes \mathbb{Q} \to \pi_*(\mathrm{map}(N, M; g)) \otimes \mathbb{Q}.$$

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Let $\phi : (\wedge V, d) \to (A, d)$ be a Sullivan model and $\rho = f \circ \phi$. We have the following commutative diagram

$$\begin{array}{ccc} H_*(\mathrm{Der}\wedge V, \delta) & \hookrightarrow & HH^*(A, A) \\ \downarrow & & \downarrow \\ H_*(\mathrm{Der}(\wedge V, B; \rho), \delta) & \hookrightarrow & HH^*(A, B), \end{array}$$

where horizontal maps are inclusions [11]. We show that the remaining vertical arrow is injective, which is a generalization of Theorem 4.1.

Theorem 4.2. *Let $g : N \to M$ be a smooth map of non zero degree between manifolds of same dimension n and $f : (A, d) \to (B, d)$ a Poincaré duality model of g. Then the induced map*

$$HH^*(A, A) \xrightarrow{HH^*(f)} HH^*(A, B)$$

is injective.

Proof. As $B = f(A) \oplus Z$, then $f(A) = \rho(\wedge V)$ is a submodule of B viewed as a $\wedge V$-module and Z is also a $\wedge V$-submodule of B. Therefore

$$\mathrm{Hom}_{\wedge V}(\wedge V \otimes \wedge \bar{V}, B) \cong \mathrm{Hom}_{\wedge V}(\wedge V \otimes \wedge \bar{V}, f(A)) \oplus \mathrm{Hom}_{\wedge V}(\wedge V \otimes \wedge \bar{V}, Z).$$

Moreover, the projection $p : B = f(A) \oplus Z \to f(A) \cong A$ is a morphism of $\wedge V$-modules. It induces a chain map

$$p_* : \mathrm{Hom}_{\wedge V}(\wedge V \otimes \wedge \bar{V}, B) \to \mathrm{Hom}_{\wedge V}(\wedge V \otimes \wedge \bar{V}, A)$$

such that $p_* \circ f_*$ is the identity. Therefore f_* is injective in homology. □

We can then deduce the following

Corollary 4.3. *Under the hypotheses of Theorem 4.2, there is an injective map $H_*(f)_! : H_*(LM, \mathbb{Q}) \to H_*(L_N M, \mathbb{Q})$*

Proof. Recall that there is an isomorphism $HH_*(A, A) \cong H^*(LM)$ [12]. Dualizing this isomorphism and using Poincaré duality yields an isomorphism $HH^*(A, A^\#) \cong H_*(LM)$. In the same way, there is an isomorphism $HH^*(A, B^\#) \cong H_*(L_N M)$. Hence $H_*(f)_!$ is given by the composition

$$HH^*(A, A^\#) \xrightarrow{(\pi_A)_*^{-1}} HH^*(A; A) \xrightarrow{f_*} HH^*(A, B) \xrightarrow{(\pi_B)_*} HH^*(A, B^\#).$$

Hence it is injective. □

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