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CENTRALIZER NEARRINGS

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Abstract. Suppose that $(G, +)$ is a group (possibly nonabelian) and that X is a submonoid of the monoid of all endomorphisms of G under the operation of composition of functions, $(\mathrm{End}(G), \circ)$. We define the X-centralizer nearring of G by X by saying that $M_X(G) := \{f : G \to G \mid f(0_G) = 0_G \text{ and } f \circ \alpha = \alpha \circ f \text{ for all } \alpha \in X\}$. This set of functions, $M_X(G)$, is a nearring under the "usual" operations of function "addition" and "composition" of functions. This paper investigates how centralizer nearrings can be defined and investigates their ideals when X is a group of automorphisms.

1. Introduction

In this paper we are always assuming that $(G, +)$ is a group (possibly nonabelian). We let 0_G denote the identity of G and we let $G^* := G \setminus \{0_G\}$. In general in this paper X denotes a submonoid of the monoid of endomorphisms of the group G under the operation of composition of functions, $(\mathrm{End}(G), \circ)$. In some cases we need to assume a little more, namely that $X \leq \mathrm{Aut}(G)$, the group of automorphisms of the group G. We make it clear when this assumption applies.

Suppose that g is an element of the group G. We let T_g denote the function $T_g : G \to G$ defined by $T_g(x) = g^{-1}xg$ for all $x \in G$. For each $g \in G, T_g$ is an automorphism of G. It is called an inner automorphism of G. The inner automorphism group of G is defined by $\mathrm{Inn}(G) := \{T_g \mid g \in G\}$. It is a normal subgroup of $\mathrm{Aut}(G)$.

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If G is a group and $x \in G$, we define the centralizer of x in G to be $C_G(x) := \{g \in G \mid x+g = g+x\}$ and the center of G to be $Z(G) := \bigcap_{g \in G} C_G(g)$, the set of all elements of G which commute with all the elements of G.

We use the following definition.

Definition 1.1. We define $M_X(G) := \{f : G \to G \mid f(0_G) = 0_G \text{ and } f \circ \alpha = \alpha \circ f \text{ for all } \alpha \in X\}$.

It is well-known that $(M_X(G), +, \circ)$ is a nearring using the "usual" operations of function "addition" and "composition" of functions. The elements of $M_X(G)$ are just functions from G to G that are zero-preserving. We call $M_X(G)$ an X-centralizer nearring or just a centralizer nearring when X is clear.

It can be seen that every nearing with identity is isomorphic to $M_X(G)$ for some G and X [4, Theorem 14.3]. For more information on nearrings, see the books [4, 9], and [10].

The notation used in this paper is standard. If X and Y are sets, we use $X \subseteq Y$ to mean that "X is a subset of Y". We use the notation $X \leq Y$ to mean that "X and Y are groups and X is a subgroup of Y". If $H \leq G$ we use $H \lhd G$ to mean that "H is a normal subgroup of G".

2. An equivalence relation

The following equivalence relation is important in what follows. It helps to determine what the images of elements of $M_X(G)$ can be.

We let R_X denote the unique smallest equivalence relation on G which contains the relation $T = \{(x, \alpha(x)) \mid x \in G, \alpha \in X\}$. It follows that $(x, y) \in R_X$ if and only if for some positive integer n, there exist $g_0, g_1, \ldots, g_{n-1}, g_n \in G$ and $\alpha_1, \beta_1, \ldots, \alpha_n, \beta_n \in X$ so that $x = g_0, y = g_n$ and $\alpha_i(g_{i-1}) = \beta_i(g_i)$ for $i = 1, 2 \ldots, n$. It is clear that if $X \leq \mathrm{Aut}(G)$, then R_X must equal T.

The next few definitions are important in determining the elements of $M_X(G)$. Note the lemma.

Lemma 2.1. *If $f \in M_X(G)$, $x, y \in G$ and xR_Xy, then $f(x)R_Xf(y)$.*

Proof. Since xR_Xy, there exist $g_0, g_1, \ldots, g_n \in G$ and $\alpha_1, \beta_1, \ldots, \alpha_n, \beta_n \in X$ so that $x = g_0, y = g_n$ and $\alpha_i(g_{i-1}) = \beta_i(g_i)$ for $i = 1, 2, \ldots, n$. It follows that $f(x) = f(g_0), f(y) = f(g_n)$ and $\alpha_i(f(g_{i-1})) = f(\alpha_i(g_{i-1})) = f(\beta_i(g_i)) = \beta_i(f(g_i))$ for $i = 1, 2, \ldots, n$. It follows that $f(x)R_Xf(y)$. □

Thus, the elements of $M_X(G)$ must fix the equivalence classes of R_X. The next sequence of lemmas is important in what follows.

Definition 2.2. We say that a subset $P \subseteq G$ is a P-pointer set for X in G provided for all non-zero $g \in G$ there is an $x \in P$ and $\theta \in X$ so that $\theta(x) = g$.

Note that if we have $X \leq \mathrm{Aut}(G)$, we can pick a pointer set for X in G, by picking an element from each R_X-equivalence class and indeed, in this case, all the pointer sets arise in this fashion.

Definition 2.3. Suppose that G is a group and X is a submonoid of
$(\mathrm{End}(G), \circ)$.

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(1) For all $x \in G, \alpha \in X$ define
 i) $L_X(\alpha; x) := \{\beta \in X \mid \alpha(x) = \beta(x)\}$ and
 ii) $C_X(\alpha; x) := \{y \in G \mid \text{ for all } \beta \in L_X(\alpha; x)\ \alpha(y) = \beta(y)\}$.

(2) For all $x \in G$, define $F_X(x) := \bigcap_{\alpha \in X} C_X(\alpha; x)$.

Note that since $C_X(\alpha; x) \leq G$ for all $x \in G$, we have that $F_X(x) \leq G$ for all $x \in G$. The next lemma shows the importance of the above definition.

Lemma 2.4. *Suppose that $f \in M_X(G)$. Then, for all $x \in G, f(x) \in F_X(x)$.*

Proof. Suppose that $\alpha \in X$. We need to prove that for all $x \in G$, $f(x) \in C_X(\alpha; x)$. Thus, suppose that $\beta \in L(\alpha; x)$. Then, $\alpha(x) = \beta(x)$, and thus

$$\begin{aligned}\alpha(f(x)) &= f(\alpha(x)) \\ &= f(\beta(x)) = \beta(f(x)).\end{aligned}$$

Hence, $f(x) \in C_X(\alpha; x)$, as required. Thus, $f(x) \in F_X(x)$. □

A few comments are in order.

Lemma 2.5. *If $X \subseteq Y$ are both submonoids of End(G), then for all $x \in G$ we have $F_Y(x) \leq F_X(x)$*

Proof. Since $X \subseteq Y$, it is clear that for all $x \in G, \alpha \in X$ we have $L_X(\alpha; x) \subseteq L_Y(\alpha, x)$ and that $C_Y(\alpha; x) \leq C_X(\alpha; x)$. It follows that $F_Y(x) \leq F_X(x)$. □

The next lemma is very useful.

Lemma 2.6. *Suppose that $X \leq$ Aut(G). Then for all $x \in G$, $F_X(x) = \bigcap_{\beta \in C_X(x)} C_G(\beta)$.*

Proof. Let $\alpha \in X$. Now

$$\begin{aligned}L_X(\alpha; x) &= \{\beta \in X \mid \alpha(x) = \beta(x)\} \\ &= \{\beta \in X \mid \alpha^{-1}\beta(x) = x\} \\ &= \{\beta \in X \mid \alpha^{-1}\beta \in C_X(x)\} \\ &= \alpha C_X(x)\end{aligned}$$

and $C_X(\alpha; x) = \bigcap_{\beta \in \alpha C_X(x)} C_G(\alpha^{-1}\beta)$. So for all $x \in G$

$$\begin{aligned}F_X(x) &= \bigcap_{\alpha \in X} C_X(\alpha; x) \\ &= \bigcap_{\alpha \in X} \bigcap_{\beta \in \alpha C_X(x)} C_G(\alpha^{-1}\beta) \\ &= \bigcap_{\gamma \in C_X(x)} C_G(\gamma),\end{aligned}$$

as required. □

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The next lemma considers the special case that $X = \mathrm{Inn}(G)$.

Lemma 2.7. *Suppose that $X = Inn(G)$. Then, $F_X(x) = Z(C_G(x))$ for all $x \in G$.*

Proof. From the above lemma we have that

$$\begin{aligned} F_X(x) &= \bigcap_{T_g \in C_X(x)} C_G(T_g) \\ &= \bigcap_{g \in C_G(x)} C_G(g) \text{ which since } x \in C_G(x) \text{ is contained in } C_G(x). \end{aligned}$$

It follows that $F_X(x) = Z(C_G(x))$. Since for every $g \in C_G(x)$ we have $Z(C_G(x)) \le C_G(g)$ and every $z \in F_X(x)$ must commute with every $g \in C_G(x)$. □

If the pointer set for X in G is $P = \{x\}$, then the elements of $M_X(G)$ depend only on the elements of $F_X(x)$. We want to extend this fact to be able to consider the cases where the X-pointer sets of G have more points. The following definition is what is needed.

Definition 2.8. Suppose that G is a group and that X is a submonoid of $(\mathrm{End}(G), \circ)$.

(1) For all $x_1, x_2 \in G, \alpha \in X$ define
  i) $L_X(\alpha; x_1, x_2) := \{\beta \in X \mid \alpha(x_1) = \beta(x_2)\}$ and
  ii) $C_X(\alpha; x_1, x_2\} := \{(y_1, y_2) \in F_X(x_1) \times F_X(x_2) \mid$
  for all $\beta \in L_X(\alpha; x_1, x_2)$ we have $\alpha(y_1) = \beta(y_2)\}$.
(2) For all $x_1, x_2 \in G$ define $F_X(x_1, x_2) := \bigcap_{\alpha \in X} C_X(\alpha; x_1, x_2)$.
(3) For all $x_1, x_2, \dots, x_m \in G$ we define $F_X(x_1, x_2, \dots, x_m) :=$
  $\{(c_1, c_2, \dots, c_m) \in F_X(x_1) \times F_X(x_2) \times \dots \times F_X(x_m) \mid$ for all
  $1 \le i < j \le m$ we have $(c_i, c_j) \in F_X(x_i, x_j)\}$.

Note that if there is no $\beta \in X$ so that $\beta(x_2) = \alpha(x_1)$, then $L_X(\alpha; x_1, x_2) = \emptyset$ and we must have $C_X(\alpha; x_1, x_2) = F_X(x_1) \times F_X(x_2) = F_X(x_1, x_2)$. The next result is similar to Lemma 2.4.

Lemma 2.9. *Suppose that $f \in M_X(G)$. Then, for all $x_1, x_2, \dots, x_m \in G$ we have $(f(x_1), f(x_2), \dots, f(x_m)) \in F_X(x_1, x_2, \dots, x_m)$.*

Proof. Suppose that for $1 \le i \le m \ \ x_i \in G$, then by Lemma 2.4, $f(x_i) \in F_X(x_i)$. Now suppose that $1 \le i < j \le m$. If $\beta \in L(\alpha; x_i, x_j)$
as in Lemma 2.4 we have

$$\begin{aligned} \alpha(f(x_i)) &= f(\alpha(x_i)) \\ &= f(\beta(x_j)) \\ &= \beta(f(x_j)). \end{aligned}$$

It follows that $(f(x_i), f(x_j)) \in F_X(\alpha; x_i, x_j)$, as required □

The next result is a generalization of Betsch's Theorem [9, Lemma 3.3].

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Theorem 2.10 (Generalization of Betsch's Theorem)**.** *Suppose that $P = \{x_1, x_2, \ldots, x_m\}$ is an X-pointer set for G. Then, for each $(c_1, c_2, \ldots, c_m) \in F_X(x_1, x_2, \ldots, x_m)$, there is a unique $f \in M_X(G)$ so that for all $1 \leq i \leq m$, $f(x_i) = c_i$.*

Proof. (uniqueness) Suppose that $f, g \in M_X(G)$ and for each $i = 1, 2, \ldots, m$
$f(x_i) = c_i = g(x_i)$. Let $z \in G$. Since P is an X-pointer set for G, there is an $x_k \in P$ and $\theta \in X$ so that $\theta(x_k) = z$. It follows that

$$\begin{aligned} f(z) &= f(\theta(x_k)) = \theta(f(x_k)) = \theta(c_k) \\ &= \theta(g(x_k)) = g(\theta(x_k)) = g(z). \end{aligned}$$

Thus, $f = g$, as required.
(existence) Suppose that $(c_1, c_2, \ldots, c_m) \in F_X(x_1, x_2, \ldots, x_m)$ and define a function $f[c_1, c_2, \ldots, c_m] : G \to G$ by

$$f[c_1, c_2, \ldots, c_m](z) = \begin{cases} 0_G & \text{if } z = 0_G \\ \theta(c_j) & \text{if } \theta(x_j) = z \text{ for some } x_j \in P, \theta \in X. \end{cases}$$

First, we need to show that $f[c_1, c_2, \ldots, c_m]$ is well-defined. Thus, we suppose that there are $\theta_1, \theta_2 \in X$ and $x_i, x_j \in P$ so that

$$\theta_1(x_i) = \theta_2(x_j) = z.$$

Now if $i < j$, as $\theta_1(x_i) = \theta_2(x_j)$, we have $\theta_2 \in L_X(\theta_1; x_i, x_j)$. Since $(c_i, c_j) \in C_X(\theta_1; x_i, x_j)$, we must have $\theta_1(c_i) = \theta_2(c_j)$, as required.

Similarly, if $i = j$, then as $\theta_1(x_i) = \theta_2(x_i)$, we must have $\theta_2 \in L_X(\theta_1, x_i)$. Again since $c_i \in C_X(\theta_1; x_i)$, we must have $\theta_1(c_i) = \theta_2(c_i)$, as required.

It follows that $f[c_1, c_2, \ldots, c_m]$ is well-defined.

Next we want to show that $f[c_1, c_2, \ldots, c_m] \in M_X(G)$.Thus, let $\beta \in X$ and $z \in G$. Now pick $\alpha \in X$ so that $\alpha(x_k) = z$ for some $x_k \in P$. It follows that

$$\begin{aligned} f[c_1, c_2, \ldots, c_m](\beta(z)) &= f[c_1, c_2, \ldots, c_m](\beta(\alpha(x_k))) \\ &= f[c_1, c_2, \ldots, c_m]((\beta \circ \alpha)(x_k)) \\ &= (\beta \circ \alpha)(c_k) = \beta(\alpha(c_k)) = \beta(f(z)) \end{aligned}$$

and hence, $f[c_1, c_2, \ldots, c_m] \in M_X(G)$.

Since $\mathrm{Id}_G \in X$ and for $i = 1, 2, \ldots, m$ $\mathrm{Id}_G(x_i) = x_i$ we have
$f[c_1, c_2, \ldots, c_m](x_i) = \mathrm{Id}_G(c_i) = c_i$, as required. □

Corollary 2.11. *Let P be an X-pointer set for G. Then,*

$$f \in M_X(G) \Leftrightarrow f = f[f(x_1), f(x_2), \ldots, f(x_m)].$$

In particular
$M_X(G) = \{f[c_1, c_2, \ldots, c_m] \mid (c_1, c_2, \ldots, c_m) \in F_X(x_1, x_2, \ldots, x_m)\}$.

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In the case that $X \leq \mathrm{Aut}(G)$ we pick an X-pointer set for G by picking exactly one element from each R_X-eqivalence class of G, as was stated above Definition 2.3.

The next result is an application of the above result.

Theorem 2.12. *Let $E := End(G)$ where $G = \langle x\rangle \times D$ and $|x| = exp(G)$. Then, $M_E(G) = \langle Id_G\rangle$.*

Proof. It is easy to see that $P = \{x\}$ is an E-pointer set for G. Thus, we only need to consider $F_E(x)$. Now $F_E(x) \geq \langle x\rangle$ so we can write $F_E(x) = \langle x\rangle \times C$ where $C = D \cap F_E(x)$.

If $C \neq \{0_G\}$, pick $0_G \neq c \in C$ and define $\alpha \in E$ by $\alpha(x) = x$ and $\alpha\mid_D = 0_G$. Now $\mathrm{Id}_G \in L_E(\alpha; x)$, so since $c \in F_E(x)$ we have $0_G = \alpha(c) = \mathrm{Id}_G(c) = c$. It follows that $F_E(x) = \langle x\rangle$ and that $M_E(G) = \langle f[x]\rangle = \langle \mathrm{Id}_G\rangle$, as required. □

Corollary 2.13. *Let G be a finite abelian group, $E = End(G)$ and suppose $e = exp(G)$. Then, $M_E(G) = \langle Id_G\rangle = \mathbb{Z}_e$.*

From the above corollary (Corollary 2.13) we see that $M_E(\mathbb{Z}_{10} \times \mathbb{Z}_2)$ is isomorphic to $(\mathbb{Z}_{10}, +, \cdot)$ which is a ring. Below we consider $M_A(\mathbb{Z}_{10} \times \mathbb{Z}_2)$ where $A := \mathrm{Aut}(\mathbb{Z}_{10} \times \mathbb{Z}_2)$. For convenience let $G := \mathbb{Z}_{10} \times \mathbb{Z}_2$.

Since G has 12 elements of order 10 and 3 elements of order 2, it is easy to see that $|\mathrm{Aut}(G)| = 24$. One can map $(1, 0)$ to any element of order 10 and there are then two possible elements of order 2 to map $(0, 1)$.

It is easy to see that $P = \{(1, 0), (2, 0), (5, 0)\}$ is an A-pointer set for G. Considering all the possibilities one can see that $F_A((1, 0)) = \langle(1, 0)\rangle$,
$F_A((2, 0) = \langle(2, 0)\rangle, F_A((5, 0)) = \langle(5, 0)\rangle$. It follows that
$|M_A(G)| = 10 \cdot 5 \cdot 2 = 100$.

Clearly, $f[c_1, c_2, c_3] \circ f[(1, 0), (2, 0), (5, 0)] = f[c_1, c_2, c_3]$, so
$f[(1, 0), (2, 0), (5, 0)]$ is a right identity for $M_X(G)$. However,
$f[(1, 0), (2, 0), (5, 0)] \circ f[(4, 0), (4, 0), (0, 0)] = f[(7, 0), (4, 0), (0, 0)]$ so
$f[(1, 0), (2, 0), (5, 0)]$ is not an identity of $M_A(G)$.
Of course if $I := \mathrm{Inn}(G)$, then for $x \in G$, we would have $F_I(x) = G$ and $M_I(G) = \{f : G \to G \mid f(0_G) = 0_G\}$ which has order 10^9.

3. Automorphic centralizer nearrings

This section is concerned mainly with automorphic centralizer nearrings. This is the case when we have $X \leq \mathrm{Aut}(G)$. The next lemma is useful.

Lemma 3.1. *Suppose $X \leq Aut(G)$. If $\alpha \in X$, then $F_X(\alpha(x)) \geq \alpha(F_X(x))$.*

Proof. Now

$$\begin{aligned}\beta \in C_X(\alpha(x)) &\Leftrightarrow \beta(\alpha(x)) = \alpha(x)\\ &\Leftrightarrow \alpha^{-1}\beta\alpha(x) = x\\ &\Leftrightarrow \alpha^{-1}\beta\alpha \in C_X(x)\end{aligned}$$

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and

$$\begin{aligned} w \in C_G(\alpha^{-1}\beta\alpha) &\Leftrightarrow \alpha^{-1}\beta\alpha(w) = w \\ &\Leftrightarrow \beta\alpha(w) = \alpha(w) \\ &\Leftrightarrow \alpha(w) \in C_G(\beta) \\ &\Leftrightarrow w \in \alpha^{-1}(C_G(\beta)) \end{aligned}$$

It follows that $\alpha(C_G(\alpha^{-1}\beta\alpha)) = C_G(\beta)$. Hence, using Lemma 2.6

$$\begin{aligned} F_X(\alpha(x)) &= \bigcap_{\beta \in C_X(\alpha(x))} C_G(\beta) \\ &= \bigcap_{\alpha^{-1}\beta\alpha \in C_X(x)} \alpha(C_G(\alpha^{-1}\beta\alpha)) \\ &= \alpha(\bigcap_{\alpha^{-1}\beta\alpha \in C_X(x)} C_B(\alpha^{-1}\beta\alpha)) \\ &\geq \alpha(\bigcap_{\gamma \in C_X(x)} C_G(\gamma)) = \alpha(F_X(x)) \end{aligned}$$

□

Now we want to specfically determine the elements of $M_X(G)$. To begin we find elements of G, $\{x_1, x_2, \cdots, x_n\}$, so that

$$G^* = \text{ orb}_X(x_1) \dot\cup \text{ orb}_X(x_2) \dot\cup \cdots \dot\cup \text{ orb}_X(x_n).$$

Thus, we have a partition of the nonidentity elements of G. Since $X \leq \text{Aut}(G)$, $P = \{x_1, x_2, \ldots, x_n\}$ is an X-pointer set for G.

Recall that by Corollary 2.11 $M_X(G) = \{f[c_1, c_2, \ldots, c_n] \mid (c_1, c_2, \ldots, c_n) \in F_X(x_1, x_2, \ldots, x_n)\}$. (We know that every element of $M_X(G)$ must take 0_G to 0_G.)

It is easy to see that $f[c_1, c_2, \cdots, c_n] + f[d_1, d_2, \ldots, d_n] = f[c_1 + d_1, \ldots, c_n + d_n]$. Hence, $(M_X(G), +)$ is an abelian group provided $\text{Inn}(G) \leq X \leq \text{Aut}(G)$ (See Lemma 2.7.).

Now we go back to Lemma 3.1. Let $\alpha \in X$ and suppose that $x \in \text{ orb}_X(x_i)$, then $\alpha(x) \in \text{ orb}_X(x_i)$. It follows that in the definition of $F_X[c_1, c_2, \cdots, c_n]$ we may replace the "x_i" by "$\alpha(x)$" or by "x". It then is easy to see that $|F_X(\alpha(x))| = |F_X(x)|$. Now using Lemma 3.1 we see that

Lemma 3.2. *Suppose that $\alpha \in X \leq Aut(G)$, then $F_X(\alpha(x)) = \alpha(F_X(x))$.*

Here are some examples.

Example 3.3. Let $G = \mathbb{Z}_p^n$ be an elementary abelian p-group of order p^n and suppose that $X = \text{Aut}(G)$. Suppose that $1 \neq x \in G$. Then if $G = \langle x \rangle$, $M_X(G) = \langle x \rangle$. Otherwise, for all $y \in G \setminus \langle x \rangle$, then we can write $G = \langle x \rangle \times \langle y \rangle \times E$ for some $E \leq G$. There is an $\alpha \in X$ so that $\alpha(x) = x$ and $\alpha(y) = x + y \neq y$. It follows that $y \notin M_X(x)$. Since y was arbitrary, not in $\langle x \rangle$, it follows that $F_X(x) = \langle x \rangle$.

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It is clear that $G^* = \text{orb}_X(x)$ and hence that, using the notation of Theorem 2.10, $M_X(G) = \{f[ix] \mid 0 \leq i < p\} = \mathbb{Z}_p$. The last equality is as rings.

Example 3.4. Let $G = Q_8 = \langle x, y \mid x^4 = 1, x^2 = y^2, y^{-1}xy = x^{-1}\rangle$, the quaternion group of order 8.

a) Suppose $X = \text{Aut}(G)$. In this case $G^* = \text{orb}_X(x^2)\dot\cup\, \text{orb}_X(x), F_X(x^2) = \langle x^2\rangle$, and $F_X(x) = \langle x\rangle$. It follows that $|M_X(G)| = 8$.

b) Suppose that $X = \text{Inn}(G)$.
Now $G^* = \text{orb}_X(x^2)\dot\cup\, \text{orb}_X(x)\dot\cup\, \text{orb}_X(y)\dot\cup\, \text{orb}_X(xy)$,
$F_X(x^2) = \langle x^2\rangle, F_X(x) = \langle x\rangle, F_X(y) = \langle y\rangle, F_X(xy) = \langle xy\rangle$. It follows that $|M_X(G)| = 2{\cdot}4{\cdot}4{\cdot}4 = 2^7 = 128$.

Example 3.5. Suppose that $G = S_5$, the symmetric group on $\{1, 2, 3, 4, 5\}$. In this case we let $X = \text{Inn}(G) = \text{Aut}(G)$. It follows from Lemma 2.7 that for each $x \in G$, $F_X(x) = Z(C_G(x))$.

nos.	x	$C_G(x)$	$M_X(G) = Z(C_G(x))$
A	(12)	$\mathbb{Z}_2 \times S_3$	$\mathbb{Z}_2$
B	(12)(34)	D_8	$\mathbb{Z}_2$
C	(123)	$\mathbb{Z}_3 \times \mathbb{Z}_2$	$\mathbb{Z}_6$
D	(123)(45)	$\mathbb{Z}_3 \times \mathbb{Z}_2$	$\mathbb{Z}_6$
E	(1234)	$\mathbb{Z}_4$	$\mathbb{Z}_4$
F	(12345)	$\mathbb{Z}_5$	$\mathbb{Z}_5$

We know that $G^* = \text{orb}_X((12))\dot\cup\, \text{orb}_X((12)(34))\dot\cup\, \text{orb}_X((123))\dot\cup$
$\text{orb}_X((123)(45))\dot\cup\, \text{orb}_X((1234))\dot\cup\, \text{orb}_X((12345))$. It follows that
$|M_X(S_5)| = 2 \cdot 2 \cdot 6 \cdot 6 \cdot 4 \cdot 5 = 2880$.

We will discuss this example more in the next section.

4. Ideals of centralizer nearrings

Supose that G is a group and $X \leq \text{Aut}(G)$. Suppose that $x_1, x_2, \ldots x_n \in G$ and $G^* = \text{orb}_X(x_1)\dot\cup\, \text{orb}_X(x_2)\dot\cup \cdots \dot\cup\, \text{orb}_X(x_n)$. We define a graph whose vertices are the set $V = \{1, 2, \ldots, n\}$ where two distinct vertices i and j are connected by an (undirected) edge provided that either $F_X(x_i) \cap \text{orb}_X(x_j) \neq \emptyset$ or $\text{orb}_X(x_i) \cap F_X(x_j) \neq \emptyset$. Let E be the set of edges. The graph is (V, E). Now let $\{C_1, C_2, \ldots, C_r\}$ be the set of connected components of this graph.

Let C be a connected component of the graph V, above and define

$$I[C] = \{f[c_1, c_2, \ldots, c_n] \mid (c_1, c_2, \ldots, c_n) \in F_X(x_1, x_2, \ldots, x_n)$$
$$\text{and for all } i = 1, 2, \ldots, n \text{ we have } c_i = 0 \text{ provided } i \notin C\}.$$

We claim that $I[C]$ is an ideal of $M_X(G)$.

Lemma 4.1. *Suppose $Inn(G) \leq X \leq Aut(G)$ and that C is a connected component of the graph (V, E). Then, $I[C]$ is an ideal of $M_X(G)$.*

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Proof. We need to verify the conditions for $I[C]$ to be an ideal.

i) Since $(M_X(G), +)$ is an abelian group, it is clear that $(I[C], +) \lhd (M_X(G), +)$.

ii) Let $g = f[c_1, c_2, \ldots, c_n], h = f[d_1, d_2, \ldots, d_n] \in M_X(G)$ and suppose that $j = f[e_1, e_2, \ldots, e_n] \in I[C]$. Now we let $k = g(h + j) - gh$. We need to show that $k \in I[C]$. We do this by showing that if $i \notin C$, then $k(x_i) = 0$. So we have

$$\begin{aligned} k(x_i) &= (g(h + j) - gh)(x_i) \\ &= g(h + j)(x_i) - gh(x_i) \\ &= g((h + j)(x_i)) - g(h(x_i)) \\ &= g(h(x_i) + j(x_i)) - g(h(x_i)) \\ &= g(h(x_i) + 0) - g(h(x_i)) \text{ since } i \notin C \\ &= g(h(x_i)) - g(h(x_i)) = 0 \end{aligned}$$

Thus, indeed $k = g(h + j) - gh \in I[C]$ whenever $j \in I[C], g, h \in M_X(G)$.

iii) Suppose that $k = jh$ where $j = f[c_1, c_2, \ldots c_n] \in I[C]$ and $h = f[d_1, d_2, \ldots, d_n] \in M_X(G)$ and suppose that $i \notin C$.

$$\begin{aligned} k(x_i) &= j(h(x_i)) \\ &= j(d_i) \\ &[\text{ Now } d_i \in F_X(x_i). \\ &\text{and } d_i \in \text{ orb}_X(x_{i'}) \text{ for some } i' \in V. \\ \text{Now, if } i = i'. &\text{ then } i' \notin C \text{ and if } i \neq i', \\ &\text{then } i \text{ and } i' \\ &\text{form an edge. Thus, in either case } i' \notin C \text{ and so }] \\ &= \alpha(0) = 0 \text{ where } \alpha(x_{i'}) = d_i \text{ for some } \alpha \in X. \end{aligned}$$

It follows that $k \in I[C]$ and hence, $I[C]M_X(G) \leq M_X(G)$.

It follows that $I[C]$ is an ideal of $M_X(G)$. □

Theorem 4.2. *Suppose Inn$(G) \leq X \leq$ Aut(G). Let (V, E) be the graph defined above and suppose that $\{C_1, C_2, \ldots, C_r\}$ is the set of connected components of the graph (V, E). Then,*

$$M_X(G) = I[C_1] \oplus I[C_2] \oplus \cdots \oplus I[C_r].$$

Proof. The proof is clear. □

Here are a few facts. Suppose that G is a group, $\text{Inn}(G) \leq X \leq \text{Aut}(G)$ and that (V, E) is the graph as defined above.

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i) If (V, E) is not a connected graph, then $M_X(G)$ is not a simple nearring. Indeed, it is decomposable as a direct sum of ideals.

ii) If $G^* = \text{ orb}_X(x)$, then all the nonidentity elements of G have the same order, which must be a prime, and G must be abelian (Since the center is a characteristic subgroup of G.). It follows that G is an elementary abelian p-group for some prime p. Now Example 3.3 implies that (in this case) if $X = \text{Aut}(G)$, then $M_X(G)$ is isomrphic to $\mathbb{Z}_p$ which is a simple ring.

iii) Returning to Example 3.5, we can see that the connected components of the graph (V, E) are $\{\{A, D, C\}, \{B, E\}, \{F\}\}$. Hence, we get

$$M_X(S_5) = I[A, D, C] \oplus I[B, E] \oplus I[F]$$

where $I[A, D, C]$ is an ideal of order $2 \cdot 6 \cdot 6 = 72$, $I[B, E]$ is an ideal of order $2 \cdot 4 = 8$, and $I[F]$ is an ideal of order 5.

One final remark.

Lemma 4.3. *Suppose that G is a finite group so that*

(i) $Inn(G) \leq X \leq Aut(G)$ and

(ii) There is $x \in G$, so that $|x| = p$, p a prime and $C_G(x) = \langle x \rangle$

Then, $C = \text{ orb}_X(x)$ is a connected component of the graph (V, E). In particular, if $M_X(G)$ is a simple nearring, then $G = \langle x \rangle$ is a cyclic group of order p and $M_X(G) = \langle Id_G \rangle$.

Proof. Suppose that $y \notin C$ and (y, x^i) is an edge. it follows that there is an $\alpha \in X$ so that either (i) $\alpha(y) \in F_X(x^i)$ or (ii) $\alpha(x^i) \in F_X(y)$.
In case (i) $\alpha(y) \in F_X(x^i) \leq Z(C_G(x^i)) = \langle x \rangle$. It follows that $y \in C$, a contradiction.
In case (ii) $\alpha(x^i) \in F_X(y) \leq Z(C_G(y))$. Thus, $y \in C_G(x^i) = C_G(x) = \langle x \rangle$. Again this is a contradiction.

It follows that C is a connected component of the graph (V, E). The result follows. □

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