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A TOPOLOGIST'S INTERACTIONS WITH DEREK J. S. ROBINSON AND HIS MATHEMATICS

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Dedicated to Professor Derek J. S. Robinson

Abstract. The paper begins with a brief history of this topologist's interactions with Derek J. S. Robinson. It continues with a topological proof of Derek's result showing that the Schur multiplier of a generalized Baumslag-Solitar group G is free abelian of rank one less than the rank of the torsion free first homology of G and that both of these ranks can be computed by inspecting a weighted directed graph associate to G. In this paper the topology of a special subclass of Seifert fibred 2-dimensional complexes is used to provide proofs of Derek's results.

My original introduction to Derek J.S. Robinson and his mathematics was via his book *A Course in the Theory of Groups* [10] while working on my Ph.D. in topology at the University of Iowa. It was one of the texts I consulted regularly during graduate school (and ever since).

A few years after finishing my Ph.D., I was investigating connected topological spaces Y, like the n-tori $T^n = S^1 \times \cdots \times S^1$, which have the property that all their connected finite sheeted covering spaces $f : X \to Y$ have total space X homeomorphic to Y. After a colloquium talk at my home institution by Nigel Boston, one of Derek's colleagues at the time at the University of Illinois, I asked Nigel about the group theoretic version of these investigations. That is, I asked Nigel about groups G, other than $G = \mathbb{Z} \times \cdots \times \mathbb{Z}$, which have the property that all their finite index subgroups are isomorphic to G. A few days later I received an email from Nigel in which he indicated that he hadn't

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been able to make much progress on the particular question I had asked and he asked if he could share the question with one of his group theorist colleagues. My answer in the affirmative led to an informative, enjoyable, and productive 20+ year long mathematical collaboration with Derek.

Derek and I have co-authored five papers. In the first of these [13] Derek proves a structure theorem for finitely generated groups all of whose finite index subgroups are isomorophic to the whole group. The other four papers [2, 3, 4, 5], co-authored with Alberto Delagado, investigate various aspects of the finitely presented generalized Baumslag-Solitar (GBS) groups – the connection with the question originally posed to Nigel Boston is that all finitely presented GBS groups G have at least one proper finite index subgroup isomorophic to the whole group G. The results in [2, 3], and [4] focus on the group theory while those in [5] look at the interactions between the group theory of GBS groups and the topology of 3-dimensional manifolds using an approach modeled after those in Heil [6] and Hempel [7].

The paper [2] introduces a class of special group homomorphisms $\varphi : G_1 \to G_2$ between GBS groups G_1 and G_2, the so called geometric homomorphisms of the title, which are induced by the weighted directed graphs, let's call them *GBS graphs*, which can be used to define the groups. In [3], these maps and the GBS graphs which induce them are used to classify the soluble GBS groups. In [4], the combinatorial properties of the GBS graph which defines a GBS group G are further exploited to arrive at an algorithm for determining the center and cyclic radical of G. It is also shown that the unimodular GBS groups are in a sense "covered," by a GBS group with particularly nice local structure around each vertex in the associated GBS graph. This "covering" property is further exploited in [4] where those GBS groups which can be fundamental groups of compact orientable 3-manifolds are determined.

Derek's approach to the GBS groups is via classical group theoretic techniques, Alberto's approach is from a geometric group theoretic perspective, mine is from the topological point of view. These disparate points of view had multiple benefits. In particular, when we had a result, they provided three frequently quite different validations of it. Also, and perhaps more importantly, they provided the opportunity for a lively, intense, and enjoyable process of discovery and discussion of the mathematics. Finally, the discussions generated by these different points of view began to suggest that the topology of a class of 2-dimensional spaces originally investigated in [1], the group theory of the GBS groups, and the combinatorics of the GBS graphs are three different manifestations of the same phenomena.

The mathematics below is another installment, see also [15], illustrating in more detail the connections between the topology, group theory, and combinatorics begun in [1]. It develops topological proofs of Derek's group theoretic results on the Schur multiplier of the GBS groups [11]. The results here and in [15] can also be thought of as a topological parallel to Derek's summary of the state of the GBS art, at the time, provided in [12].

The paper is organized as follows. Section 1 contains basic definitions and results on the combinatorics, group theory, and topology needed in the sequel sections. It closes with statements of Derek's results on the Schur multiplier and first homology of a generalized Baumslag-Solitar group. Sections 2 and 3 continue with additional details on the topology and combinatorics, respectively, needed in

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the topological proofs of Derek's results. Sections 4 and 5 contain the details of these proofs. Section 6 contains statements of some open problems resulting from my long association with Derek and his mathematics.

1. **Preliminaries**

The space $\tau_n = \{r\exp(i\theta) : 0 \leq r \leq 1; \theta = \frac{2\pi k}{n}, k = 0, \ldots, n-1\}$ is an *n-od*. We can also write τ_n as the union $\tau_n = \bigcup_{i=1}^{n}[0,1]_i$ of n intervals $[0,1]_i$ with the n copies of the origin identified. The 2-dimensional space $S^1 \times \tau_n$ is a disjoint union $S^1 \times \tau_n = \cup\{S^1 \times t : t \in \tau_n\}$ of circles $S^1 \times t$ called *Seifert fibres* and there is a quotient map $\eta : S^1 \times \tau_n \to \tau_n$ given by $S^1 \times t \mapsto t$. Consequently $S^1 \times \tau_n$ is a *Seifert fibred 2-complex with base space τ_n and projection η.*

In general a *Seifert fibred 2-dimensional complex* is a 2-dimensional metric space $K = \bigcup_{j\in J} S^1_j$ which can be (1) written as a union of a pairwise disjoint collection of circles $\{S^1_j : j \in J\}$, called *fibres*, together with a quotient map $\eta : K \to B$ mapping each S^1_j to a point in B and (2) for which for each S^1_j there is an $n_j \in \mathbb{N}$ and a neighborhood $N(S^1_j)$ which is a union of fibres and which is finitely-to-one covered by a Seifert fibration preserving covering projection from the product $S^1 \times \tau_{n_j}$, with the Seifert fibration above, onto $N(S^1_j)$. The quotient map η is called the *projection* of the Seifert fibration and the space B is its *base space*.

A special class of Seifert fibred 2-complexes, those exhibiting a certain "rigidity," call them *generalized Baumslag-Solitar (GBS) complexes*, are studied in [15]. This note continues the exploration of the relationships between the various classes of GBS objects by providing proofs of the homological results of Robinson [11] using the topology of these Seifert fibred spaces. Note that Levitt [8] also produces a version of the result about the first homology of a GBS group. He does so by using geometric group theoretic methods.

Formally, let Γ be a finite connected graph, loops and multiple edges between pairs of vertices allowed, with vertex set $V(\Gamma)$ and directed edge set $E(\Gamma)$. For each edge e, assign an initial vertex e^- and terminal vertex e^+ indicating the direction on e. See, for example, Figure 1. The directed edge e with initial vertex $e^- = u$ and terminal vertex $e^+ = v$ is sometimes denoted $e = [u, v]$ and the weighted edge (e, ω) with weights $\omega(e) = (m, n)$ can be written $(e, \omega) = [u^m, v^n]$. Associate to e, e^- and e^+ infinite cyclic groups $\langle u_e\rangle$, $\langle g_{e^-}\rangle$, and $\langle g_{e^+}\rangle$. Monomorphisms from $\langle u_e\rangle$, into $\langle g_{e^-}\rangle$, and $\langle g_{e^+}\rangle$ are determined by the assignments $u_e \mapsto g_{e^-}^{\omega^-(e)}$ and $u_e \mapsto g_{e^+}^{\omega+(e)}$ where $\omega^{\pm}(e) \in \mathbb{Z}^* = \mathbb{Z} \setminus 0$. These data constitute a graph of groups (Γ, ω) which is completely determined by the finite connected graph together with the *weight function* $\omega : E(\Gamma) \to \mathbb{Z}^* \times \mathbb{Z}^*$ whose values will be written $\omega(e) = (\omega^-(e), \omega^+(e))$. The pair (Γ, ω) is a *generalized Baumslag-Solitar (GBS) graph.* Its *fundamental group* is $G = \pi_1(\Gamma, \omega)$. Note that $G = \pi_1(\Gamma, \omega)$ is not the topological fundamental group of the graph Γ. The topological fundamental group of Γ is $\pi_1(\Gamma, x_0)$ with base point $x_0 \in \Gamma$.

To obtain a presentation for G, choose a maximal (spanning) subtree T of Γ. The group G has (1) a *vertex generator* g_v for each vertex v of Γ; (2) an *edge generator* t_e for each edge $e \in E(\Gamma \setminus T)$;

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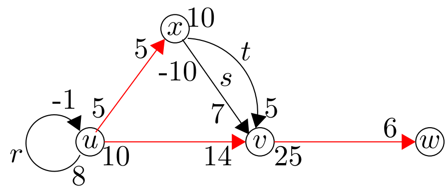

FIGURE 1. An example of a GBS graph (Γ, ω).

(3) *T-relations*, $g_{e^-}^{\omega^-(e)} = g_{e^+}^{\omega^+(e)}$ for $e \in E(T)$; and (4) *non-T relations*, $t_e^{-1} g_{e^-}^{\omega^-(e)} t_e = g_{e^+}^{\omega^+(e)}$ for $e \in E(\Gamma \setminus T)$. A presentation for a GBS group which has generators and relations as indicated is called a *GBS presentation*. For example, consider the GBS graph (Γ, ω) of Figure 1. Let T be the spanning subtree of Γ formed by the red edges in the figure. The non-T edges of Γ are the black edges. Their labels are r, s, and t. We use the vertex labels themselves to represent the generators associated to each vertex and the non-T edge labels to represent the corresponding edge generators. Then, $\pi_1(\Gamma, \omega)$ has a presentation with four vertex generators x, u, v, w; three non-T edge generators r, s, t; and six relations

$$u^5 = x^5, u^{10} = v^{14}, v^{25} = w^6, r^{-1}u^8 r = u^{-1}, s^{-1}x^{-10}s = v^7, t^{-1}x^{10}t = v^5.$$

While the presentation depends on T, it is well known, e.g., Serre [14], or for a topological approach [1] or [15], that $\pi_1(\Gamma, \omega)$ is independent of T. Note also that by collapsing T to a point, it is easy to see that the topological fundamental group $\pi_1(\Gamma, x_0)$ is the free group F_3 on three generators. Thus, it is clear that, in general, the GBS group associated to a GBS graph (Γ, ω) is different then the topological fundamental group of the graph Γ.

The GBS graph (Γ, ω), recall Γ is connected, also defines a connected 2-dimensional complex, a *generalized Baumslag-Solitar (GBS) complex* $K(\Gamma, \omega)$. Think of the 2-complex as being embedded in some high dimensional Euclidean space. It is accordingly a metric space. It is constructed as follows. See Figures 2, 3.

Let (Γ, ω) be a GBS graph. For each $v \in V(\Gamma)$, let

$$S_v^1 = \{\exp(i\theta) : 0 \leq \theta \leq 2\pi\}$$

be a copy of the oriented unit circle in the complex plane with the orientation induced by the parametrization. The S_v^1, $v \in V(\Gamma)$ are called *vertex circles*. For each $e \in E(\Gamma)$, there is an annulus

$$A_e = S_e^1 \times [0,1]_e$$

where S_e^1 is again a copy of the oriented unit circle with the orientation induced by the parametrization above and $[0,1]_e$ is a copy of the unit interval parametrized in the usual way. Both boundary circles of A_e are oriented by the parametrizations they inherent from S_e^1. For each e define the oriented degree $\omega^{\mp}(e)$ attaching maps to be the $\omega^{\mp}(e)$-cyclic covers

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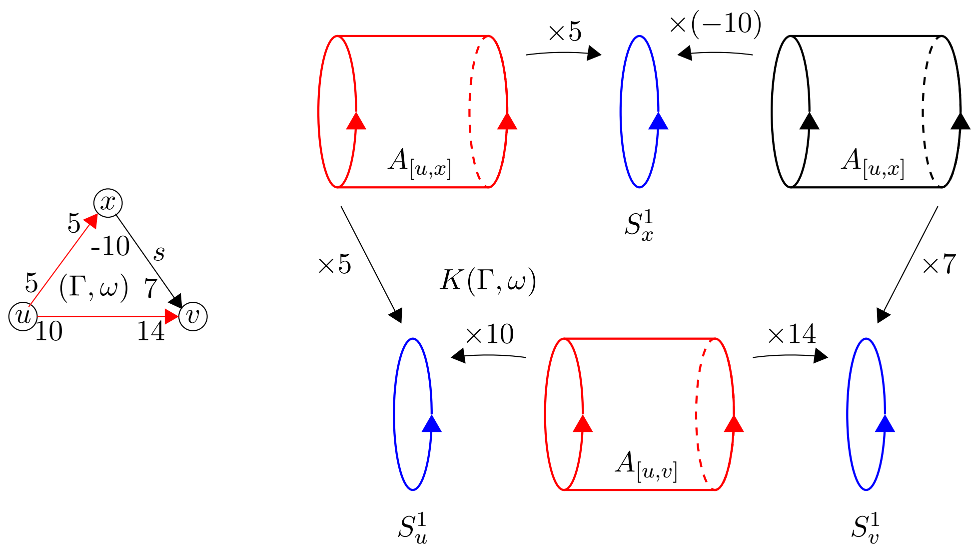

FIGURE 2. A GBS graph (Γ, ω) and the associated GBS complex $K(\Gamma, \omega)$.

$$p_{e^-} : S^1_e \times 0 \to S^1_{e^-}\textbf{:} \text{ given by } (\exp(i\theta), 0) \mapsto \exp(i \cdot \omega^-(e)\theta), \text{ and}$$
$$p_{e^+} : S^1_e \times 1 \to S^1_{e^+}\textbf{:} \text{ given by } (\exp(i\theta), 1) \mapsto \exp(i \cdot \omega^+(e)\theta).$$

Let $K(\Gamma, \omega)$ be the resulting 2-complex. The collection of quintuples

$$\mathcal{A} = \mathcal{A}(\Gamma, \omega) = \left\{ \left(A_e, S^1_{e^-}, S^1_{e^+}, p_{e^-}, p_{e^+} \right) : e \in E(\Gamma) \right\}$$

forms the *annular decomposition for* $K(\Gamma, \omega)$ *associated to* (Γ, ω). The collection of parametrized circles and annuli in the annular decomposition for a GBS complex together with the specified attaching maps form the *rigidity conditions* mentioned above.

We will not normally distinguish between the annulus A_e as an element of $\mathcal{A}$ and its image, which may be a mapping cylinder or mapping torus, in $K(\Gamma, \omega)$. Both will be called an *annulus*. Similarly, the circles $S^1_e \times 0$ and $S^1_e \times 1$ and their images $S^1_{e^-}$ and $S^1_{e^+}$ in $K(\Gamma, \omega)$ will all be called *vertex circles*. The context should make our intentions clear. $K(\Gamma, \omega)$ inherits a Seifert fibration from the product Seifert fibration on each of the annuli $A_e \in \mathcal{A}$. The fibres of the Seifert fibration of $K(\Gamma, \omega)$ are the vertex circles and the $S^1_e \times t$, where $t \in (0, 1)_e$. The vertex circles are also called *exceptional circles* because the locally topology around points on them is (usually) different then the points in the interiors of the annuli A_e. The circles $S^1_e \times t$, $t \in (0, 1)_e$, which fibre the interior of the annulus A_e are *ordinary circles*.

Of fundamental importance in [1, 15], and below is that the rigidity conditions imply that the arc $1_e \times [0, 1]_e \subset S^1_e \times [0, 1]_e$ forms a loop in $K(\Gamma, \omega)$ if and only if $e \in E(\Gamma)$ forms a loop in Γ. More generally the rigidity conditions imply that the 1-dimensional subspace $\cup\{1_e \times [0, 1,]_e \subset K(\Gamma, \omega) : e \in E(\Gamma)\}$ is homeomorphic to Γ and sits in $K(\Gamma, \omega)$ in a natural way. More is said about this below.

Since the GBS group $\pi_1(\Gamma,\omega)$ and the GBS complex $K(\Gamma,\omega)$ are determined by the same GBS graph, it is natural to investigate and exploit connections between the group theory of $\pi_1(\Gamma,\omega)$, the topology of $K(\Gamma,\omega)$, and the combinatorics of (Γ,ω). This note does so by producing topological proofs of Derek's results on the homology of the GBS groups. These results determine the structure of the Schur multiplier, i.e., the second homology, of a GBS group and the rank of the torsion free part of the first homology of the group. Specifically, let $\beta_i(G)$ denote the i^{th}-betti number of the group G, that is, the rank of the free abelian summand of the i-th homology group of G. Derek's theorems are below. Gilbert Levitt's version of Theorem 1.2 appears in [8], Proposition 1.1. Note, all homology computations use $\mathbb{Z}$ coefficients.

Theorem 1.1. *Let (Γ,ω) be a GBS graph with associated GBS group $G=\pi_1(\Gamma,\omega)$. Then $H_2(G)$ is free abelian of rank $\beta_1(G)-1$.*

Theorem 1.2. *Let (Γ,ω) be a GBS graph with associated GBS group G and $\epsilon=\epsilon(\Gamma,\omega)\in\{0,1\}$. Then*

$$\beta_1(G)=|E(\Gamma)|-|V(\Gamma)|+1+\epsilon.$$

Consequently, $\beta_2(G)=|E(\Gamma)|-|V(\Gamma)|+\epsilon$.

The value $\epsilon=\epsilon(\Gamma,\omega)$ is a function which encodes combinatorial relationships among the weights on Γ. Thus, the above results show that the ranks of the Schur multiplier and the torsion free portion of the first homology of a GBS group can be read off from the combinatorial structure of a GBS graph associated to the group. In the many conversations Derek, Alberto, and I had about GBS groups, I always thought of $\epsilon(\Gamma,\omega)$ as "Robinson's ϵ." It's precise meaning is given below. An immediate interesting corollary to Theorem 1.2, pointed out by the referee, is

Corollary 1.3. *If (Γ,ω) is a GBS graph with associated GBS group G, then the first betti number, $\beta_1(G)$ is strictly positive.*

To see this, note that for any connected graph Γ, $E(\Gamma)-V(\Gamma)\geq -1$ with equality exactly when Γ is a tree (start with a maximal subtree T of Γ which has $E(T)-V(T)=-1$ and add the remaining edges). When Γ is not a tree, $E(\Gamma)-V(\Gamma)+1>0$. When Γ is a tree, $\epsilon(\Gamma)=1$, and it follows that $E(\Gamma)-V(\Gamma)+1+\epsilon=1$.

2. GBS complexes

Given a finite connected GBS graph (Γ,ω) it is known [1] that the topological fundamental group $\pi_1(K(\Gamma,\omega),x_0)$ of the GBS complex $K(\Gamma,\omega)$ is isomorphic to the group theoretic fundamental group $\pi_1(\Gamma,\omega)$. As mentioned above, the rigidity conditions on $K(\Gamma,\omega)$ allow one to show that there is a copy of Γ which sits in $K(\Gamma,\omega)$ in a natural way. In fact, let $\eta: K(\Gamma,\omega)\to\Gamma$ be the projection of the of Seifert fibration to its base space Γ. Then, by [1, 15], the Seifert fibration of $K(\Gamma,\omega)$ has a *section*. That is, there is an embedding $\iota:\Gamma\to K(\Gamma,\omega)$ such that $\eta\circ\iota=id_\Gamma$. Immediate consequences of the existence of this section are that the topological fundamental group $\pi_1(\Gamma,x_0)$ embeds into the

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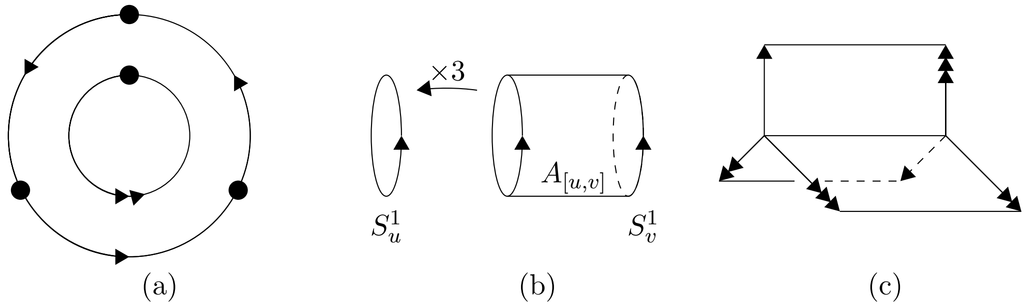

FIGURE 3. Three illustrations of $K([u^3, v^1])$. The drawings in (a) and (b) emphasize that $K([u^3, v^1])$ has an annular decompostion. Because the degree of the attaching map $p_{[u,v]^+} : S^1 \times 1 \to S^1_v$ is 1, S^1_v is identified with $S^1 \times 1$ in (b). Part (c) emphasises that it is a locally trivial 3-od bundle over S^1.

group theoretic fundamental group $\pi_1(\Gamma, \omega) \cong \pi_1(K(\Gamma, x_0))$ and that there is an epimorphism from the group theoretic $\pi_1(\Gamma, \omega) \cong \pi_1(K(\Gamma, x_0))$ onto the topological $\pi_1(\Gamma, x_0)$.

The topological fundamental group $\pi_1(\Gamma, x_0)$ of a finite connected graph Γ is a free group F_n on some number n of generators. Consequently, when Γ contains two or more distinct simple circuits, the topology gives that the group theoretic $\pi_1(\Gamma, \omega)$ both contains a free group F_n, $n \geq 2$ and has it as a quotient. Thus, the topology implies that the GBS groups whose associated GBS graphs contain at least two simple circuits have structures at least as complicated as the free group on 2 generators. At the homological level, the existence of this section immediately provides a lower bound on the rank of the first betti number of the GBS group $\pi_1(\Gamma, \omega)$ [15].

In fact, by choosing a maximal subtree $T \subset \Gamma$, this embedded copy of Γ can then be used to define a natural or *standard system* [15] of generators for $\pi_1(K(\Gamma, \omega), x_0)$. It consists of the homotopy classes of the vertex circles together with the homotopy classes of the simple circuits determined by the non-T edges of Γ: each non-T edge determines the simple circuit in Γ consisting of e and the unique simple path in T from e^- to e^+. Repeated applications of Van Kampen's Theorem to obtain a presentation for $\pi_1(K(\Gamma, \omega), x_0)$ produces a GBS presentation which is the same presentation one obtains by using the same maximal subtree T of Γ to produce the GBS presentation for the group theoretic $\pi_1(\Gamma, \omega)$.

We also use the result in [1] that a GBS complex $K(\Gamma, \omega)$ is aspherical. Then, applying standard results, e.g., Mac Lane [9, p.344], we have

Theorem 2.1. (The Homology Isomorphism Theorem) *Let (Γ, ω) be a GBS-graph. Then for each $i \geq 0$ the integral homology groups $H_i(\pi_1(\Gamma, \omega))$ and $H_i(K(\Gamma, \omega))$ are isomorphic.*

Thus, topological methods can be used to compute all of the homology groups of a GBS group. For example, because $K(\Gamma, \omega)$ is 2-dimensional, it immediately follows that

Corollary 2.2. *The higher homology groups $H_i(\pi_1(\Gamma, \omega)) \cong H_i(K(\Gamma, \omega))$, $i \geq 3$ are all trivial.*

The fact that Γ sits in $K(\Gamma, \omega)$ in a natural way and that it has a natural CW structure allows one to compute the Euler characteristic $\chi(K(\Gamma, \omega)) = \chi(\pi_1(\Gamma, \omega))$ quite easily. $K(\Gamma, \omega)$ inherits a CW

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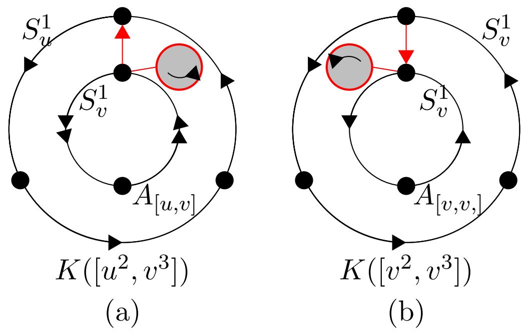

FIGURE 4. The base case of the induction for the proof of the Presentation Theorem.

structure via the following construction. Γ has a CW structure consisting of one 0-cell for each vertex $u \in V(\Gamma)$ and one 1-cell for each edge $e \in E(\Gamma)$. Then attach to Γ an addition 1-cell $c_u^{(1)} = [0,1]_u$ by identifying 0 and 1 with u to form a vertex circle S_u^1. Finally, for each $e \in E(\Gamma)$, attach a 2-cell $c_e^{(2)}$ to the 1-skeleton $\Gamma \cup \left(\cup\{S_u^1 : u \in V(\Gamma)\}\right)$ via instructions provided by the weight function $\omega(e)$. Count cells in each dimension and form their alternating sum. This produces

Corollary 2.3. $\chi(\pi_1(\Gamma, \omega)) = \chi(K(\Gamma, \omega)) = 0$.

2.1. **A simplicial structure on $K(\Gamma, \omega)$ and its immediate consequences.** We henceforth denote the standard copy of Γ in K by the same symbol. We assume that $K = K(\Gamma, \omega)$ is triangulated so that for each edge e of Γ the annulus A_e in the annular decomposition is triangulated. Assume that each vertex v in the standard copy of Γ in K is a 0-simplex in the triangulation. Also assume that for each vertex v and each edge e, the associated vertex circle S_v^1 in K and the embedded copy of e in K are unions of 1-simplices in the given triangulation. Assume that the triangulation is fine enough that for each edge e there is a closed disk $D_e \subset int(A_e)$ such that both D_e and its closed complement $A_e \setminus Int(A_e)$ are unions of 2-simplices of the triangulation. Finally, for each e choose some point $P_e \in int(D_e)$. We assume each P_e is a 0-simplex in the triangulation. We also require that when T is a maximal subtree of Γ, v a vertex of Γ and e a non-T edge of Γ, the standard generators x_v and t_e are also unions of 0- and 1-simplices of the triangulation.

Let $\emptyset \neq S \subset E(\Gamma)$. Let $\mathcal{D}_S = \cup\{D_e : e \in S\}$ and let $\mathcal{C}_S$ be the closed complement of $\mathcal{D}_S$. That is, $\mathcal{C}_S \equiv_{def} K \setminus (\cup\{int(D_e) : e \in S\})$. Let Γ_S be the graph with vertex set $V(\Gamma_S) = V(\Gamma)$ and edge set $E(\Gamma_S) = E(\Gamma) \setminus E(S)$. We also write $\Gamma \setminus S$ for Γ_S. Note that $V(\Gamma_S) = V(\Gamma)$. Let $K_S = K(\Gamma_S, \omega)$. In particular, $K_{E(\Gamma)} = \cup\{S_v^1 : v \in V(\Gamma)\}$ is the disjoint union of the vertex circles.

We use integral simplicial homology. The notation $[*]$ denotes the homology class of $*$ of the appropriate dimension. We use the notation $\mathbb{Z}\langle\alpha\rangle$ when we need to specify a particular generator α for the infinite cyclic group $\mathbb{Z}$.

Now consider the upper end of the Mayer-Vietoris sequence for the simplicial triple $(K, \mathcal{D}_S, \mathcal{C}_S)$. Observe that for each $e \in S$, the disk D_e strong deformation retracts to the point P_e. Therefore, $\mathcal{D}_S$ strong deformation retracts to the finite point-set $\{P_e : e \in S\}$. Also, for each $e \in S$, there is a strong deformation retract of $A_e \setminus int(D_e)$ onto the 1-complex $S_{e^-}^1 \cup e \cup S_{e^+}^1$. This collection of

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strong deformation retracts induces a strong deformation retract of $\mathcal{C}_S$ onto the (connected) complex $K_S \cup \left(\cup\{S^1_{e^-} \cup e \cup S^1_{e^+} : e \in S\}\right) \subset K$.

Since all of K, $\mathcal{D}_S$, and $\mathcal{C}_S$ are 2-dimensional, all of the higher homology groups for all three are trivial for every $\emptyset \neq \mathcal{S} \subset E(\Gamma)$. This produces the Mayer-Vietoris sequence

$$\begin{aligned} 0 \to \quad & H_2(\mathcal{C}_S \cap \mathcal{D}_S) \to H_2(\mathcal{C}_S) \oplus H_2(\mathcal{D}_S) \\ \to \quad & H_2(K) \xrightarrow{\partial} H_1(\mathcal{C}_S \cap \mathcal{D}_S) \xrightarrow{\varphi} H_1(\mathcal{C}_S) \oplus H_1(\mathcal{D}_S) \to \cdots \end{aligned} \tag{2.1}$$

But, $\mathcal{D}_S \cap \mathcal{C}_S = \partial \mathcal{D}_S$ is a collection of $|S|$ circles and $\mathcal{D}_S$ has the homotopy type of a collection of $|S|$ points. Thus, all of $H_2(\mathcal{C}_S \cap \mathcal{D}_S)$, $H_2(\mathcal{D}_S)$, and $H_1(\mathcal{D}_S)$ are trivial for dimensional reasons and $H_1(\mathcal{D}_S \cap \mathcal{C}_S) = \overset{|S|}{\underset{i=1}{\oplus}} \mathbb{Z}$. Therefore, equation (2.1) simplifies to

$$0 \to \quad H_2(\mathcal{C}_S) \oplus 0 \to H_2(K) \xrightarrow{\partial} \overset{|S|}{\underset{i=1}{\oplus}} \mathbb{Z} \xrightarrow{\varphi} H_1(\mathcal{C}_S) \oplus 0 \to \cdots \tag{2.2}$$

Now set $S = E(\Gamma)$. In this case $K_S = K_{E(\Gamma)}$ is just the union of the vertex circles of K and $\mathcal{C}_S = \mathcal{C}_{E(\Gamma)}$ strong deformation retracts to the 1-complex

$$\cup\{S^1_{e^-} \cup e \cup S^1_{e^+} : e \in E(\Gamma)\} = \Gamma \cup \left(\cup\{S^1_v : v \in V(\Gamma)\}\right).$$

Therefore, again for dimensional reasons, equation (2.2), simplifies to

$$0 \to \quad H_2(K) \xrightarrow{\partial} \overset{|E(\Gamma)|}{\underset{i=1}{\oplus}} \mathbb{Z} \xrightarrow{\varphi} H_1(\mathcal{C}_{E(\Gamma)}) \oplus 0 \to \cdots \tag{2.3}$$

From the exactness of equation (2.3), we easily obtain part of Theorem 1.1, also noted in [15] that $H_2(K)$ is free abelian. We have, in fact, the following:

Proposition 2.4. *If $K = K(\Gamma, \omega)$, then $H_2(K) \cong H_2(\pi_1(\Gamma, \omega))$ is free abelian of rank $\beta_2(K) \leq |E(\Gamma)|$. Consequently, since $\chi(K) = 0$, it follows that $|E(\Gamma \backslash T)| - 1 \leq \beta_2(K) \leq |E(\Gamma)|$ and $|E(\Gamma \backslash T)| \leq \beta_1(K) \leq |E(\Gamma)| + 1$.*

Given a maximal subtree T of Γ, each non-T edge e determines two important equations in $\pi_1(K)$. For $u, v \in V(\Gamma)$ we have $\gamma(u, v)$, the unique simple directed path in T from u to v. If $u = v$, then $\gamma(u, v)$ is a point and let $(p^-(u, v), p^+(u, v)) = (1, 1)$. Otherwise, let $p^-(u, v)$ be the product of all the initial weights of the edges in $\gamma(u, v)$ between u and v and $p^+(u, v)$ be the product of all the terminal weights of the edges of $\gamma(u, v)$ between u and v. For the maximal subtree T of Γ and $e \in E(\Gamma \setminus T)$, define the *weight product function* $p(e) = p_T(e) = (M, N)$ where $M = p^-(e^-, e^+)$ and $N = p^+(e^-, e^+)$. Then $\gamma(e^-, e^+)$ determines the relation $x^M_{e^-} = x^N_{e^+}$ in the GBS group $\pi_1(\Gamma, \omega)$. Because e is a non-tree edge, setting $\omega(e) = (m, n)$ produces the relation $t_e x^m_{e^-} t^{-1}_e = x^n_{e^+}$ in $\pi_1(K(\Gamma, \omega), x_0)$. Since $[x_{e^\pm}] = [S^1_{e^\pm}]$ these equations produce the important homological equations in the next proposition.

Proposition 2.5. *Let $K = K(\Gamma, \omega)$. In $H_1(K)$ there are the equations $M[S^1_{e^-}] = N[S^1_{e^+}]$ and $m[S^1_{e^-}] = n[S^1_{e^+}]$.*

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3. Tree and non-tree dependent GBS graphs

We say the non-T edge e is T *dependent* if the weight product pair $p(e) = (M, N)$ from above is a rational multiple of $\omega(e) = (\omega^-(e), \omega^+(e)) = (m, n)$. Note that if e is loop, then e is T dependent if and only if $\omega^-(e) = \omega^+(e)$. A GBS graph (Γ, ω) is *tree dependent*, if it is itself a tree or there is some maximal subtree T of Γ such that every edge e of $E(\Gamma \setminus T)$ is T dependent. A group theoretic version of tree dependence is that of *unimodularity* developed in [8]. Define *Robinson's* ϵ by

$$\epsilon = \epsilon(\Gamma, \omega) = \begin{cases} 1, & \text{if } \Gamma \text{ is tree dependent;} \\ 0, & \text{otherwise.} \end{cases}$$

Observe that while the weight-product function p above is dependent upon the particular maximal sub-tree T, results in [11] show that tree dependence of a GBS graph is not. Thus, tree dependence is a combinatorial invariant of the GBS graph (Γ, ω).

Also, note the following:

$$\begin{aligned} (M, N) = \tfrac{P}{Q}(m, n) \quad &\Leftrightarrow \quad \tfrac{M}{m} = \tfrac{P}{Q} = \tfrac{N}{n} \\ &\Leftrightarrow \quad \tfrac{M}{m} = \tfrac{N}{n} \\ &\Leftrightarrow \quad Mn = Nm \end{aligned}$$

Consequently, when m, n, M, and N are the values defined above for cycles in a GBS graph, it follows that $p^-(e)\omega^+(e) = p^+(e)\omega^-(e)$. Denote e with the reverse orientation by $\overline{e}$. Then, by changing the orientation of e (in the case e is not itself a loop) so that $\gamma(e^-, e^+) \cup e$ becomes a directed simple cycle, we see that

Proposition 3.1. *Given a maximal subtree T of Γ and a non-T edge e, e is tree dependent if and only if the product of the initial weights on the simple directed cycle $\gamma(e^-, e^+) \cup \overline{e}$ equals the product of the terminal weights on its edges.*

4. The proof for (Γ, ω) tree dependent.

It is interesting to note the degree to which the results in this and the next section depend upon the fact that $K(\Gamma, \omega)$ contains a copy of Γ which sits in it in a natural way.

We begin with the case where Γ is itself a tree.

Lemma 4.1. *Assume $\Gamma = T$ is a tree. If $K = K(\Gamma, \omega)$, then $H_2(K)$ is trivial. Therefore, $\beta_2(K) = 0$ and $\beta_1(K) = 1$. Furthermore, for some, and hence every, vertex $v \in V(\Gamma)$, $|[S^1_v]|$ is infinite in $H_1(K)$.*

Proof. We induct on $|E(\Gamma)|$. When $|E(\Gamma)| = 0$, Γ is a single vertex and consequently K is a circle. It is clear that the result holds in this case.

Next, suppose the result holds when Γ is a tree and $|E(\Gamma)| = n \geq 0$. Let Γ be a tree with $n + 1$ edges.

Choose leaf $e \in E(\Gamma)$. Set $S = \{e\}$. In this case $\Gamma_S = \Gamma \setminus e \equiv_{def} \Gamma_e$ is a tree with n edges. Futhermore, $\mathcal{C}_S = \mathcal{C}_e$ strong deformation retracts onto $K_e \cup S^1_{e^-} \cup e \cup S^1_{e^+}$ where $K_e = K(\Gamma_e, \omega)$ and

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exactly one of $S^1_{e^-}$ or $S^1_{e^+}$, say $S^1_{e^-}$, is contained in K_e. The induction hypothesis applies to K_e. Consequently, $H_2(K_e) = 0$. Furthermore, by contracting the edge e, we see that $K_e \cup S^1_{e^-} \cup e \cup S^1_{e^+}$ has the homotopy type of $K_e \underset{e^-=e^+}{\cup} S^1_{e^+}$, the one point join of the GBS complex K_e and vertex circle $S^1_{e^+}$. Consequently, $H_2(\mathcal{C}_e) = H_2(K_e) \oplus H_2(S^1_{e^+}) = 0 \oplus 0$, $H_1(\mathcal{C}_S \cap \mathcal{D}_S) = \mathbb{Z}\langle[\partial D_e]\rangle$, and $H_1(C_e) \cong H_1(K_e) \oplus \mathbb{Z}\langle[S^1_{e^+}]\rangle$. Therefore, equation (2.2) becomes

$$\begin{aligned} 0 \to\ & 0 \oplus 0 \to H_2(K) \\ \xrightarrow{\partial}\ & \mathbb{Z}\langle[\partial D_e]\rangle \xrightarrow{\varphi} H_1(K_e) \oplus \mathbb{Z}\langle[S^1_{e^+}]\rangle \oplus 0 \to \cdots \end{aligned} \tag{4.1}$$

But in (4.1), with the appropriate orientations on ∂D_e, $S^1_{e^-}$, and $S^1_{e^+}$,

$$\varphi\left([\partial D_e]\right) = (\omega^-(e)[S^1_{e^-}], -\omega^+(e)[S^1_{e^+}], 0).$$

Since the direct summand $\mathbb{Z}\langle[S^1_{e^+}]\rangle$ is free ablian with generator $[S^1_{e^+}]$ and $\omega^+(e) \neq 0$, it follows that $\omega^+(e)[S^1_{e^+}]$ is non-trivial. Therefore, φ is a monomorphism. By exactness of equation (4.1), $ker(\varphi) = im(\partial) = 0$. Again, by exactness of (4.1), it follows that $H_2(K) = 0$. Consequently, $\beta_2(K) = 0$ and $\beta_1(K) = 1$.

Next, suppose that for each vertex we have that $[S^1_v]$ has finite order in $H_1(K)$. Since Γ is a tree, [1] gives that the homotopy classes of the standard generators x_v, for $v \in V(\Gamma)$, generate $\pi_1(K)$. Therefore, their homology classes generate $H_1(K)$. But $[S^1_v] = [x_v]$ for $v \in V(\Gamma)$ and therefore $\{[S^1_v] : v \in V(\Gamma)\}$ generate $H_1(K)$. But, with the order of each generator finite (and $H_1(K)$ abelian), it follows that $\beta_1(K) = 0$ which contradicts the preceding paragraph. Therefore, there is some vertex $u \in V(\Gamma)$ such that $[S^1_u]$ has infinite order in $H_1(K)$.

Finally, suppose that $v \neq u$ is another vertex of Γ. Since Γ is a tree there is a (unique) simple directed path $\gamma(u, v)$ from u to v. This path determines the relation $x_u^{p^-(u,v)} = x_v^{p^+(u,v)}$ in $\pi_1(K)$ and, consequently, the equation $p^-(u, v)[S^1_u] = p^+(u, v)[S^1_v]$ in $H_1(K)$, for the non-zero integer products $p^-(u, v)$, $p^+(u, v)$. Therefore, $[S^1_v]$ has infinite order for every vertex $v \in V(\Gamma)$. □

Theorem 4.2. *Suppose that* (Γ, ω) *is a tree dependent GBS graph and* $K = K(\Gamma, \omega)$. *Then* $\beta_2(K) = |E(\Gamma)| - |V(\Gamma)| + \epsilon(\Gamma, \omega)$ *and* $\beta_1(K) = |E(\Gamma)| - |V(\Gamma)| + 1 + \epsilon(\Gamma, \omega)$

Proof. Since (Γ, ω) is tree dependent, $\epsilon(\Gamma, \omega) = 1$. We induct on the number $n = n(E(\Gamma) \setminus E(T))$ of non-tree edges in Γ where T is a maximal subtree of Γ. Note that this number, like $\epsilon(\Gamma, \omega)$ is independent of the particular maximal subtree T chosen.

When $n = 0$, Γ is a tree. By Lemma (4.1), $\beta_2(K) = 0$. Since Γ is a tree, $|E(\Gamma)| - |V(\Gamma)| = -1$. Since $\epsilon(\Gamma, \omega) = 1$, the result follows.

Now assume that the result holds when there are n non-tree edges. Suppose (Γ, ω) is a GBS graph with $n + 1$ non-tree edges.

Let T be a maximal subtree of Γ. Choose a non-T edge e of Γ. Then (Γ_e, ω) is still T dependent but with only n non-T edges. We have K_e the GBS complex associated to (Γ_e, ω). Observe that the induction hypothesis applies to (Γ_e, ω) and K_e.

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In this case $\mathcal{D}_{\{e\}} \cap \mathcal{C}_{\{e\}} = \partial D_e$ is a single circle and $\mathcal{C}_{\{e\}} = C_e$ strong deformation retracts to to the connected complex $K_e \cup (S^1_{e^-} \cup e \cup S^1_{e^+})$ where both vertex circles $S^1_{e^-}$ and $S^1_{e^+}$ are contained in K_e. Therefore, $K_e \cup (S^1_{e^-} \cup e \cup S^1_{e^+}) = K_e \cup e$. Observe also that e^- can be homotoped along $\gamma(e^-, e^+)$ until e^- and e^+ coincide. Therefore $K_e \cup e$ has the homotopy type of $K_e \cup [v, v]$ where $[v, v]$ is a loop attached to K_e based at the vertex $v = e^+$. Consequently, $H_i(\mathcal{C}_e) \cong H_i(K_e) \oplus H_i(S^1)$. In particular, $H_2(\mathcal{C}_e) \cong H_2(K_e) \oplus 0$.

Since the induction hypothesis applies to K_e, $\beta_2(K_e) = |E(\Gamma_e)| - |V(\Gamma_e)| + \epsilon(\Gamma_e)$. But $|E(\Gamma_e)| = |E(\Gamma)| - 1$, $|V(\Gamma_e)| = |V(\Gamma)|$ and $\epsilon(\Gamma_e) = 1$ since Γ_e is T dependent. Therefore, equation (2.2) becomes

$$\begin{aligned} 0 \to\ & \left(\bigoplus_{i=1}^{|E(\Gamma)|-|V(\Gamma)|} \mathbb{Z}\right) \oplus 0 \to H_2(K) \\ \xrightarrow{\partial}\ & \mathbb{Z}\langle[\partial D_e]\rangle \xrightarrow{\varphi} H_1(\mathcal{C}_e) \oplus 0 \to \cdots \end{aligned} \tag{4.2}$$

We want to show that $ker(\varphi) \neq 0$. Set

$$p(e) = (p^-(e), p^+(e)) = (M, N) \text{ and } (\omega^-(e), \omega^+(e)) = (m, n).$$

Since (Γ, ω) is T dependent, there are non-zero integers P,Q such that $(M, N) = \frac{P}{Q}(m, n)$, i.e, such that, $(QM, QN) = (Pm, Pn)$. Then, an application of Proposition 2.5 to $H_1(K_e \cup e) \cong H_1(\mathcal{C}_e)$ produces the equation $M \cdot [S^1_{e^-}] = N \cdot [S^1_{e^+}]$ in $H_1(\mathcal{C}_S)$. Consequently,

$$\begin{aligned} \varphi(P \cdot [\partial D_e]) &= Pm \cdot [S^1_{e^-}] + [e] - Pn \cdot [S^1_{e^+}] - [e] \\ &= Pm \cdot [S^1_{e^-}] - Pn \cdot [S^1_{e^+}] \\ &= QM \cdot [S^1_{e^-}] - QN \cdot [S^1_{e^+}] \\ &= Q \cdot \big(M \cdot [S^1_{e^-}] - N \cdot [S^1_{e^+}]\big) \\ &= 0. \end{aligned}$$

Since $P \neq 0$ and $[\partial D_e]$ generates a copy of $\mathbb{Z}$, the exactness of equation (4.2), produces a short exact sequence

$$0 \to \left(\bigoplus_{i=1}^{|E(\Gamma)|-|V(\Gamma)|} \mathbb{Z}\right) \oplus 0 \to H_2(K) \xrightarrow{\partial} \mathbb{Z} \to 0 \tag{4.3}$$

But (4.3) is a short exact sequence of abelian groups with a free abelian group in the penultimate slot. As such, it splits. Consequently, $H_2(K)$ is free abelian of rank $\beta_2(K) = |E(\Gamma)| - |V(\Gamma)| + 1$. Since (Γ, ω) is T dependent, we have $\epsilon(\Gamma, \omega) = 1$ and the result follows. □

By choosing a maximal subtree of Γ we can apply the argument in Lemma 4.1 to obtain

Proposition 4.3. *If (Γ, ω) is a tree dependent GBS graph with associated GBS complex K and $v \in V(\Gamma)$, then $[S^1_v]$ has infinite order in $H_1(K)$.*

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5. The proof for (Γ, ω) non-tree dependent

Theorem 5.1. *Suppose that (Γ, ω) is a non-tree dependent GBS graph with associated GBS complex K. Then $H_2(K)$ is free abelian of rank $\beta_2(K) = |E(\Gamma)| - |V(\Gamma)| + \epsilon(\Gamma, \omega)$. Consequently $\beta_1(K) = |E(\Gamma)| - |V(\Gamma)| + 1 + \epsilon(\Gamma, \omega)$. Furthermore, for each vertex v of Γ, $[S^1_v]$ has finite order in $H_1(K)$.*

Proof. We use the notation developed in the previous section. Choose a maximal subtree T of Γ. Since tree dependence is independent of the maximal subtree of Γ, there is at least one non-T edge e of Γ such that for every pair P, Q of non-zero integers,

$$\begin{aligned} \tfrac{P}{Q}(\omega^-(e), \omega^+(e)) &= \tfrac{P}{Q}(m, n) \\ &\neq (p^-(e), p^+(e)) = (M, N) \end{aligned} \tag{5.1}$$

Consequently, for every such P and Q, $(QM - Pm, QN - Pn) \neq (0, 0)$. In particular, for every $0 \neq \alpha \in \mathbb{Z}$, setting $Q = \alpha M$ and $P = \alpha m$

$$\begin{aligned} (0,0) &\neq Q(m,n) - P(M,N) \\ &= (\alpha Mm - \alpha mM, Qn - PN) \\ &= (0, Qn - PN) \\ &= (0, \alpha Mn - \alpha mN) \end{aligned} \tag{5.2}$$

Therefore, equation (5.2) implies $0 \neq Qn - PN = \alpha Mn - \alpha mN$.

We induct on the number n of non-T edges e for which the cycle $(\gamma(e^-), \gamma(e^+)) \cup e$ is not T dependent. That is, we induct on the number of non-T edges for which (5.1) and (5.2) hold.

Assume the number of non-T edges for which (5.1) holds is $n = 1$ and let e be the only such edge. Then (Γ_e, ω) is T dependent and Theorem 4.2 applies to K_e. In fact, the conclusions about the terms of Mayer-Vietoris Sequence for $(K, \mathcal{C}_S, \mathcal{D}_S) = (K, \mathcal{C}_e, D_e)$ leading to the long exact sequence (4.2) are valid and, consequently, we again must understand φ in (4.2).

We claim that $ker(\varphi) = 0$. Assume to the contrary that it does not. Then there is a non-zero integer α for which $ker(\varphi) = \mathbb{Z}\langle \alpha \cdot [\partial D_e] \rangle$. Observe that e^+ can be homotoped along $\gamma(e)$ to $v = e^-$ to produce the loop $[v, v]$. Then it follows that $\mathcal{C}_e$ is then homotopic to $K_e \cup$loop. But with appropriately chosen orientations for ∂D_e, $S^1_{e^\pm}$, and t_e, we have

$$\begin{aligned} 0 &= \varphi(\alpha[\partial D_e]) \\ &= \alpha \left(m[S^1_{e^-}] + [e] - n[S^1_{e^+}] - [e] \right) \\ &= \alpha m[S^1_{e^-}] - \alpha n[S^1_{e^+}]. \end{aligned} \tag{5.3}$$

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But, by Proposition 2.5 there is the equation $M[S^1_{e^-}] - N[S^1_{e^+}] = 0$ in $H_1(K_e)$. We have, as a consequence, the system

$$
(5.4)\qquad \begin{gathered} Q = \alpha M,\ P = \alpha m,\ Qn - PN \neq 0 \\ \alpha m[S^1_{e^-}] - \alpha n[S^1_{e^+}] = 0 \\ M[S^1_{e^-}] - N[S^1_{e^+}] = 0 \end{gathered}
$$

Multiply the second line in (5.4) by M, the third line in (5.4) by αm, and subtract. This produces

$$0 = (\alpha m N - \alpha M n)[S^1_{e^+}] = -(Qn - PN)[S^1_{e^+}].$$

Then, from the first line in (5.4), $(Qn - PN) \neq 0$. Therefore $[S^1_{e^+}]$ has finite order in $H_1(K_e)$.

But, by Proposition 4.3, the homology class of every vertex circle $[S^1_u]$ of K_e has infinite order in $H_1(K_e)$. Contradiction. Therefore, $ker(\varphi) = im(\partial) = 0$. Therefore, we can extract the exact sequence

$$(5.5)\qquad 0 \to \left(\bigoplus_{i=1}^{|E(\Gamma)|-|V(\Gamma)|} \mathbb{Z} \right) \oplus 0 \to H_2(K) \xrightarrow{\partial} 0$$

from (4.2). It follows that $\beta_2(K) = |E(\Gamma)| - |V(\Gamma|$. Since (Γ, ω) is not tree dependent, $\epsilon(\Gamma, \omega) = 0$ and the formulas for β_1 and β_2 given in theorem hold for the base case of the induction.

Finally, consider the vertex e^- of the non-T edge e and turn the cycle $\gamma(e^-, e^+) \cup e$ into an oriented simple cycle. Reading along this path produces the relation $x_{e^-}^{p^-(e)\omega^+(e)} = x_{e^-}^{p^+(e)\omega^-(e)}$. Consequently, $[x_{e^-}]$ has finite order in $H_1(K)$. Because there is a path in the maximal subtree T from e^- to every other vertex v of Γ, it follows that every vertex generator x_v has finite order in $H_1(K)$ and the result follows for the base case of our induction.

We now prove the general case. Assume that the result holds when Γ is T dependent with $n \geq 1$ tree dependent cycles.

Suppose Γ is T dependent with $n+1$ non-T dependent edges. Deleting one of these non-T dependent edges e from Γ produces the non-T dependent graph (Γ_e, ω) with maximal subtree T, the associated GBS complexes $K(e, \omega)$, $K(\Gamma_e, \omega)$, the associated disk D_e, and its associated closed complement $\mathcal{C}_e = K \setminus (intD_e)$.

But $D_e \cap \mathcal{C}_e = \partial D_e$ and D_e strong deformation retracts to a point. With $\mathcal{C}_S = \mathcal{C}_e$, equation (2.2) becomes

$$(5.6)\qquad 0 \to \quad H_2(\mathcal{C}_e) \oplus 0 \to H_2(K) \xrightarrow{\partial} \mathbb{Z}\langle \partial D_e \rangle \xrightarrow{\varphi} H_1(\mathcal{C}_e) \oplus 0 \to \cdots$$

Now $\mathcal{C}_e$ strong deformation retracts to $K_e \cup e = K_e \cup t_e$ and e^+ can be homotoped along $\gamma(e^-, e^+) \subset t_e$ to produce a loop $[v, v]$ based at $v = e^-$. It follows that $\mathcal{C}_e$ is homotopy equivalent to $K_e \cup [v, v]$. Consequently, $H_i(\mathcal{C}_e) \cong H_i(K_e) \oplus H_i([v, v])$. But, $H_2(t_e) = 0$ and $H_1([v, v]) = \mathbb{Z}\langle [t_e] \rangle$.

The induction hypothesis applies to K_e. Consequently,

$$H_2(K_e) \cong \bigoplus_{i=1}^{|E(\Gamma_e)|-|V(\Gamma_e)|+\epsilon(\Gamma_e,\omega)} \mathbb{Z} = \bigoplus_{i=1}^{|E(\Gamma)|-1-|V(\Gamma_e)|} \mathbb{Z}$$

since $|E(\Gamma_e)| = |E(\Gamma)| - 1$ and $\epsilon(\Gamma_e, \omega) = 0$.

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Therefore, equation (5.6) becomes

$$0 \to \bigoplus_{i=1}^{|E(\Gamma)|-1-|V(\Gamma_e)|} \mathbb{Z} \oplus 0 \to H_2(K) \xrightarrow{\partial} \mathbb{Z}\langle[\partial D_e]\rangle \xrightarrow{\varphi} H_1(K_e) \oplus \mathbb{Z}\langle t_e\rangle \oplus 0 \to \cdots \tag{5.7}$$

We again need to understand $ker(\varphi) = im(\partial)$. Specifically, we want to show $ker(\varphi)$ is nontrivial.

Now, in $H_1(\mathcal{C}_e) \cong H_1(K_e) \oplus \mathbb{Z}\langle[\partial D_e]\rangle$, with appropriate orientations on ∂D_e, $S^1_{e^-}$, and $S^1_{e^+}$,

$$\begin{aligned} \varphi([\partial D_e]) &= m[S^1_{e^-}] + [e] - n[S^1_{e^+}] - [e] \\ &= m[S^1_{e^-}] - n[S^1_{e^+}]. \end{aligned}$$

But K_e satisfies the induction hypothesis. Therefore, there are nonzero integers, R and S, such that $R \cdot [S^1_{e^-}] = 0$ and $S \cdot [S^1_{e^+}] = 0$. Setting $\lambda = lcm\{R, S\}$, it follows that $\varphi(\lambda \cdot [\partial D_e]) = \lambda \cdot (m[S^1_{e^-}] - n[S^1_{e^+}]) = 0$. Thus some nonzero multiple $k \cdot [\partial D_e]$ generates $\ker(\varphi)$.

Thus, from 5.7 we extract the short exact sequence

$$0 \to \bigoplus_{i=1}^{|E(\Gamma)|-1-|V(\Gamma_e)|} \mathbb{Z} \oplus 0 \to H_2(K) \xrightarrow{\partial} \mathbb{Z}\langle k \cdot [\partial D_e]\rangle \xrightarrow{\varphi} 0$$

Because it is a short exact sequence of abelian groups ending in a free group, it splits and this gives that $H_2(K)$ is free abelian of rank $\beta_2(K) = |E(\Gamma)| - |V(\Gamma)|$. Since (Γ, ω) is non-T dependent, $\epsilon(\Gamma, \omega) = 0$ and the formula for $\beta_2(K)$ is valid for non-tree dependent GBS graphs. Note that by reading around the GBS circuit t_e in (Γ, ω), it is easy to see that $[S^1_{e^-}]$ has finite order in $H_1(K)$. Since there is a GBS path from e^- to every other vertex v of the GBS graph (Γ, ω), it follows that the homology class of every vertex circle $[S^1_v]$ has finite order. $\square$

6. **Two open problems**

The Hawaian Earring is the subspace $\mathcal{H} = \{(x, y) : x^2 + (y - \frac{1}{n})^2 = \left(\frac{1}{n}\right)^2\} \subset \mathbb{R}^2$, a countable infinite sequence of circles with radus converging to 0, all of which are tangent to the x-axis at $(0, 0)$. Setting $x_0 = (0, 0)$, its fundamental group $G = \pi_1(\mathcal{H}, x_0)$ has the property that every finite index subgroup N of G is isomorphic to G. However, Derek's structure theorem in [13] does not apply to G because it is (uncountably) infinitely generated. One wonders if it is possible to extend the structure theorem to infinitely generated groups which have the property that all their finite index subgroups are isomorphic to the whole group. If so, what does this extension tell us about $G = \pi_1(\mathcal{H}, x_0)$.

A group G is *non-cohopfian* if it has a proper finite index subgroup H isomorphic to the whole group G. Thus, the groups for which Derek's structure theorem [13] applies and those which have at least one proper finite index subgroup isomorphic to the whole group, including the GBS groups, are non-cohopfian. In this context, the GBS groups are exceptionally interesting because they have a graph of groups description in which all the vertex and edge groups are also non-cohopfian (they are all copies of $\mathbb{Z}$) and the edge maps are monomorphisms of the edge groups onto finite index subgroups of the vertex groups which are also isomorphic to the entire vertex group. This suggests the next problem. Assume G_0 is a non-cohopfian group. It then has proper finite index subgroups isomorphic to all of G_0. Let G be a group which has a graph of groups description in which all vertex and edge groups are copies of G_0 and all edge maps are monomorphisms onto a subgroup of each vertex group which is isomorphic to G_0. Is G also non-cohopfian? This problem also appears in [15].

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