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BOHR CONDITIONS AND ALMOST PERIODIC MEANS IN QUASI-COMPLETE SPACES

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Dedicated to professor A. T.-M. Lau for his lifetime contributions to mathematics.

Abstract. We study Bohr conditions for functions on topological groups taking values in locally convex spaces. We show that functions satisfying Bohr conditions are uniformly continuous. We show that in quasi-complete spaces, Bohr conditions are equivalent to Bochner's characterizations of almost periodicity. We prove the existence of invariant mean for almost periodic functions with values in quasi-complete spaces.

1. Introduction

In [2], Bochner and von Neumann studied almost periodic functions with values in locally convex spaces E satisfying the following two conditions: (1) $0 \in E$ is a G_δ-set, and (2) every closed totally bounded subset of E is limit-point compact (detailed properties of these spaces were developed by von Neumann [13], and the above conditions can be found in Definitions 2b, 8, and 10 of this reference). As noted by these authors, condition (2) is necessary for a "smooth working" of the theory. One of the aims of this paper is to show that condition (1) is not needed for a development of the theory, and in particular, this condition is not needed to prove the existence of almost periodic invariant mean.

In Section 2 we discuss some preliminaries from topological vector spaces and quasi-complete spaces. In Section 3 we give a general formulation of Bohr conditions of almost periodicity for functions on topological groups taking values in locally convex spaces (Definition 3.2). Earlier studies of this

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condition for functions on commutative groups can be found in Bochner [1], N'Guérékata [9, 10], and Simon [11, §6.6]. In Theorem 3.4 we show that the left Bohr condition implies right uniform continuity, and the right Bohr condition implies left uniform continuity. In Theorem 3.5 we show the equivalence of Bohr conditions to Bochner conditions of almost periodicity for functions taking values in quasi-complete spaces. This theorem extends and unifies earlier results on the equivalence of these conditions found in the above references.

An important tool for the study of vector-valued almost periodic functions is the vector-valued invariant mean (see, for example, Chou–Lau [5] for a study of vector-valued invariant means for almost periodic operators arising from Hilbert space representations of locally compact groups). For functions taking values in locally convex spaces, the first proof of the existence of invariant mean was given in [2, Theorem 12, p. 27]. However, an inspection of this proof shows that it relies in an essential way on properties derived from condition (1) imposed on locally convex spaces. For this reason, much of the theory of vector-valued almost periodic functions in [2] is not applicable to functions taking values in nonseparable Hilbert spaces with weak topologies, or to functions taking values in dual Banach spaces with w^*-topologies. More important, the theory developed in [2] is not applicable to almost periodic representations $\pi\colon G \longrightarrow \mathcal{B}(X)$, where X is a Banach space and $\mathcal{B}(X)$ has the strong operator topology (since in general this space does not satisfy the condition (1)).

To remove the limitations imposed by the condition (1), we shall prove in Section 4 the existence of invariant mean for almost periodic functions taking values in quasi-complete spaces (Theorem 4.6). Quasi-complete spaces automatically satisfy the condition (2) required by Bochner and von Neumann, but they need not satisfy the condition (1). These spaces include all the examples of locally convex spaces mentioned above, and thus allow a wider applicability of the theory developed in [2]. The proof of the existence of mean given here is an adaptation to vector-valued functions of a proof due to Maak [8] (see also Hewitt and Ross [7, §18]).

2. Preliminaries

Throughout this paper G denotes a Hausdorff topological group (not necessarily locally compact), and E denotes a Hausdorff locally convex topological vector space. We shall assume that E is equipped with a fixed local base $\mathscr{U}$ of neighborhoods of $0 \in E$ consisting of absorbing, balanced, closed, and convex sets. Two standard properties of such a local base are:

(a) If $U \in \mathscr{U}$ and $0 < r < s$, then $\overline{rU} \subset \operatorname{int}(sU)$.
(b) If $U \in \mathscr{U}$, $r > 0$, $n \in \mathbb{N}$, there exists $V \in \mathscr{U}$ such that

$$\alpha_1 V + \cdots + \alpha_n V \subset U \qquad \text{if } |\alpha_i| \leq r,\ i = 1, \ldots, n.$$

Let X be a Hausdorff topological space and E a locally convex space. A function $f\colon X \longrightarrow E$ is called bounded if its range $f(X)$ is a bounded subset of E. We denote the set of all such bounded functions by $\mathcal{F}^b(X, E)$. If $\mathscr{U}$ is a local base of $0 \in E$ and $U \in \mathscr{U}$, we define $U' \subset \mathcal{F}^b(X, E)$ to be the set of all $f \in \mathcal{F}^b(X, E)$ such that $f(X) \subset U$. We denote the set of all such U' by $\mathscr{U}'$. With $\mathscr{U}'$ as a local base, $\mathcal{F}^b(X, E)$ is a locally convex space.

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The convergence in $\mathcal{F}^b(X,E)$ is *uniform* in the following sense. If f_α is a net in $\mathcal{F}^b(X,E)$ such that $f_\alpha \to f$, then for every $U \in \mathscr{U}$, there is some α_0 such that if $\alpha \geq \alpha_0$, then $f_\alpha - f \in U'$, equivalently, $f_\alpha(x) - f(x) \in U$, for all $x \in X$ if $\alpha \geq \alpha_0$.

The space $C^b(X,E)$ of all continuous bounded functions on X with values in E, is a closed linear subspace of $\mathcal{F}^b(X,E)$.

Let $\mathscr{U}$ be a local base at $0 \in E$, and $\mathscr{U}'$ be the corresponding local base at 0 in $\mathcal{F}^b(X,E)$. For each $U \in \mathscr{U}$, let $\|\cdot\|_U$ and $\|\cdot\|_{U'}$ be the corresponding Minkowski functionals on E and $\mathcal{F}^b(X,E)$, respectively. Since U and U' are convex balanced neighborhoods, $\|\cdot\|_U$ and $\|\cdot\|_{U'}$ are seminorms. The following properties of the seminorms $\|\cdot\|_U$ and $\|\cdot\|_{U'}$ are easy to check: for every $U \in \mathscr{U}$, $f \in \mathcal{F}^b(X,E)$, and $t > 0$,

(i) $tU' = (tU)'$, and $U' + U' \subset (U+U)'$,
(ii) $\|f\|_{U'} = \inf\{t > 0 \colon f(X) \subset tU\}$,
(iii) $\|f\|_{U'} = \sup_{x\in X} \|f(x)\|_U$.

A function $f\colon G \longrightarrow E$ is *left* [resp., *right*] *uniformly continuous* if for every $U \in \mathscr{U}$, there is a neighborhood W of $e \in G$ such that if $x, y \in G$ and $xy^{-1} \in W$ [resp., $y^{-1}x \in W$], then $f(x) - f(y) \in U$. As for scalar-valued functions, it can be shown that $f \in C^b(G,E)$ is left uniformly continuous if and only if the function $x \mapsto L_x f$, $G \longrightarrow C^b(G,E)$ is continuous (similar result holds for right uniform continuity if $L_x f$ is replaced with $R_x f$).

A locally convex space E is called *quasi-complete* if every bounded Cauchy net in E is convergent. The interest in quasi-complete spaces arises from the property that relatively compact subsets in E are exactly totally bounded sets; and therefore, a set in E is compact if and only if it is closed and totally bounded. A great number of locally convex spaces of interest in analysis are quasi-complete: these include (i) Hilbert, Banach, Fréchet, and LF-spaces, together with their dual spaces under w^*-topologies; (ii) all complete locally convex spaces; (iii) all operator spaces $\mathcal{B}(X)$ under the strong operator topology when X is a Banach space. For more details and additional examples see Trèves [12, Section 19], Bourbaki [4, pp. III.9–III.11].

Since the equivalence of total boundedness and relative compactness plays an important role in the main results of this paper (see Theorem 3.5 and Theorem 4.6), we shall be mainly interested in almost periodic functions taking values in quasi-complete spaces. If E is quasi-complete, then it is easy to show that $\mathcal{F}^b(X,E)$ and all its closed linear subspaces (such as $C^b(G,E)$) are also quasi-complete.

If $f \in C^b(G,E)$, and $a \in G$, the left and right translations of f are denoted by $L_a f$ and $R_a f$, respectively. The function $D_a f \in C^b(G \times G, E)$ is defined by $D_a f(x,y) = f(xay)$ $(x, y \in G)$. We put $\mathfrak{L}_f = \{L_a f \colon a \in G\}$, $\mathfrak{R}_f = \{R_a f \colon a \in G\}$, and $\mathfrak{D}_f = \{D_a f \colon a \in G\}$. If E is quasi-complete, then the relative compactness of $\mathfrak{L}_f$, $\mathfrak{R}_f$, and $\mathfrak{D}_f$ are equivalent (*Bochner conditions*), and when these conditions hold, f is called *almost periodic*. The space of all almost periodic functions with values in E is denoted by $AP(G,E)$. Since $AP(G,E)$ is a closed linear subspace of $C^b(G,E)$, it is quasi-complete.

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3. **Bohr conditions and almost periodicity**

Bohr condition for a continuous function $f\colon \mathbb{R} \longrightarrow \mathbb{C}$ is as follows: for every $\epsilon > 0$ there exists some $\ell > 0$ such that every interval of length ℓ contains an element τ with the property that $|f(x+\tau) - f(x)| < \epsilon$, for every $x \in \mathbb{R}$ ([3, p. 30]). We may think of τ as a 'period up to the accuracy of ϵ', or simply, an 'ϵ-almost period' of the function f. In this section we show that Bohr condition can be adapted to functions on topological groups with values in locally convex spaces E. We will then show that when E is quasi-complete, this condition is equivalent to Bochner conditions.

Definition 3.1. Let G be a topological group, E a locally convex space, and $\mathscr{U}$ a local base at $0 \in E$. Let $f\colon G \longrightarrow E$ be a function and $U \in \mathscr{U}$. An element $a \in G$ is a *left U-almost period* of f if $f(ax) - f(x) \in U$ $(x \in G)$. Similarly, an element $b \in G$ is a *right U-almost period* of f if $f(xb) - f(x) \in U$ $(x \in G)$.

Uniformly continuous functions have plenty U-almost periods. To see this, suppose $f\colon G \longrightarrow E$ is left uniformly continuous and $U \in \mathscr{U}$. Then there exists a neighborhood W of $e \in G$ such that $xy^{-1} \in W$ implies $f(x) - f(y) \in U$. In that case, every $a \in W$ is a left U-almost period of f, since for every $x \in G$, $(ax)x^{-1} \in W$ and therefore $f(ax) - f(x) \in U$. A similar situation holds for right uniformly continuous functions.

Definition 3.2. Let G be a topological group, E a locally convex space, and $f\colon G \longrightarrow E$, be a continuous function.

(a) f satisfies the *left Bohr condition* if for every $U \in \mathscr{U}$, there is a compact set $K \subset G$ such that yK contains a left U-almost period for all $y \in G$.

(b) f satisfies the *right Bohr condition* if for every $U \in \mathscr{U}$, there is a compact set $K \subset G$ such that Ky contains a right U-almost period for all $y \in G$.

Remark 3.3. (a) Since the relations $a \in yK$ and $f(ax) - f(x) \in U$ for all $x \in G$, are equivalent to $a^{-1} \in K^{-1}y^{-1}$ and $f(a^{-1}x) - f(x) \in U$ for all $x \in G$ (just replace x with $a^{-1}x$ and note that $-U = U$), and since K^{-1} is compact, it follows that the left Bohr condition can be rephrased as follows: for every $U \in \mathscr{U}$, there is a compact set $K \subset G$ such that Ky contains a left U-almost period for all $y \in G$. Similarly for the right Bohr condition.

(b) If G is compact, then every continuous function $f\colon G \longrightarrow E$ satisfies both left and right Bohr conditions trivially (by taking $K = G$, and $e \in G$ as U-almost period).

Theorem 3.4. *Let G be a topological group and E be a locally convex space. Suppose that $f\colon G \longrightarrow E$ is a continuous function.*

(a) If f satisfies the left Bohr condition, then it is bounded and right uniformly continuous.

(b) If f satisfies the right Bohr condition, then it is bounded and left uniformly continuous.

Proof. (a) Let f satisfy the left Bohr condition. First we show that f must be bounded. Let $x \in G$, and $U \in \mathscr{U}$. Corresponding to U, there exists a compact set $K_0 \subset G$ such that K_0x^{-1} contains a left

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U-almost period, say k_0x^{-1} ($k_0 \in K_0$). Thus

$$f(k_0x^{-1}y) - f(y) \in U \qquad \text{for all } y \in G.$$

Therefore, for $y = x$, we find $f(k_0) - f(x) \in U$, and hence $\|f(x) - f(k_0)\|_U \leq 1$. It follows that

$$\|f(x)\|_U = \|f(x) - f(k_0) + f(k_0)\|_U \leq \|f(x) - f(k_0)\|_U + \|f(k_0)\|_U \leq 1 + \sup_{k \in K_0} \|f(k)\|_U.$$

Since K_0 is compact and the map $y \mapsto \|f(y)\|_U$ is continuous, it follows that $\sup_{k\in K_0} \|f(k)\|_U < \infty$. However, K_0 is independent of x, and thus f is bounded in the seminorm $\|\cdot\|_U$. Since $U \in \mathscr{U}$ is arbitrary, it follows that f is bounded.

Next, we show that f is right uniformly continuous. Given $U \in \mathscr{U}$, we need to find a neighborhood W of $e \in G$ such that $y^{-1}x \in W$ implies that $f(y) - f(x) \in U$. Choose $U_1 \in \mathscr{U}$ such that

$$U_1 + U_1 + U_1 + U_1 \subset U.$$

For each $y \in G$, $L_y f$ is continuous at e, and so there is a neighborhood V_y of e such that if $z \in V_y$, then

$$L_y f(z) - L_y f(e) \in U_1.$$

Therefore if $z, z' \in V_y$, then

$$L_y f(z) - L_y f(z') = L_y f(z) - L_y f(e) + L_y f(e) - L_y f(z') \in U_1 + U_1.$$

Choose a neighborhood W_y of e such that $W_y^2 \subset V_y$. Then it follows from the above that for all $z, z', z'' \in W_y$:

$$L_{yz''} f(z) - L_{yz''} f(z') \in U_1 + U_1. \tag{3.1}$$

By the assumption that f satisfies the left Bohr condition, we know that corresponding to U_1 there exists a compact set $K \subset G$ so that for all $y \in G$, there exists some $w \in K$ such that $wy^{-1} \in Ky^{-1}$ is a left U_1-almost period of f, i.e.,

$$L_{wy^{-1}} f(a) - f(a) \in U_1 \qquad \text{for all } a \in G. \tag{3.2}$$

Hence for the same choices of y and w, we have

$$L_w f(a) - L_y f(a) = L_{wy^{-1}} f(ya) - f(ya) \in U_1 \qquad \text{for all } a \in G. \tag{3.3}$$

Since the family $\{yW_y\}_{y\in K}$ is a covering of K, there exists $y_1, \ldots, y_\ell \in K$ so that

$$K \subset \bigcup_{j=1}^{\ell} y_j W_{y_j}.$$

Let us define a neighborhood W of $e \in G$ by putting

$$W = \bigcap_{j=1}^{\ell} W_{y_j}.$$

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If $y \in K$, then for some $1 \le j \le \ell$, $y = y_j z_j$, with $z_j \in W_{y_j}$, and hence for all $z, z' \in W$, it follows from (3.1) that

$$L_y f(z) - L_y f(z') = L_{y_j z_j} f(z) - L_{y_j z_j} f(z') \in U_1 + U_1. \tag{3.4}$$

Now suppose that $x, y \in G$ are such that $y^{-1}x \in W$, then with $w \in K$ chosen as in the paragraph preceding to (3.2) (w depending on y), it follows from (3.3) and (3.4) that

$$\begin{aligned} f(y) - f(x) &= L_y f(e) - L_y f(y^{-1}x) \\ &= L_y f(e) - L_w f(e) + L_w f(e) - L_w f(y^{-1}x) + L_w f(y^{-1}x) - L_y f(y^{-1}x) \\ &\in U_1 + (U_1 + U_1) + U_1 \subset U, \end{aligned}$$

where the first and the last inclusions have followed from (3.3) because of our choice of w, and the middle inclusion $L_w f(e) - L_w f(y^{-1}x) \in U_1 + U_1$ has followed from (3.4) since we know that $w \in K$ and $e, y^{-1}x \in W$. This proves that f is right uniformly continuous.

The proof of (b) is similar to the above argument with natural modifications. □

Theorem 3.5. *Let G be a topological group, E be a quasi-complete space, and let $f \in C^b(G, E)$. Then $f \in AP(G, E)$ if and only if f satisfies both left and right Bohr conditions.*

Proof. Suppose that $f \in AP(G, E)$. We will show that f satisfies both left and right Bohr conditions. Given $U \in \mathscr{U}$, the total boundedness of $\mathfrak{L}_f = \{L_y f \colon y \in G\}$ implies that we can pick $y_1, \ldots, y_\ell \in G$ so that for every $L_y f$ there corresponds some $L_{y_j} f$ $(1 \le j \le \ell)$ such that

$$L_y f(a) - L_{y_j} f(a) \in U \qquad \text{for all } a \in G.$$

Let $K = \{y_1^{-1}, \ldots, y_\ell^{-1}\}$. For a given $x \in G$, pick y_j $(1 \le j \le \ell)$ so that

$$L_x f(a) - L_{y_j} f(a) \in U \qquad \text{for all } a \in G.$$

Then

$$L_{xy_j^{-1}} f(a) - f(a) = L_x f(y_j^{-1}a) - L_{y_j} f(y_j^{-1}a) \in U \qquad \text{for all } a \in G.$$

This means that the element $xy_j^{-1} \in xK$ is a left U-almost period of f. Since $x \in G$ and $U \in \mathscr{U}$ are both arbitrary, it follows that f satisfies the left Bohr condition.

To prove that f satisfies the right Bohr condition, given $U \in \mathscr{U}$, the total boundedness of $\mathfrak{R}_f = \{R_y f \colon y \in G\}$ implies that we can pick $y_1, \ldots, y_\ell \in G$ so that for every $R_y f$ there corresponds some $R_{y_j} f$ $(1 \le j \le \ell)$ such that

$$R_y f(a) - R_{y_j} f(a) \in U \qquad \text{for all } a \in G.$$

Let $K = \{y_1^{-1}, \ldots, y_\ell^{-1}\}$. For a given $x \in G$, pick y_j $(1 \le j \le \ell)$ so that

$$R_x f(a) - R_{y_j} f(a) \in U \qquad \text{for all } a \in G.$$

Then

$$R_{y_j^{-1}x} f(a) - f(a) = R_x f(ay_j^{-1}) - R_{y_j} f(ay_j^{-1}) \in U \qquad \text{for all } a \in G.$$

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This means that the element $y_j^{-1}x \in Kx$ is a right U-almost period of f. Since $x \in G$ and $U \in \mathscr{U}$ are both arbitrary, it follows that f satisfies the right Bohr condition.

Conversely, suppose that f satisfies both left and right Bohr conditions and we will show that $\mathfrak{L}_f = \{L_y f \colon y \in G\}$ is totally bounded in $C^b(G,E)$ (and hence f is almost periodic). Given $U \in \mathscr{U}$, we want to find finitely many $y_1, \ldots, y_n \in G$ so that for every $L_y f$ there corresponds some $L_{y_j} f$ $(1 \le j \le n)$ such that

$$L_y f(a) - L_{y_j} f(a) \in U \qquad \text{for all } a \in G. \tag{3.5}$$

We choose $U_1 \in \mathscr{U}$ such that $U_1 + U_1 \subset U$. Since f satisfies the left Bohr condition, there exists a compact subset K of G (depending on U_1 and f) such that for all $y \in G$, there is some $x \in K$ so that

$$L_{xy^{-1}} f(a) - f(a) \in U_1 \qquad \text{for all } a \in G. \tag{3.6}$$

Since f also satisfies the right Bohr condition, by Theorem 3.4 it is left uniformly continuous, and therefore the mapping

$$G \longrightarrow C^b(G,E), \quad x \mapsto L_x f,$$

is continuous. Thus compactness of K implies that $\{L_x f \colon x \in K\}$ is a compact subset of $C^b(G,E)$, and in particular, it is totally bounded in $C^b(G,E)$. As a result, we can find $y_1, \ldots, y_n \in K$ so that for every $L_x f$ $(x \in K)$ there corresponds some $L_{y_j} f$ $(1 \le j \le n)$ such that

$$L_x f(y) - L_{y_j} f(y) \in U_1 \qquad \text{for all } y \in G. \tag{3.7}$$

We claim that the elements $y_1, \ldots, y_n$ satisfy the required condition in (3.5). In fact, for $y \in G$, we first choose $x \in K$ such that (3.6) holds, and subsequently, we choose $y_j \in K$ so that (3.7) holds, in which case for all $a \in G$, we can write

$$\begin{aligned} L_y f(a) - L_{y_j} f(a) &= L_y f(a) - L_x f(a) + L_x f(a) - L_{y_j} f(a) \\ &= L_y (f - L_{xy^{-1}} f)(a) + L_x f(a) - L_{y_j} f(a) \in U_1 + U_1 \subset U. \end{aligned}$$

This proves that $\mathfrak{L}_f = \{L_y f \colon y \in G\}$ is totally bounded in $C^b(G,E)$, and hence $f \in AP(G,E)$. □

Combining Theorems 3.4 and 3.5 gives:

Corollary 3.6. *If G is a topological group and E is a quasi-complete space, then each $f \in AP(G,E)$ is uniformly continuous.*

By applying Remark 3.3(b) and Theorem 3.5 we obtain

Corollary 3.7. *If G is a compact topological group and E is a quasi-complete space, then $AP(G,E) = C^b(G,E)$.*

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4. **Invariant mean on** $AP(G,E)$

In this section we give a proof for the existence of the invariant mean for almost periodic functions taking values in quasi-complete spaces. Throughout this section, E is a quasi-complete locally convex space, $\mathscr{U}$ is a local base at $0 \in E$, and G is a Hausdorff topological group.

Definition 4.1. If $A \subset E$ and $U \in \mathscr{U}$, then a U-*mesh* of A is a finite set $\{u_1, \dots, u_n\}$ in E such that for every $u \in A$ there exists some u_i, $1 \le i \le n$, such that $u - u_i \in U$.

Thus a set $A \subset E$ is totally bounded if and only if A has a U-mesh for every $U \in \mathscr{U}$. In particular, if $f \in AP(G,E)$ then $\{D_a f \colon a \in G\} \subset C^b(G \times G, E)$ is totally bounded, and hence has a U'-mesh for every $U \in \mathscr{U}$, where

$$U' = \{F \in C^b(G \times G, E) \colon F(G \times G) \subset U\}, \tag{4.1}$$

(cf. Section 2).

The following combinatoric lemma is due to Hall [6, Theorem 1, p. 27].

Lemma 4.2. *Suppose that P and Q are nonempty sets, P is finite, and $\mathscr{P}(Q)$ is the collection of all nonempty subsets of Q. Let $\rho \colon P \longrightarrow \mathscr{P}(Q)$ be a function such that for every $S \subset P$,*

$$\left| \bigcup_{x \in S} \rho(x) \right| \ge |S|,$$

where $|\cdot|$ denotes the cardinality. Then there is an injective map $\sigma \colon P \longrightarrow Q$ such that $\sigma(x) \in \rho(x)$, for all $x \in P$.

Lemma 4.3. *Let $A \subset E$, $U \in \mathscr{U}$, and $\{u_1, \dots, u_n\}$ be a U-mesh of A such that the number n of elements of the mesh is minimum for the given U. Let B be any subset of E such that for each $u \in A$ there is some $v \in B$ with $u - v \in U$. Then there is an injective map $\sigma \colon \{u_1, \dots, u_n\} \longrightarrow B$ such that*

$$u_k - \sigma(u_k) \in U + U \qquad (k = 1, \dots, n).$$

Proof. For every $k = 1, \dots, n$, let

$$\rho(u_k) = \{v \in B \colon \text{there exists } w \in A \text{ with } w - v \in U \text{ and } w - u_k \in U\}.$$

Consider an arbitrary subset $\{u_{j_1}, \dots, u_{j_r}\}$ of $\{u_1, \dots, u_n\}$ and write the remaining elements as $u_{j_{r+1}}, \dots, u_{j_n}$. We will verify that

$$|\rho(u_{j_1}) \cup \cdots \cup \rho(u_{j_r})| \ge r. \tag{4.2}$$

If the set $\rho(u_{j_1}) \cup \cdots \cup \rho(u_{j_r})$ is infinite, then (4.2) trivially holds. So, suppose that $\rho(u_{j_1}) \cup \cdots \cup \rho(u_{j_r})$ is finite, and write $\rho(u_{j_1}) \cup \cdots \cup \rho(u_{j_r}) = \{v_1, \dots, v_s\}$, where the v's are distinct. Form the set

$$S = \{v_1, \dots, v_s, u_{j_{r+1}}, \dots, u_{j_n}\}.$$

The set S is a U-mesh of A, for if $w \in A$ and

$$w - u_{j_p} \notin U \quad \text{ for } p = r+1, \dots, n,$$

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then since $\{u_1, \dots, u_n\}$ is a U-mesh of A, we must have

$$w - u_{j_p} \in U \quad \text{for some } p = 1, \dots, r.$$

But, by assumption, there is a $v \in B$ such that

$$w - v \in U,$$

and so by the definition of ρ, we have $v \in \rho(u_{j_p})$, i.e., $v = v_\ell$ for some $\ell = 1, 2, \dots, s$. This proves that S is a U-mesh of A. By the choice of n, we have $n \le |S| \le s + n - r$, so that $s \ge r$. This establishes (4.2).

Now we apply Lemma 4.2 with $P = \{u_1, \dots, u_n\}$, $Q = B$, and ρ as defined above. So, there is an injective map $\sigma\colon P \longrightarrow B$, $u_k \mapsto \sigma(u_k)$, such that $\sigma(u_k) \in \rho(u_k)$. By definition of $\rho(u_k)$, we can find $w_k \in A$ such that

$$w_k - u_k \in U, \quad w_k - \sigma(u_k) \in U,$$

and therefore

$$u_k - \sigma(u_k) = u_k - w_k + w_k - \sigma(u_k) \in U + U,$$

since $-U = U$. This complete the proof of the lemma. □

Lemma 4.4. *Let $f \in AP(G, E)$, $U \in \mathscr{U}$, and U' be the corresponding neighborhood of 0 in $C^b(G \times G, E)$ defined in (4.1). Let $\{D_{a_1} f, \dots, D_{a_n} f\}$ and $\{D_{b_1} f, \dots, D_{b_n} f\}$ be U'-meshes of $\{D_a f : a \in G\} \subset C^b(G \times G, E)$, both having the least cardinal number n among all U'-meshes. Then*

$$\frac{1}{n}\sum_{k=1}^{n} D_{a_k} f - \frac{1}{n}\sum_{k=1}^{n} D_{b_k} f \in (U + U)'. \tag{4.3}$$

Proof. Applying Lemma 4.3 to the U'-meshes $\{D_{a_1} f, \dots, D_{a_n} f\}$ and $\{D_{b_1} f, \dots, D_{b_n} f\}$ of $\{D_a f : a \in G\}$, we find an injective map $\sigma\colon \{1, \dots, n\} \longrightarrow \{1, \dots, n\}$ such that

$$D_{a_k} f - D_{b_{\sigma(k)}} f \in U' + U' \qquad (k = 1, \dots, n).$$

Multiplying these relations by $1/n$ and adding over $k = 1, \dots, n$, we obtain

$$\sum_{k=1}^{n}\left(\frac{1}{n} D_{a_k} f - \frac{1}{n} D_{b_{\sigma(k)}} f\right) \in \frac{1}{n}(U' + U') + \cdots + \frac{1}{n}(U' + U') = U' + U' \subset (U + U)',$$

where the equality on the right side has followed from properties of convex sets and the fact that $U' + U'$ is convex. Since σ is a permutation of $\{1, \dots, n\}$, (4.3) follows. □

Lemma 4.5. *Let $f \in AP(G, E)$, $U \in \mathscr{U}$, and U' be the corresponding neighborhood of 0 in $C^b(G \times G, E)$. Let $\{D_{a_1} f, \dots, D_{a_n} f\}$ be a U'-mesh of $\{D_a f : a \in G\}$, having the least cardinal number n. Then*

$$\frac{1}{n}\sum_{k=1}^{n} f(a_k) - \frac{1}{n}\sum_{k=1}^{n} D_{a_k} f \in (U + U)', \tag{4.4}$$

where we regard the first sum on the left side as a constant function on $G \times G$.

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Proof. Let $c, d \in G$ be arbitrary. We claim that $\{D_{ca_1d}f, \dots, D_{ca_nd}f\}$ is a U'-mesh of $\{D_a f : a \in G\}$. To prove this, using the assumption that $\{D_{a_1}f, \dots, D_{a_n}f\}$ is a U'-mesh, we find that given $a \in G$, there is some $1 \le k \le n$ such that

$$D_{c^{-1}ad^{-1}}f - D_{a_k}f \in U'. \tag{4.5}$$

Now, for $x, y \in G$, letting $x' = xc$ and $y' = dy$, we get

$$\begin{aligned} D_a f(x,y) - D_{ca_kd}f(x,y) &= f(xay) - f(xca_kdy) \\ &= f(x'c^{-1}ad^{-1}y') - f(x'a_ky') \\ \text{by (4.5)} \quad &= D_{c^{-1}ad^{-1}}(x',y') - D_{a_k}f(x',y') \in U. \end{aligned}$$

Since $x, y \in G$ are arbitrary, $D_a f - D_{ca_kd}f \in U'$, proving that $\{D_{ca_1d}f, \dots, D_{ca_nd}f\}$ is a U'-mesh of $\{D_a f : a \in G\}$.

For $\{D_{a_k}f\}$ and $\{D_{b_k}f\}$ with $b_k = ca_kd$, computing the function in (4.3), at the point $(e, e) \in G \times G$, we obtain

$$\frac{1}{n}\sum_{k=1}^{n} f(a_k) - \frac{1}{n}\sum_{k=1}^{n} f(ca_kd) \in U + U.$$

Since $(c, d) \in G \times G$ is arbitrary, we find

$$\frac{1}{n}\sum_{k=1}^{n} f(a_k) - \frac{1}{n}\sum_{k=1}^{n} D_{a_k}f \in (U + U)',$$

which is what we needed to show. □

We are now ready to prove the existence of an invariant mean for $f \in AP(G, E)$.

Theorem 4.6. *Let G be a topological group and E be a quasi-complete space. If $f \in AP(G, E)$, there exists a unique vector $M(f) \in E$ (the mean of f) such that for every $U \in \mathscr{U}$, there is a sequence $\{a_1, \dots, a_n\}$ in G, for which*

$$M(f) - \frac{1}{n}\sum_{k=1}^{n} f(xa_ky) \in U \qquad \textit{for all } x, y \in G. \tag{4.6}$$

Moreover, $M(f) \in \overline{ch}(\mathfrak{L}_f) \cap \overline{ch}(\mathfrak{R}_f)$ (the intersection of closed convex hulls of $\mathfrak{L}_f$ and $\mathfrak{R}_f$).

Proof. Let $f \in AP(G, E)$ and $U \in \mathscr{U}$ be given. Let $ch(f(G))$ be the convex hull of $f(G)$ in E, and let E_U be the set of all vectors $u \in ch(f(G))$, such that for some sequence $c_1, \dots, c_p$ in G,

$$u - \frac{1}{p}\sum_{j=1}^{p} f(xc_jy) \in U \qquad \text{for all } x, y \in G. \tag{4.7}$$

If we choose $V \in \mathscr{U}$ such that $V + V \subset U$, then applying Lemma 4.5 to the function $f \in AP(G, E)$ and the neighborhood V, we find that $E_U \neq \varnothing$. Suppose that $u_1, u_2 \in E_U$, so that for suitable c_j and

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d_k in G:

$$u_1 - \frac{1}{p}\sum_{j=1}^{p} f(xc_j y) \in U \qquad \text{for all } x, y \in G, \tag{4.8}$$

$$u_2 - \frac{1}{q}\sum_{k=1}^{q} f(xd_j y) \in U \qquad \text{for all } x, y \in G. \tag{4.9}$$

Setting $x = e$ and $y = d_k$ in (4.8), adding over $k = 1, \ldots, q$ and dividing by q we obtain

$$u_1 - \frac{1}{pq}\sum_{k=1}^{q}\sum_{j=1}^{p} f(c_j d_k) \in U.$$

Setting $y = e$ and $x = c_j$ in (4.9), we obtain similarly

$$u_2 - \frac{1}{pq}\sum_{j=1}^{p}\sum_{k=1}^{q} f(c_j d_k) \in U.$$

Hence

$$u_1 - u_2 \in U + U. \tag{4.10}$$

Since f is almost periodic, $\mathfrak{R}_f$ is totally bounded. By evaluating the functions in $\mathfrak{R}_f$ at $x = e$, we find that $f(G) \subset E$ must be totally bounded. Therefore $\overline{ch(f(G))}$ is a compact subset of E since E is quasi-complete. Thus $\overline{E_U}$ is a compact subset of E.

If $U_1, U_2 \in \mathscr{U}$, $U_1 \subset U_2$, then obviously $E_{U_1} \subset E_{U_2}$, and so $\overline{E_{U_1}} \subset \overline{E_{U_2}}$. Thus the relation

$$\overline{E_{U_1}} \cap \cdots \cap \overline{E_{U_n}} \supset \overline{E_{U_1 \cap \cdots \cap U_n}} \neq \varnothing,$$

shows that $\bigcap_{U \in \mathscr{U}} \overline{E_U} \neq \varnothing$. Now, the relation (4.10) shows that $\bigcap_{U \in \mathscr{U}} \overline{E_U}$ contains exactly one point. In fact, if $v_1, v_2 \in \bigcap_{U \in \mathscr{U}} \overline{E_U}$, then for every $U \in \mathscr{U}$, $v_1, v_2 \in \overline{E_U}$, and thus there are $u_1, u_2 \in E_U$ such that

$$u_1 - v_1 \in U, \quad u_2 - v_2 \in U.$$

Therefore,

$$v_1 - v_2 = v_1 - u_1 + u_1 - u_2 + u_2 - v_2 \in U + U + U + U,$$

and since U is arbitrary, $v_1 = v_2$.

Let the point in $\bigcap_{U \in \mathscr{U}} \overline{E_U}$ be denoted by $M(f)$. If $U, V \in \mathscr{U}$ are such that $U + U \subset V$, then $\overline{E_U} \subset E_V$. To see this, let $u \in \overline{E_U}$ and find $u_0 \in E_U$ such that $u - u_0 \in U$. Since $u_0 \in E_U$, there are $c_1, \ldots, c_p \in G$ such that

$$u_0 - \frac{1}{p}\sum_{j=1}^{p} f(xc_j y) \in U \quad \text{for all } x, y \in G.$$

Thus

$$u - \frac{1}{p}\sum_{j=1}^{p} f(xc_j y) = u - u_0 + u_0 - \frac{1}{p}\sum_{j=1}^{p} f(xc_j y) \in U + U \subset V,$$

thus $u \in E_V$. It follows that $M(f)$ lies in all E_U $(U \in \mathscr{U})$, and therefore $M(f)$ is an invariant mean of f.

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Next we show the uniqueness of $M(f)$. Suppose $M(f)' \in E$ is another invariant mean of f. Let $U \in \mathscr{U}$, and choose $V \in \mathscr{U}$ such that $V + V \subset U$. The characteristic property of invariant mean in (4.6), implies that there are $c_1, \ldots, c_n$, $d_1, \ldots, d_m$ in G such that for all $x \in G$:

$$M(f) - \frac{1}{n}\sum_{i=1}^{n} f(c_i x) \in V \tag{4.11}$$

$$M(f)' - \frac{1}{m}\sum_{j=1}^{m} f(x d_j) \in V. \tag{4.12}$$

We multiply (4.11) by $1/m$, replace x with xd_j and add over $j = 1, \ldots, m$, to obtain

$$M(f) - \frac{1}{nm}\sum_{j=1}^{m}\sum_{i=1}^{n} f(c_i x d_j) \in V.$$

Similarly, We multiply (4.12) by $1/n$, replace x with $c_i x$ and add over $i = 1, \ldots, n$, to obtain

$$M(f)' - \frac{1}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m} f(c_i x d_j) \in V.$$

Subtracting the above expressions,

$$M(f) - M(f)' \in V + V \subset U.$$

Since U is arbitrary, $M(f) = M(f)'$.

The fact that $M(f) \in \overline{ch}(\mathfrak{L}_f) \cap \overline{ch}(\mathfrak{R}_f)$ is immediate from (4.6).

□

For completeness we summarize the properties of almost periodic invariant means in quasi-complete spaces. The proof which is similar to the proof of [2, Theorem 16, p. 29] is omitted.

Theorem 4.7. *Let G be a Hausdorff topological group, E a quasi-complete space, and $M\colon AP(G,E) \longrightarrow E$, $f \mapsto M(f)$, be the mean defined in Theorem 4.6. Then:*

(i) *M is linear.*

(ii) *M is two-sided invariant: $M(L_a f) = M(R_a f) = M(f)$, for all $a \in G$.*

(iii) *If $f(x) = u$ $(u \in E)$ is a constant function, then $M(f) = u$.*

(iv) *If $\check{f}(x) = f(x^{-1})$, then $M(\check{f}) = M(f)$.*

(v) *If $U \in \mathscr{U}$, $\|M(f)\|_U \le m(\|f\|_U) \le \|f\|_{U'}$, where m is the scalar-valued almost periodic invariant mean.*

(vi) *If $U \in \mathscr{U}$, and $f - g \in U'$, then $M(f - g) \in 2U$.*

(vii) *M is the unique left [or right] translation invariant, linear map satisfying properties* (iii) *and* (vi).

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